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ROTUND RENORMINGS IN SPACES OF BOCHNER INTEGRABLE FUNCTIONS

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Dedicated to Václav Zizler on the occasion of his 70th birthday

Abstract. We show that if μ is a probability measure and X is a Banach space, then the space Lebesgue-Bochner space $L^1(\mu, X)$ admits an equivalent norm which is rotund (uniformly rotund in every direction, locally uniformly rotund, or midpoint locally uniformly rotund) if X does. We also prove that if X admits a uniformly rotund norm, then the space $L^1(\mu, X)$ has an equivalent norm whose restriction to every reflexive subspace is uniformly rotund. This is done via the Luxemburg norm associated to a suitable Orlicz function.

1. Introduction and statement of the results

In the paper [14], Smith and Turett proved that if μ is a probability measure and X is a Banach space, then the canonical norm of the Lebesgue-Bochner function space $L^p(\mu, X)$ inherits the classical properties of convexity of the norm of X whenever $1 < p < +\infty$. Some results of the same nature for various smoothness properties were previously established in [9] and [10]. We notice that, since the canonical norm of the space $L^1(\mu)$ is not rotund, neither is it smooth, and this space isometrically embeds into $L^1(\mu, X)$, the analogues of the aforementioned results for $p = 1$ do not make sense. However, it is possible to lift some geometrical properties of the space X to $L^1(\mu, X)$ under a renorming. In [5], we constructed an equivalent norm (namely, an Orlicz-Bochner norm) on $L^1(\mu, X)$ which preserves several properties of smoothness of the space X. The purpose of this work is to show that a renorming of this type also permits to transfer the relevant properties of rotundity of the norm of X to the space $L^1(\mu, X)$.

Our notation is standard and can be found, for instance, in [2] and [3]. From now on, let $(X, \|\cdot\|)$ be a real Banach space. The symbols S_X and B_X stand for the unit sphere and the closed unit ball in it, respectively. If (Ω, Σ, μ) is a probability space, we denote by $L^1(\Omega, \Sigma, \mu; X)$ or simply by $L^1(\mu, X)$ the Banach space made up of all (equivalence classes of) Bochner integrable functions $f : \Omega \longrightarrow X$, endowed with the norm $\|f\|_{L^1(\mu,X)} = \int_\Omega \|f(\omega)\| \mathrm{d}\mu(\omega)$. The norm $\|\cdot\|$ is called *rotund (R)* if the unit sphere S_X does not contain any non-degenerate segment, that is, if $x = y$ whenever $x, y \in X$ and $\|(x+y)/2\| = \|x\| = \|y\|$. The norm $\|\cdot\|$ is called *uniformly rotund (UR)* if, for any two sequences (x_n) and (y_n)

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in S_X such that

$$\lim_{n\to\infty} \|x_n + y_n\| = 2$$

we have $\lim_{n\to\infty} \|x_n - y_n\| = 0$. If $\lim_{n\to\infty} \|x_n - x\| = 0$ for every $x \in S_X$ and every sequence (x_n) in S_X such that

$$\lim_{n\to\infty} \|x_n + x\| = 2,$$

we say that $\|\cdot\|$ is *locally uniformly rotund (LUR)*. The norm $\|\cdot\|$ is called *midpoint locally uniformly rotund (MLUR)* if for each $x \in S_X$ and any two sequences (u_n) and (v_n) in S_X such that

$$\lim_{n\to\infty} \|(u_n + v_n)/2 - x\| = 0,$$

we have $\lim_{n\to\infty} \|u_n - v_n\| = 0$. Finally, we say that $\|\cdot\|$ is *uniformly rotund in every direction (URED)* if, for any two sequences (x_n) and (y_n) in X and every $z \in X$ such that

$$\lim_{n\to\infty} \|x_n\| = 1, \quad \lim_{n\to\infty} \|y_n\| = 1, \quad \lim_{n\to\infty} \|x_n + y_n\| = 2 \text{ and } x_n - y_n = \mathbb{R}z \text{ for all } n \in \mathbb{N},$$

we have $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

It is easy to see that UR $\Rightarrow$ LUR $\Rightarrow$ MLUR $\Rightarrow$ R and that UR $\Rightarrow$ URED $\Rightarrow$ R. It is well known (see e.g. [2, Section IV.4]) that a Banach space admits an equivalent UR norm if, and only if, X is super-reflexive. The class of spaces that admit an equivalent LUR norm is very large. It includes in particular weakly countably determined spaces (see e.g. [2, Section VII.1]). Therefore, separable spaces, reflexive spaces, and spaces of the types $c_0(\Gamma)$ and $L^1(\mu)$, where Γ is a set and μ is a probability measure, are LUR renormable. An easy convexity argument (see e.g. [2, Theorem II.6.8]) yields that a Banach space X has an equivalent URED norm if there exist a Hilbert space H and a bounded, linear and one-to-one operator $T : X \to H$. Thus, every separable space, the dual of every separable space and the long James space $J(\eta)$ (for every ordinal η) admit renormings of this kind. Another important example of Banach space with an equivalent URED norm is $L^1(\mu)$, where μ is a probability measure (c.f. [2, Theorem II.7.16]).

The converses of the implications above do not hold in general, even up to renorming. The first example of a MLUR space with no equivalent LUR renorming (a space of continuous functions on a scattered compact space) was provided in [6]. In [1] it was shown that the space ℓ^∞, which admits an equivalent URED norm (being the dual of a separable space), does not have any equivalent MLUR renorming. A LUR renormable space with no equivalent URED norm is $c_0(\Gamma)$, where Γ is any uncountable set (c.f. [2, Prop. II.7.9]). Another example of LUR space that admits no equivalent URED renorming (a nonseparable reflexive space with an unconditional basis) was constructed in [7] (c.f. [2, Theorem IV.6.5]). For a detailed information on rotundity properties, and the applications of these notions to the Geometry of Banach spaces we refer to the monographs [2] and [4].

In this paper, we show that the properties of rotundity, uniform rotundity in every direction, locally uniform rotundity, and midpoint locally uniform rotundity of the norm on X lift to $L^1(\mu, X)$ under a suitable renorming. More precisely, we have the following result.

Theorem 1. *Let (Ω, Σ, μ) be a probability space and let $(X, \|\cdot\|)$ be a Banach space. Then, there exists an equivalent norm $|||\cdot|||$ on $L^1(\mu, X)$ (dependent on $\|\cdot\|$), such that, if $\|\cdot\|$ is rotund, URED, LUR, or MLUR, then so is $|||\cdot|||$, respectively.*

Since the (non-reflexive) space $L^1([0,1])$ does not admit any uniformly rotund norm, an analogue of the theorem above for the property of uniform rotundity can not be achieved. We have, however, the following result.

Theorem 2. *Let (Ω, Σ, μ) be a probability space and let $(X, \|\cdot\|)$ be a (superreflexive) Banach space. If $\|\cdot\|$ is uniformly rotund, then the norm $|||\cdot|||$ on $L^1(\mu, X)$ constructed in Theorem 1 is such that its restriction to every reflexive subspace of it is uniformly rotund.*

We notice that the LUR case of Theorem 1 was previously established in [13] by a different method. The approach employed there was based on the technique of Troyanski and Zizler about LUR renorming in Banach spaces with projectional resolutions of the identity; cf. [2, Chapter VII]. Our method provides an explicit formula for the new norm, namely, an Orlicz-Bochner norm. Moreover, this construction preserves the lattice structure whenever X is a Banach lattice (see [5, Remark 2.6]). Further, from the former theorems and our previous results in [5] we can deduce that, if the norm $\|\cdot\|$ of the space space X has simultaneously any of the rotundity properties considered above and some property of smoothness, then the Orlicz-Bochner norm on $L^1(\mu, X)$ is such that this norm, or the restriction of it to the reflexive subspaces of $L^1(\mu, X)$, shares the same properties of rotundity and smoothness, and no extra averaging process is needed to get the combination of such properties. This fact applies in particular if X is super-reflexive or if X is Asplund and simultaneously weakly compactly generated.

2. Proofs

As we mentioned before, the renorming in the theorems above is the Orlicz-Bochner norm associated to a suitable Orlicz function and the norm on X with the relevant properties of rotundity. In what follows, we shall consider an Orlicz function $M : \mathbb{R} \longrightarrow [0, +\infty)$, that is, M is a convex, even function such that M increases on $[0, +\infty)$, $M(t) \longrightarrow +\infty$ whenever $t \to +\infty$ and $M(0) = 0$. Further, we shall assume that M is Lipschitzian and twice differentiable, and that $\lim\limits_{t\to+\infty} t^2 M''(t) \in (0, +\infty)$. Examples of such functions are

$$M_1(t) = |t| - \log(1 + |t|), \;\; t \in \mathbb{R}$$

and

$$M_2(t) = \int_0^{|t|} \arctan(s)\,\mathrm{d}s, \;\; t \in \mathbb{R}.$$

An argument like in [8] (inequality (7) and Lemma 6) yields that, if M is an Orlicz function satisfying the properties above, then

$$M(\alpha t) \geq \alpha^2 M(t) \quad \text{for all} \quad 0 < \alpha < 1 \text{ and all } t \in \mathbb{R}, \tag{2.1}$$

and

$$M(a) + M(b) - 2M\left(\frac{a+b}{2}\right) \geq \frac{1}{4} M''(\max\{a,b\})(b-a)^2 \quad \text{for all} \quad a, b \geq 0. \tag{2.2}$$

Now, consider the set $B = \left\{f \in L^1(\mu, X) : \int_\Omega M(\|f(\omega)\|)\mathrm{d}\mu(\omega) \leq 1\right\}$. From the properties of M it follows that B is convex, closed, bounded, symmetric and has non-empty interior. In particular, the Minkowski functional of B, that is, the function

$$|||f||| = \inf\left\{\rho > 0 : \int_\Omega M\left(\|f(\omega)/\rho\|\right)\mathrm{d}\mu(\omega) \leq 1\right\}, \quad f \in L^1(\mu, X) \tag{2.3}$$

defines a norm on $L^1(\mu, X)$, the Luxemburg norm associated to M and $\|\cdot\|$. In [5, page 250], it is shown that, if C is the Lipschitz constant of M, then

$$\frac{1}{C}|||f||| \leq \|f\|_{L^1(\mu,X)} \leq \frac{2M'(1)+1}{M'(1)}|||f||| \quad \text{for all} \quad h \in L^1(\mu, X). \tag{2.4}$$

In particular, $|||\cdot|||$ is an equivalent norm on $L^1(\mu, X)$. We notice that if $f \in L^1(\mu, X)$, then $|||f||| = 1$ if, and only if, $\int_\Omega M(\|f(\omega)\|)\mathrm{d}\mu(\omega) = 1$. As an easy consequence of this fact and inequality (2.1) it follows that if (f_n) is a sequence in $L^1(\mu, X)$ such that $|||f_n||| \leq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} |||f_n||| = 1$, then $\lim_{n\to\infty} \int_\Omega M(\|f_n(\omega)\|)\mathrm{d}\mu(\omega) = 1$.

We are ready to prove our theorems.

Proof of Theorem 1. We shall prove that the norm given by equality (2.3) satisfies the required properties. First, assume that the norm $\|\cdot\|$ on X is rotund. We shall show that $|||\cdot|||$ is rotund as well. Pick any $f, g \in L^1(\mu, X)$ such that $|||f||| = |||g||| = \frac{1}{2}|||f+g|||$. The convexity and monotonicity of M, together with inequality (2.2) yield

$$\begin{aligned}
0 &= \int_\Omega M\big(\|f(\omega)\|\big)\mathrm{d}\mu(\omega) + \int_\Omega M\big(\|g(\omega)\|\big)\mathrm{d}\mu(\omega) - 2\int_\Omega M\Big(\frac{\|f(\omega)+g(\omega)\|}{2}\Big)\mathrm{d}\mu(\omega) \\
&= \int_\Omega \Big[M\big(\|f(\omega)\|\big) + M\big(\|g(\omega)\|\big) - 2M\Big(\frac{\|f(\omega)\|+\|g(\omega)\|}{2}\Big)\Big]\mathrm{d}\mu(\omega) \\
&\quad +2\int_\Omega \Big[M\Big(\frac{\|f(\omega)\|+\|g(\omega)\|}{2}\Big) - M\Big(\frac{\|f(\omega)+g(\omega)\|}{2}\Big)\Big]\mathrm{d}\mu(\omega) \\
&\geq \frac{1}{4}\int_\Omega M''(\max\{\|f(\omega)\|, \|g(\omega)\|\})\cdot\big(\|f(\omega)\| - \|g(\omega)\|\big)^2\mathrm{d}\mu(\omega) \\
&\quad +2\int_\Omega \Big[M\Big(\frac{\|f(\omega)\|+\|g(\omega)\|}{2}\Big) - M\Big(\frac{\|f(\omega)+g(\omega)\|}{2}\Big)\Big]\mathrm{d}\mu(\omega).
\end{aligned}$$

Since the integrands in the last member of the inequality above are both non-negative,

$$M''(\max\{\|f(\omega)\|, \|g(\omega)\|\})\cdot\big(\|f(\omega)\| - \|g(\omega)\|\big)^2 = 0$$

and

$$M\Big(\frac{\|f(\omega)\|+\|g(\omega)\|}{2}\Big)-M\Big(\frac{\|f(\omega)+g(\omega)\|}{2}\Big)=0$$

for almost all $\omega\in\Omega$. Bearing in mind that M'' does not vanish and that M is strictly increasing on $[0,+\infty)$ we can conclude that

$$\|f(\omega)\|=\|g(\omega)\| \quad \text{and} \quad \|f(\omega)\|+\|g(\omega)\|=\|f(\omega)+g(\omega)\|$$

for almost all $\omega\in\Omega$. These equalities and the rotundity of the norm $\|\cdot\|$ guarantee that $f(\omega)=g(\omega)$ for almost all $\omega\in\Omega$. Thus, the norm $|||\cdot|||$ is rotund.

Now, suppose that the norm $\|\cdot\|$ on X is URED. Assume that $h\in L^1(\mu,X)$ is given and that (f_n), (g_n) are two sequences in $L^1(\mu,X)$ such that $\lim_{n\to\infty}|||f_n|||=\lim_{n\to\infty}|||g_n|||=1$, $\lim_{n\to\infty}|||f_n+g_n|||=2$, and that $f_n-g_n\in\mathbb{R}h$ for every $\mathbb{N}$. Since M is Lipschitzian, from (2.4) it easily follows that $\int_\Omega M\big(\|f_n(\omega)\|\big)\mathrm{d}\mu(\omega)\longrightarrow 1$, $\int_\Omega M\big(\|g_n(\omega)\|\big)\mathrm{d}\mu(\omega)\longrightarrow 1$ and $\int_\Omega M\Big(\frac{\|f_n(\omega)+g_n(\omega)\|}{2}\Big)\mathrm{d}\mu(\omega)\longrightarrow 1$ as $n\to\infty$. If $h(\cdot)=0$ almost everywhere, there is nothing to prove. From now on assume that $\mu\big(\{\omega\in\Omega:\ h(\omega)\neq 0\}\big)>0$; then, necessarily, there is $\eta>0$ such that the set $\{\omega\in\Omega:\ \|h(\omega)\|>\eta\}$ has positive measure; denote this set by Ω_0. For every $n\in\mathbb{N}$ find a suitable $\lambda_n\in\mathbb{R}$ such that $f_n-g_n=\lambda_n h$. We have to show that $\lim_{n\to\infty}\lambda_n=0$. Assume this is not the case. Then there exists $\varepsilon>0$ such that $|\lambda_n|>\varepsilon$ (for simplicity) for every $n\in\mathbb{N}$.

Bearing in mind that $\int_\Omega M(\|u(\omega)\|)\mathrm{d}\mu(\omega)=1$ whenever $u\in L^1(\mu,X)$ and $|||u|||=1$, we get

$$\begin{aligned}
0 &= \int_\Omega\left[M(\|f_n(\omega)\|)+M(\|g_n(\omega)\|)-2M\left(\frac{\|f_n(\omega)+g_n(\omega)\|}{|||f_n+g_n|||}\right)\right]\mathrm{d}\mu(\omega)\\
&= \int_\Omega\left[M(\|f_n(\omega)\|)+M(\|g_n(\omega)\|)-2M\left(\frac{\|f_n(\omega)\|+\|g_n(\omega)\|}{2}\right)\right]\mathrm{d}\mu(\omega)\\
&\quad +2\int_\Omega\left[M\left(\frac{\|f_n(\omega)\|+\|g_n(\omega)\|}{2}\right)-M\left(\frac{\|f_n(\omega)+g_n(\omega)\|}{2}\right)\right]\mathrm{d}\mu(\omega)\\
&\quad +2\int_\Omega\left[M\left(\frac{\|f_n(\omega)+g_n(\omega)\|}{2}\right)-M\left(\frac{\|f_n(\omega)+g_n(\omega)\|}{|||f_n+g_n|||}\right)\right]\mathrm{d}\mu(\omega).
\end{aligned}\tag{2.5}$$

For every $n\in\mathbb{N}$ and every $\omega\in\Omega$ we put

$$a_n(\omega):=M(\|f_n(\omega)\|)+M(\|g_n(\omega)\|)-2M\left(\frac{\|f_n(\omega)\|+\|g_n(\omega)\|}{2}\right)$$

and

$$b_n(\omega):=M\left(\frac{\|f_n(\omega)+g_n(\omega)\|}{2}\right)-M\left(\frac{\|f_n(\omega)+g_n(\omega)\|}{|||f_n+g_n|||}\right).$$

According to inequality (2.2) we have for all $n\in\mathbb{N}$

$$a_n(\omega)\geq\frac{1}{4}M''(\max\{\|f_n(\omega)\|,\|g_n(\omega)\|\})(\|f_n(\omega)\|-\|g_n(\omega)\|)^2 \quad \text{for all} \quad \omega\in\Omega.$$

and integrating in this inequality, we obtain

$$\int_\Omega a_n(\omega)\mathrm{d}\mu(\omega) \geq \frac{1}{4}\int_\Omega M''\left(\max\{\|f_n(\omega)\|,\|g_n(\omega)\|\}\right)\cdot\left(\|f_n(\omega)\|-\|g_n(\omega)\|\right)^2 \mathrm{d}\mu(\omega) \tag{2.6}$$

On the other hand, as M is Lipschitz we get

$$b_n(\omega) \geq -C(\|f_n(\omega)\|+\|g_n(\omega)\|)\left|\frac{1}{|||f_n+g_n|||}-\frac{1}{2}\right|$$

for some constant $C>0$, for all $n\in\mathbb{N}$ and all $\omega\in\Omega$. Thus,

$$\int_\Omega b_n(\omega)\mathrm{d}\mu(\omega) \geq -C\left|\frac{1}{|||f_n+g_n|||}-\frac{1}{2}\right|\int_\Omega(\|f_n(\omega)\|+\|g_n(\omega)\|)\mathrm{d}\mu(\omega). \tag{2.7}$$

Plugging (2.6) and (2.7) in (2.5) we get

$$\begin{aligned} 0 \;\geq\; & \frac{1}{4}\int_\Omega M''\left(\max\{\|f_n(\omega)\|,\|g_n(\omega)\|\}\right)\cdot\left(\|f_n(\omega)\|-\|g_n(\omega)\|\right)^2 \mathrm{d}\mu(\omega) \\ & +2\int_\Omega\left[M\left(\frac{\|f_n(\omega)\|+\|g_n(\omega)\|}{2}\right)-M\left(\frac{\|f_n(\omega)+g_n(\omega)\|}{2}\right)\right]\mathrm{d}\mu(\omega) \\ & -C\left|\frac{1}{|||f_n+g_n|||}-\frac{1}{2}\right|\int_\Omega\left(\|f_n(\omega)\|+\|g_n(\omega)\|\right)\mathrm{d}\mu(\omega). \end{aligned}$$

But, by our assumption and (2.4), we have

$$\left|\frac{1}{|||f_n+g_n|||}-\frac{1}{2}\right|\int_\Omega\left(\|f_n(\omega)\|+\|g_n(\omega)\|\right)\mathrm{d}\mu(\omega) \longrightarrow 0 \quad \text{as} \quad n\to\infty.$$

Hence, taking into account that the two first integrals in the last member of the former inequality are non-negative, it follows that

$$\int_\Omega M''\left(\max\{\|f_n(\omega)\|,\|g_n(\omega)\|\}\right)\cdot\left(\|f_n(\omega)\|-\|g_n(\omega\|\right)^2 \mathrm{d}\mu(\omega) \longrightarrow 0 \quad \text{as} \quad n\to\infty \tag{2.8}$$

and

$$\int_\Omega\left[M\left(\frac{\|f_n(\omega)\|+\|g_n(\omega)\|}{2}\right)-M\left(\frac{\|f_n(\omega)+g_n(\omega)\|}{2}\right)\right]\mathrm{d}\mu(\omega) \longrightarrow 0 \quad \text{as} \quad n\to\infty\,. \tag{2.9}$$

Now, a standard trick applied to the convergences above yields an infinite set $N\subset\mathbb{N}$ such that for almost all $\omega\in\Omega$,

$$M''\left(\max\{\|f_n(\omega)\|,\|g_n(\omega)\|\}\right)\cdot\left(\|f_n(\omega)\|-\|g_n(\omega\|\right)^2 \longrightarrow 0 \quad \text{as} \quad N\ni n\to\infty \tag{2.10}$$

and

$$M\left(\frac{\|f_n(\omega)\|+\|g_n(\omega)\|}{2}\right)-M\left(\frac{\|f_n(\omega)+g_n(\omega)\|}{2}\right) \longrightarrow 0 \quad \text{as} \quad N\ni n\to\infty\,. \tag{2.11}$$

Further, using Fatou's lemma, we have

$$\begin{aligned} & \int_\Omega \liminf_{N\ni n\to\infty}\left(M\left(\|f_n(\omega)\|\right)+M\left(\|g_n(\omega)\|\right)\right)\mathrm{d}\mu(\omega) \\ \leq\; & \liminf_{N\ni n\to\infty}\int_\Omega\left(M\left(\|f_n(\omega)\|\right)+M\left(\|g_n(\omega)\|\right)\right)\mathrm{d}\mu(\omega)=2. \end{aligned}$$

Thus $\liminf_{N\ni n\to\infty} [M(\|f_n(\omega)\|) + M(\|g_n(\omega)\|)] < +\infty$, and hence also

$$\liminf_{N\ni n\to\infty} (\|f_n(\omega)\| + \|g_n(\omega)\|) < +\infty \tag{2.12}$$

for almost all $\omega \in \Omega$. Now, fix an $\omega \in \Omega_0$ such that (2.10), (2.11) and (2.12) hold. From (2.12) we can find an infinite set $N_1 \subset N$ such that the limit $\lim_{N_1\ni n\to\infty} (\|f_n(\omega)\| + \|g_n(\omega)\|)$ exists and is equal to $\liminf_{N\ni n\to\infty} (\|f_n(\omega)\| + \|g_n(\omega)\|)$. Thus, $\sup_{n\in N_1} (\|f_n(\omega)\| + \|g_n(\omega)\|) =: c < +\infty$. Then (2.10) implies that

$$M''(c)\big(\|f_n(\omega)\| - \|g_n(\omega)\|\big)^2 \longrightarrow 0 \quad \text{as} \quad N_1 \ni n \to \infty,$$

and hence, bearing in mind that $M''(c) > 0$, we get $\lim_{N_1\ni n\to\infty} (\|f_n(\omega)\| - \|g_n(\omega)\|) = 0$. Find now an infinite set $N_2 \subset N_1$ such that the limit $\lim_{N_2\ni n\to\infty} \|f_n(\omega)\| =: a$ exists; note that $a < +\infty$ and that $\lim_{N_2\ni n\to\infty} \|g_n(\omega)\| = a$. Then, (2.11) yields that $M\left(\frac{1}{2}\|f_n(\omega) + g_n(\omega)\|\right) \longrightarrow M(a)$ as $N_2 \ni n \to \infty$, and since M is increasing and continuous on the interval $[0, +\infty)$ it follows that $\lim_{N_2\ni n\to\infty} \|f_n(\omega) + g_n(\omega)\| = 2a$. If $a = 0$, we would have that $0 = \lim_{n\to\infty} \|f_n(\omega) - g_n(\omega)\| = \lim_{n\to\infty} |\lambda_n| \|h(\omega)\| \geq \eta\varepsilon > 0$, a contradiction; hence $a > 0$. Now, putting $x_n = f_n(\omega)/a,\ y_n = g_n(\omega)/a$ for $n \in N_2$, we get that $\|x_n\| \longrightarrow 1,\ \|y_n\| \longrightarrow 1$ and $\|x_n + y_n\| \longrightarrow 2$ as $N_2 \ni n \to \infty$, and $x_n - y_n \in \mathbb{R}\, h(\omega)$ for every $n \in \mathbb{N}$. Then the URED property of the norm $\|\cdot\|$ implies that

$$\|x_n - y_n\| \longrightarrow 0, \quad \text{and so} \quad \|f_n(\omega) - g_n(\omega)\| \longrightarrow 0 \quad \text{as} \quad N_2 \ni n \to \infty.$$

But, simultaneously, we know that $\|f_n(\omega) - g_n(\omega)\| = |\lambda_n| \|h(\omega)\| > \varepsilon\eta > 0$ for every $n \in \mathbb{N}$; again a contradiction. Consequently, the norm $|||\cdot|||$ is URED.

Next, assume that the norm $\|\cdot\|$ on X is LUR. Consider any $f \in L^1(\mu, X)$ and any sequence $(f_n) \subset L^1(\mu, X)$ such that $|||f_n||| = |||f||| = 1$ for all $n \in \mathbb{N}$ and $|||f + f_n||| \longrightarrow 2$ as $n \to \infty$. We have to show that $|||f_n - f||| \longrightarrow 0$ as $n \to \infty$. Assume that this is not so. Then there exists $\varepsilon > 0$ so that $|||f_n - f||| > \varepsilon$ (for simplicity) for every $n \in \mathbb{N}$. Then, using inequality (2.1) we obtain

$$\begin{aligned}
\int_\Omega M\left(\frac{\|f_n(\omega) - f(\omega)\|}{2}\right) \mathrm{d}\mu(\omega) &= \int_\Omega M\left(\frac{|||f_n - f|||}{2} \cdot \frac{\|f_n(\omega) - f(\omega)\|}{|||f_n - f|||}\right) \mathrm{d}\mu(\omega) \\
&\geq \int_\Omega \frac{1}{4} |||f_n - f|||^2 M\left(\frac{\|f_n(\omega) - f(\omega)\|}{|||f_n - f|||}\right) \mathrm{d}\mu(\omega) \\
&= \frac{1}{4} |||f_n - f|||^2 > \frac{\varepsilon^2}{4}
\end{aligned} \tag{2.13}$$

for each $n \in \mathbb{N}$. On the other hand, using the general assertions (2.10) and (2.11) for the URED case, where g_n is replaced by f, we have for almost all $\omega \in \Omega$,

$$M''(\max\{\|f(\omega)\|, \|f_n(\omega)\|\}) \cdot (\|f(\omega)\| - \|f_n(\omega\|)^2 \longrightarrow 0 \ \text{ as } \ N \ni n \to \infty$$

and

$$M\left(\frac{\|f(\omega)\|+\|f_n(\omega)\|}{2}\right)-M\left(\frac{\|f(\omega)+f_n(\omega)\|}{2}\right)\longrightarrow 0 \ \text{ as } \ N\ni n\to\infty \tag{2.14}$$

where N is a suitable fixed infinite subset of $\mathbb{N}$. Fix for a while any $\omega\in\Omega$ for which the convergences above occur. Let $\lambda(\omega)\in[0,+\infty]$ be an accumulation point of the sequence $(\|f_n(\omega)\|)_{n\in N}$. Assume that $\lambda(\omega)=+\infty$. Then, as $\lim_{t\to+\infty}M''(t)\,t^2\in(0,+\infty)$, we have

$$\begin{aligned} 0 &= \lim_{N_1\ni n\to\infty} M''\big(\max\{\|f(\omega)\|,\|f_n(\omega)\|\}\big)\cdot\big(\|f(\omega)\|-\|f_n(\omega\|\big)^2 \\ &= \lim_{N_1\ni n\to\infty} M''\big(\|f_n(\omega)\|\big)\cdot\big(\|f(\omega)\|-\|f_n(\omega)\|\big)^2=\lim_{t\to+\infty}M''(t)\big(\|f(\omega)\|-t\big)^2>0, \end{aligned}$$

where N_1 is a suitable infinite subset of N. We obtained a contradiction. Therefore $\lambda(\omega)$ must be a finite number. Then we get $M''\big(\max\{\|f(\omega)\|,\lambda(\omega)\}\big)\ \big(\|f(\omega)\|-\lambda(\omega)\big)^2=0$, and so $\lambda(\omega)=\|f(\omega)\|$. Consequently,

$$\|f_n(\omega)\|\longrightarrow\|f(\omega)\| \ \text{ as } \ N\ni n\to\infty \tag{2.15}$$

for almost all $\omega\in\Omega$. Thus, using (2.14), we obtain

$$M\left(\frac{\|f(\omega)+f_n(\omega)\|}{2}\right)\longrightarrow M\left(\|f(\omega)\|\right) \ \text{ as } \ N\ni n\to\infty.$$

But M is continuous and strictly increasing on $[0,+\infty)$, so

$$\|f(\omega)+f_n(\omega)\|\longrightarrow 2\|f(\omega)\| \ \text{ as } \ N\ni n\to\infty \ \text{ for almost every } \ \omega\in\Omega.$$

Since the norm $\|\cdot\|$ on X is LUR, from the latter convergence and (2.15) we deduce that

$$\|f_n(\omega)-f(\omega)\|\longrightarrow 0 \ \text{ as } \ N\ni n\to\infty \ \text{ for almost every } \ \omega\in\Omega. \tag{2.16}$$

It remains to perform a final attack. In what follows we shall use ideas from the proof of Vitali's convergence theorem (see e.g. [11, Exercise 12.9]). Put

$$\varphi_n(\omega):=M\big(\|f(\omega)\|\big)+M\big(\|f_n(\omega)\|\big)-2M\left(\frac{\|f_n(\omega)-f(\omega)\|}{2}\right),\quad \omega\in\Omega,\quad n\in N\,.$$

The convexity and monotonicity of M yields that

$$\varphi_n(\omega)\ge M\big(\|f(\omega)\|\big)+M\big(\|f_n(\omega)\|\big)-2M\left(\frac{\|f_n(\omega)\|+\|f(\omega)\|}{2}\right)\ge 0$$

for every $\omega\in\Omega$ and every $n\in N$. Further, (2.15) and (2.16) imply that $\varphi_n(\omega)\longrightarrow 2M\big(\|f(\omega)\|\big)$ as $N\ni n\to\infty$ for almost all $\omega\in\Omega$. Hence, by Fatou's lemma,

$$\begin{aligned} 2 &= \int_\Omega 2M\left(\|f(\omega)\|\right)\mathrm{d}\mu(\omega)\le\liminf_{N\ni n\to\infty}\int_\Omega\varphi_n(\omega)\mathrm{d}\mu(\omega) \\ &= 2-\limsup_{N\ni n\to\infty}\int_\Omega 2M\left(\frac{\|f_n(\omega)-f(\omega)\|}{2}\right)\mathrm{d}\mu(\omega), \end{aligned}$$

and therefore

$$\lim_{N\ni n\to\infty}\int_\Omega M\left(\frac{\|f_n(\omega)-f(\omega)\|}{2}\right)\mathrm{d}\mu(\omega)=0.$$

But this contradicts inequality (2.13). We thus proved that the norm $|||\cdot|||$ is LUR.

It remains to show that if the norm $\|\cdot\|$ on X is MLUR, then so is $|||\cdot|||$. Pick $h \in L^1(\mu, X)$ and two sequences (f_n), (g_n) in $L^1(\mu, X)$ such that $|||f_n||| = |||g_n||| = |||h||| = 1$ for all $n \in \mathbb{N}$ and $|||(f_n + g_n)/2 - h||| \longrightarrow 0$ as $n \to \infty$. Arguing by contradiction, suppose that there is $\varepsilon > 0$ such that $|||f_n - g_n||| > \varepsilon$ (for simplicity) for every $n \in \mathbb{N}$. Then, thanks to inequality (2.1), we have

$$\begin{aligned}\int_\Omega M\left(\frac{\|f_n(\omega) - g_n(\omega)\|}{2}\right) \mathrm{d}\mu(\omega) &= \int_\Omega M\left(|||(f_n - g_n)/2|||\frac{\|f_n(\omega) - g_n(\omega)\|}{|||f_n - g_n|||}\right) \mathrm{d}\mu(\omega) \\ &\geq |||(f_n - g_n)/2|||^2 > \varepsilon^2/4\end{aligned} \tag{2.17}$$

for all $n \in \mathbb{N}$.

Now, for $n \in \mathbb{N}$ set $h_n = \frac{1}{2}(f_n + g_n)$. We already know that the norm $|||\cdot|||$ is rotund. Hence $|||h - h_n||| \geq 1 - |||h_n||| > 0$ for all $n \in \mathbb{N}$. We may and do assume that $|||h_n - h||| < 1$ for every $n \in \mathbb{N}$. Bearing in mind that M is convex we deduce that

$$\int_\Omega M\left(\|h_n(\omega) - h(\omega)\|\right) \mathrm{d}\mu(\omega) \leq |||h_n - h||| \int_\Omega M\left(\frac{\|h_n(\omega) - h(\omega)\|}{|||h_n - h|||}\right) \mathrm{d}\mu(\omega) = |||h_n - h|||$$

for all $n \in \mathbb{N}$. Hence, using the fact that $|||h_n - h||| \longrightarrow 0$ as $n \to \infty$, it follows that

$$\int_\Omega M\left(\|h_n(\omega) - h(\omega)\|\right) \mathrm{d}\mu(\omega) \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Thus, there is an infinite set $N \subset \mathbb{N}$ such that

$$M\left(\|h_n(\omega) - h(\omega)\|\right) \longrightarrow 0 \quad \text{as} \quad N \ni n \to \infty,$$

for almost all $\omega \in \Omega$. Consequently

$$\|(f_n(\omega) + g_n(\omega))/2 - h(\omega)\| \longrightarrow 0 \quad \text{as} \quad N \ni n \to \infty, \tag{2.18}$$

for almost all $\omega \in \Omega$. For each $n \in N$ and each $\omega \in \Omega$ put

$$a_n(\omega) := M(\|f_n(\omega)\|) + M(\|h(\omega)\|) - 2M\left(\frac{\|f_n(\omega)\| + \|h(\omega)\|}{2}\right)$$

and

$$b_n(\omega) := M(\|g_n(\omega)\|) + M(\|h(\omega)\|) - 2M\left(\frac{\|g_n(\omega)\| + \|h(\omega)\|}{2}\right).$$

The convexity and monotonicity of M guarantees that $a_n(\omega) \geq 0,\ b_n(\omega) \geq 0$, and

$$M\left(\frac{\|(f_n(\omega)\| + \|h(\omega)\|}{2}\right) + M\left(\frac{\|g_n(\omega)\| + \|h(\omega)\|}{2}\right) \geq 2M\left(\frac{\|(f_n(\omega) + g_n(\omega)\| + 2\|h(\omega)\|}{4}\right)$$

for all $n \in N$ and all ω. Thus

$$\begin{aligned}0 &\leq \int_\Omega a_n(\omega)\mathrm{d}\mu(\omega) + \int_\Omega b_n(\omega)\mathrm{d}\mu(\omega) \\ &\leq 4 - 4\int_\Omega M\left(\frac{\|(f_n(\omega) + g_n(\omega)\| + 2\|h(\omega)\|}{4}\right) \mathrm{d}\mu(\omega)\end{aligned} \tag{2.19}$$

for all $n \in N$. On the other hand, from (2.18) we get

$$M\left(\frac{\|(f_n(\omega)+g_n(\omega)\|+2\|h(\omega)\|}{4}\right) \longrightarrow M(\|h(\omega)\|) \quad \text{as} \quad N \ni n \to \infty$$

for almost all $\omega \in \Omega$. Fatou's Lemma and (2.19) then yield that

$$(1=) \ \int_\Omega M\big(\|h(\omega)\|\big)\mathrm{d}\mu(\omega) \leq \liminf_{n\longrightarrow\infty} \int_\Omega M\Big(\frac{\|f_n(\omega)+g_n(\omega)\|+2\|h(\omega)\|}{4}\Big)\mathrm{d}\mu(\omega).$$

This equality and (2.19) imply that $\lim_{N\ni n\to\infty}\int_\Omega a_n(\omega)\mathrm{d}\mu(\omega) = \lim_{N\ni n\to\infty}\int_\Omega b_n(\omega) = 0$. Thus, there exists an infinite set $N_1 \subset N$ such that $a_n(\omega) \to 0$ and $b_n(\omega) \to 0$ as $N_1 \ni n \to \infty$ for almost all $\omega \in \Omega$, and by (2.2),

$$\lim_{N_1\ni n\to\infty} M''(\max\{\|f_n(\omega)\|, \|h(\omega)\|\})\left(\|f_n(\omega)\| - \|h(\omega)\|\right)^2 = 0$$

and

$$\lim_{N_1\ni n\to\infty} M''(\max\{\|g_n(\omega)\|, \|h(\omega)\|\})\left(\|g_n(\omega)\| - \|h(\omega)\|\right)^2 = 0.$$

Using the same argument as in the LUR case, we deduce that

$$\lim_{N_1\ni n\to\infty} \|f_n(\omega)\| = \lim_{N_1\ni n\to\infty} \|g_n(\omega)\| = \|h(\omega)\| \quad \text{for almost all} \quad \omega \in \Omega.$$

Since the norm $\|\cdot\|$ is MLUR, from these equalities and (2.18) we easily deduce that

$$\lim_{N_1\ni n\to\infty} \|f_n(\omega) - g_n(\omega)\| = 0 \quad \text{for almost all} \quad \omega \in \Omega.$$

Now, imitating again Vitali's argument, for each $n \in N$ and each $\omega \in \Omega$ we put

$$\varphi_n(\omega) := M(\|f_n(\omega)\|) + M(\|g_n(\omega)\| - 2M\left(\|f_n(\omega) - g_n(\omega)\|/2\right).$$

Then $\varphi_n(\omega) \geq 0$ for all $n \in N_1$ and $\lim_{N_1\ni n\to\infty}\varphi_n(\omega) = 2M(\|h(\omega)\|)$ for almost all $\omega \in \Omega$. Therefore, according to Fatou's lemma, we have

$$\begin{aligned}(2=) \ \int_\Omega 2M(\|h(\omega)\|)\mathrm{d}\mu(\omega) &\leq \liminf_{N_1\ni n\to\infty} \int_\Omega \varphi_n(\omega)\mathrm{d}\mu(\omega) \\ &= 2 - 2\limsup_{N_1\ni n\to\infty} \int_\Omega M\left(\|f_n(\omega) - g_n(\omega)\|/2\right)\mathrm{d}\mu(\omega).\end{aligned}$$

We thus arrived at the equality

$$\lim_{N_1\ni n\to\infty} \int_\Omega M\left(\|f_n(\omega) - g_n(\omega)\|/2\right)\mathrm{d}\mu(\omega) = 0,$$

which contradicts (2.17). Therefore, the norm $|||\cdot|||$ is MLUR, as we wanted to show. ☐

Now, we proceed with the proof of Theorem 2.

Proof of Theorem 2. On $L^1(\mu, X)$, we still consider the norm $|||\cdot|||$ constructed in the proof of Theorem 1. Let Y be a reflexive subspace of $L^1(\mu, X)$. Pick any two sequences (f_n) and (g_n) in $L^1(\mu, X)$ such that $|||f_n||| = |||g_n||| = 1$, $|||f_n+g_n||| \longrightarrow 2$ as $n \to \infty$, and that $f_n,\ g_n \in Y$ (or more generally that just $f_n - g_n \in Y$) for all $n \in \mathbb{N}$. Arguing by contradiction, assume that there is $\varepsilon > 0$ such that $|||f_n - g_n||| > \varepsilon$ (for simplicity) for all $n \in \mathbb{N}$. This inequality and (2.4) imply that

$$\int_\Omega ||f_n(\omega) - g_n(\omega)||\mathrm{d}\mu(\omega) > \tfrac{\varepsilon}{C}\,. \tag{2.20}$$

On the other hand, because of the reflexivity of Y it follows that the set $\{f_1 - g_1, f_2 - g_2, \ldots\}$ is uniformly integrable (see e.g. [3, p. 104]), that is, there is a $\delta > 0$ so small that

$$\int_A \|f_n(\omega) - g_n(\omega)\|\mathrm{d}\mu(\omega) < \tfrac{\varepsilon}{2C} \quad \text{whenever} \quad n \in \mathbb{N},\ A \in \Sigma, \quad \text{and} \quad \mu(A) \le \frac{2(M'(1)+1)}{M'(1)}\delta.$$

For $n \in \mathbb{N}$ put

$$A_n := \left\{\omega \in \Omega :\ \|f_n(\omega)\| + \|g_n(\omega)\| > \frac{1}{\delta}\right\}.$$

According to Chebyshev's inequality, from (2.4) it follows that

$$\begin{aligned}\mu(A_n) &\le \delta \int_{A_n} \left(\|f_n(\omega)\| + \|g_n(\omega)\|\right)\mathrm{d}\mu(\omega) \le \frac{2M'(1)+1)}{M'(1)}\left(|||f_n||| + |||g_n|||\right)\delta \\ &= \frac{2(2M'(1)+1)}{M'(1)}\delta.\end{aligned}$$

Thus,

$$\int_{A_n} \|f_n(\omega) - g_n(\omega)\|\mathrm{d}\mu(\omega) < \tfrac{\varepsilon}{2C} \quad \text{for each} \quad n \in \mathbb{N}\,. \tag{2.21}$$

Now, define

$$\tilde{f}_n(\omega) = f_n(\omega),\ \tilde{g}_n(\omega) = g_n(\omega) \text{ if } \omega \in \Omega \setminus A_n, \text{ and } \tilde{f}_n(\omega) = \tilde{g}_n(\omega) = 0 \text{ if } \omega \in A_n.$$

It is clear that $\tilde{f}_n, \tilde{g}_n \in L^1(\mu, X)$ for each $n \in \mathbb{N}$. Further, we observe that for all $n \in \mathbb{N}$ and for all $\omega \in \Omega$ we have

$$\begin{aligned} 0 &\le M\big(\tfrac{1}{2}(\|\tilde{f}_n(\omega)\| + \|\tilde{g}_n(\omega)\|)\big) - M\big(\tfrac{1}{2}\|\tilde{f}_n(\omega) + \tilde{g}_n(\omega)\|\big) \\ &\le M\big(\tfrac{1}{2}(\|f_n(\omega)\| + \|g_n(\omega)\|)\big) - M\big(\tfrac{1}{2}\|f_n(\omega) + g_n(\omega)\|\big) \end{aligned}$$

and

$$0 \le M''(\tfrac{1}{\delta})\big(\|\tilde{f}_n(\omega)\| - \|\tilde{g}_n(\omega)\|\big)^2 \le M''\big(\max\{\|f_n(\omega\|, \|g_n(\omega)\|\}\big)\big(f_n(\omega)\| - \|g_n(\omega)\|\big)^2.$$

Hence, taking into account (2.8) and (2.9) (which are at hand also in the UR case), we can find an infinite set $N \subset \mathbb{N}$ such that for almost all $\omega \in \Omega$ we have

$$M\big(\tfrac{1}{2}(\|\tilde{f}_n(\omega)\| + \|\tilde{g}_n(\omega)\|)\big) - M\big(\tfrac{1}{2}\|\tilde{f}_n(\omega) + \tilde{g}_n(\omega)\|\big) \longrightarrow 0 \quad \text{as} \quad N \ni n \to \infty$$

and

$$M''(\tfrac{1}{\delta})\big(\|\tilde{f}_n(\omega)\| - \|\tilde{g}_n(\omega)\|\big)^2 \longrightarrow 0 \quad \text{as} \quad N \ni n \to \infty,$$

and so

$$\|\tilde{f}_n(\omega)\| - \|\tilde{g}_n(\omega)\| \longrightarrow 0 \quad \text{as} \quad N \ni n \to \infty$$

Thus, for almost all $\omega \in \Omega$ we have that $M\big(\|\tilde{f}_n(\omega)\|\big) - M\big(\frac{1}{2}\|\tilde{f}_n(\omega) + \tilde{g}_n(\omega)\|\big) \longrightarrow 0$ as $N \ni n \to \infty$, and, since $\|\tilde{f}_n(\omega)\| + \|\tilde{g}_n(\omega)\| \le \frac{1}{\delta}$ and M is increasing and continuous on $[0, +\infty)$, we get

$$\|\tilde{f}_n(\omega) + \tilde{g}_n(\omega)\| - 2\|\tilde{f}_n(\omega)\| \longrightarrow 0 \text{ and } \|\tilde{f}_n(\omega)\| - \|\tilde{g}_n(\omega)\| \longrightarrow 0 \text{ as } N \ni n \to \infty \tag{2.22}$$

On the other hand, it is easy to see that *if* $(x_n), (y_n)$ *are bounded sequences in a Banach space with UR norm* $\|\cdot\|$ *and* $\|x_n + y_n\| - 2\|x_n\| \longrightarrow 0$ *and* $\|x_n\| - \|y_n\| \longrightarrow 0$ *as* $n \to \infty$, *then* $\|x_n - y_n\| \longrightarrow 0$ *as* $n \to \infty$. Substituting here $x_n := \tilde{f}_n(\omega)$ and $y_n := \tilde{g}_n(\omega)$, $n \in N$, and using (2.22) we can conclude that

$$\|\tilde{f}_n(\omega) - \tilde{g}_n(\omega)\| \longrightarrow 0 \quad \text{as} \quad N \ni n \to \infty$$

for almost all $\omega \in \Omega$. Then the Lebesgue's dominated convergence theorem guarantees that

$$\int_\Omega \|\tilde{f}_n(\omega) - \tilde{g}_n(\omega)\| \mathrm{d}\mu(\omega) \longrightarrow 0 \quad \text{as} \quad N \ni n \to \infty. \tag{2.23}$$

But, because of (2.20) and (2.21) we have

$$\begin{aligned} \tfrac{\varepsilon}{C} &< \int_\Omega \|f_n(\omega) - g_n(\omega)\| \mathrm{d}\mu(\omega) = \int_{A_n} \|f_n(\omega) - g_n(\omega)\| \mathrm{d}\mu(\omega) + \int_\Omega \|\tilde{f}_n(\omega) - \tilde{g}_n(\omega)\| \mathrm{d}\mu(\omega) \\ &< \tfrac{\varepsilon}{2C} + \int_\Omega \|\tilde{f}_n(\omega) - \tilde{g}_n(\omega)\| \mathrm{d}\mu(\omega) \end{aligned}$$

for all $n \in N$. And this is in contradiction with (2.23). □

We end this paper with some comments about the possibility to achieve a combination of properties of rotundity and smoothness on the space $L^1(\mu, X)$.

Remarks:

(1) In [5, Theorem 2.1] it is shown that if M is the function M_2 considered at page 4 and the norm $\|\cdot\|$ on the Banach space X is Gâteaux (uniformly Gâteaux) smooth, then so is the corresponding Orlicz-Bochner norm $|||\cdot|||$ on $L^1(\mu, X)$. Thus, as an immediate consequence of this fact and Theorem 1, we deduce that *if* $(X, \|\cdot\|)$ *is a Banach space whose norm is Gâteaux smooth or uniformly Gâteaux smooth, and simultaneously rotund or URED or LUR or MLUR, then so is the norm* $|||\cdot|||$ *on* $L^1(\mu, X)$ *constructed in Theorem 1.*

(2) From Theorem 3.2 in [5] we have that if the norm $\|\cdot\|$ on X is Fréchet (uniformly Fréchet) smooth, then the restriction of $|||\cdot|||$ to every reflexive subspace of $L^1(\mu, X)$ is Fréchet (uniformly Fréchet) smooth. Combining the parenthetic part of this assertion with Theorem 2 we get that, *if* X *is super-reflexive, then the space* $L^1(\mu, X)$ *admits an equivalent norm whose restriction to every reflexive subspace is simultaneously UR and uniformly Fréchet smooth.* This fact provides a (new) proof of the Lebesgue-Bochner counterpart of Rosenthal's result in [12] that, if X is superreflexive, then every reflexive subspace of $L^1(\mu, X)$ is super-reflexive.

(3) Taking into account the norm $|||\cdot|||$ preserves the LUR property, the Fréchet version of Theorem 3.2 in [5] and [2, Theorem VII.1.14] it follows that, *if the space X is both Asplund and weakly compactly generated (in particular, if X is reflexive), then $L^1(\mu, X)$ has an equivalent LUR norm whose restriction to every reflexive subspace is Fréchet smooth.*

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