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SEPARABLE REDUCTIONS AND RICH FAMILIES IN THEORY OF FRÉCHET SUBDIFFERENTIALS

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Abstract. In [Fabian-Ioffe13], we presented the separable reduction for a general statement covering practically all important properties of Fréchet subdifferentials, in particular: the non-emptiness of subdifferentials, the non-zeroness of normal cones, the fuzzy calculus, and the extremal principle; all statements being considered in Fréchet sense. As in earlier studies of various separable reduction techniques, this was done with help of suitable cofinal families of separable subspaces. In this paper we show that such reductions can be done with the help of a subclass of cofinal families known as rich families, recently articulated (and used) in [Borwein-Moors00], [Lindenstrauss-Preiss-Tišer12, page 37]. The most advantageous feature of rich families is that the intersection of even countably many of them is again a rich family. This means that in case we need separable reduction of a combination of properties and know that each of them is reducible by elements of a certain rich family, then all we need to do is to take the intersection of rich families associated with the properties instead of devising a new (and typically fairly complicated) proof.

1. Prologue/warming up

It is said that a certain property $\mathcal{P}$ that may or may not be satisfied at elements of e.g. a certain Banach space X is *separably reducible* (or *separably determined*) if there is a family $\mathcal{F}$ of separable subspaces of X such that $\bigcup \mathcal{F} = X$ and for any $Y \in \mathcal{F}$ the property $\mathcal{P}$ is satisfied at $x \in Y$ if the "restriction" of $\mathcal{P}$ to Y is satisfied at x. (Usually it is not a problem to define the restriction of a property to a subspace.)

The need for separable reducibility is not rare in analysis — it often leads to substantial simplifications and sometimes is the only way to obtain a result. We refer to [Fabian-Ioffe13] for (certainly incomplete) list of separably reducible properties considered in the literature. Recently it has been realized that separable reduction can be implemented with different types of families and there is one class especially attractive as it allows to simultaneously deal with a number of different properties or objects.

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We shall illustrate this with one of the simplest properties: continuity of a function on a metric space. Let X be a metric space, preferably non-separable. For $x \in X$ and $r > 0$ the symbol $B(x, r)$ will denote the *open ball* with center x and radius r. Let $f : X \longrightarrow (-\infty, +\infty)$ be an arbitrary function. We shall focus on recognizing the points of X where f is continuous by using the continuity of the restrictions $f\restriction_Y$ of f to some separable subsets Y of X. Obviously, given a fixed point $x \in X$, then f is continuous at x if (and only if) the restriction $f|_Y$ is continuous at x for every separable, even just for every countable, subset $x \in Y \subset X$. But statements so simple like that is not an objective of this paper. We shall rather look for separable subsets $Y \subset X$ such that for every $x \in Y$, if the restriction $f|_Y$ is continuous at x, then the whole f is continuous at x. In order to recognize points of the continuity of f we concentrate on the following statement: Given any infinite countable subset $C \subset X$ we look for a closed separable set $C \subset Y \subset X$ such that for every $x \in Y$ the function f is continuous at x if (and only if) the restriction $f|_Y$ is continuous at x. The latter statement can be called a *separable reduction for the continuity of f*.

Let us construct such a Y. Denote by $\mathbb{N}$ the set of natural numbers, by $\mathbb{Q}$ the set of rational numbers, and by $\mathbb{Q}_+ := \mathbb{Q} \cap (0, +\infty)$ the set of positive rationals. Denote $C_0 := C$ and assume that we already have found countable sets $C_0 \subset C_1 \subset \cdots \subset C_{n-1} \subset X$ for some $n \in \mathbb{N}$. For every $c \in C_{n-1}$ and for every $r \in \mathbb{Q}_+$ we find a countable set $c \in D(c, r) \subset B(c, r)$ such that $\operatorname{diam} f(B(c, r)) = \operatorname{diam} f(D(c, r))$. Define

$$C_n := \bigcup \big\{ D(c, r) : \ c \in C_{n-1}, \ r \in \mathbb{Q}_+ \big\};$$

which is clearly a countable set. Do so for every $n \in \mathbb{N}$ and set finally $Y := \overline{\bigcup_{n=0}^{\infty} C_n}$; the latter set is clearly separable. We shall show that this Y "works". So, pick any $x \in Y$ (if there is any) such that $f|_Y$ is continuous at x. (If $x \in C_n$ for some $n \in \mathbb{N}$, then it is rather easy to verify that f is continuous at x. However it may happen that $x \in Y \setminus \bigcup_{n=0}^{\infty} C_n$ and then we have to work a bit harder.) Let any $\varepsilon > 0$ be given. The continuity of $f|_Y$ yields an $r > 0$ such that $\operatorname{diam} f\big(B(x, r) \cap Y\big) < \varepsilon$. We shall show that $\operatorname{diam} f(B(x, r)) < \varepsilon$ and so the continuity of f at x will be proved. So, fix any $x_1, x_2 \in B(x, r)$. Since the latter ball is open, there is $\gamma > 0$ such that $2\gamma < r$, that $r - \gamma \in \mathbb{Q}_+$, and that $B(x, r - 2\gamma) \ni x_1, x_2$. Find $n \in \mathbb{N}$ so big that $\operatorname{dist}(x, C_{n-1}) < \gamma$; find then $c \in C_{n-1}$ such that the distance between c and x is less than γ. Thus $x_1, x_2 \in B(c, r - \gamma)$, and so

$$\begin{aligned} |f(x_1) - f(x_2)| &\leq \operatorname{diam} f(B(c, r - \gamma)) = \operatorname{diam} f(D(c, r - \gamma)) \\ &\leq \operatorname{diam} f(B(c, r - \gamma) \cap Y) \leq \operatorname{diam} f(B(x, r) \cap Y) < \varepsilon. \end{aligned}$$

Therefore $\operatorname{diam} f(B(x, r)) < \varepsilon$, and the continuity of f at x is proved.

Keeping still X and f fixed, let $\mathcal{S}$ be the family of all closed separable subsets in X. Further, let $\mathcal{C}$ consist of all $Y \in \mathcal{S}$ such that for every $x \in Y$ the function f is continuous at x if (and only if) so is the restriction $f|_Y$ at x. The statement proved in the preceding paragraph can be reformulated as that *the family $\mathcal{C}$ is cofinal in $\mathcal{S}$, i.e., for every $S \in \mathcal{S}$ there is $Y \in \mathcal{C}$ such that $Y \supset S$.*

If, instead of one f, we have two functions $f_1, f_2 : X \longrightarrow (-\infty, +\infty)$, we can elaborate a bit the proof above to get a cofinal family $\mathcal{C}_{1,2} \subset \mathcal{S}$ such that for every $Y \in \mathcal{C}_{1,2}$, for every $x \in Y$, and for every $i \in \{1, 2\}$ the function f_i is continuous at x if (and only if) the restriction $f_i\restriction_Y$ is continuous at x. However it is not clear at first glance if we can for $\mathcal{C}_{1,2}$ take just the intersection of the two cofinal families found for f_1 and f_2; we do not even know if this intersection is non-empty. This obstacle, and other reasons, lead to consider a stronger concept, introduced in [Borwein-Moors00], see also [Lindenstrauss-Preiss-Tišer12, Section 3.6].

Definition 1.1. Let X be a (rather) non-separable metric [or normed] space. Let $\mathcal{S}(X)$ denote the family of all closed separable subsets [closed separable subspaces] of X. A subfamily $\mathcal{R}$ of $\mathcal{S}(X)$ is called *rich* if

(i) for every $S \in \mathcal{S}(X)$ there is $Y \in \mathcal{R}$ so that $S \subset Y$ and

(ii) whenever $Y_1, Y_2, \ldots \in \mathcal{R}$ and $Y_1 \subset Y_2 \subset \cdots$, then $Y := \overline{\bigcup_{n=1}^{\infty} Y_n} \in \mathcal{R}$.

More abstractly, let T be a set and let $\prec$ be a *(partial) order* on it, i.e. $\prec$ is a subset of $T \times T$ which is reflexive, symmetric and transitive, see [Engelking, page 21]. We agree that, instead of "$s, t \in \prec$" we will rather write "$s \prec t$". Moreover, assume that T is *(up)-directed by* $\prec$, i.e., for every $t_1, t_2 \in T$ there is $t_3 \in T$ such that $t_1 \prec t_3$ and $t_2 \prec t_3$. (An example of this is $T := \mathcal{S}$ and $\prec :=$"$\subset$", see Definition 1.1.) A subset $R \subset T$ is called *cofinal/dominating* if for every $t \in T$ there is $r \in R$ such that $t \prec r$. R is called *σ-complete/closed* if, whenever $r_1 \prec r_2 \prec \cdots$ is an increasing sequence in R, then there is $r \in R$ such that $r_i \prec r$ for every $i \in \mathbb{N}$ and $r \prec t$ whenever $t \in T$ and $r_i \prec t$ for every $i \in \mathbb{N}$. The set R is called *rich/a club set* if it is both cofinal and σ-complete. (Note that the whole T is rich if T is σ-complete.) Having these concepts introduced, then we can easily see that *a subfamily $\mathcal{R} \subset \mathcal{S}$ is rich in the sense of Definition 1.1 if and only if $\mathcal{R}$ is rich in the partially ordered up-directed family $(\mathcal{S}(X), \subset)$.*

The power of rich families is demonstrated by the following fundamental fact (see [Borwein-Moors00] and also [Lindenstrauss-Preiss-Tišer12, page 37])

Proposition 1.2. *The intersection of countably many rich families of a given space is (not only non-empty but even) rich.*

In the paper, we shall be dealing with families of closed subspaces of a Banach space ordered by inclusion.

We do not know if the family $\mathcal{C}$ defined above is rich, that is, if $\mathcal{C}$ is σ-complete. (We already showed that it is cofinal.) This can be remedied by shrinking $\mathcal{C}$ as follows. Given X and f as in the beginning,

we define $\mathcal{R}$ as the family consisting of all $Y \in \mathcal{S}$ such that for every $x \in Y$ and for every $r > 0$ we have that $\operatorname{diam} f(B(x,r)) = \operatorname{diam} f(B(x,r) \cap Y)$.

Clearly, $\mathcal{R} \subset \mathcal{C}$. The cofinality of $\mathcal{R}$ can be verified similarly as that of $\mathcal{C}$. It remains to check the σ-completeness of $\mathcal{R}$. Let $Y_1, Y_2, \ldots$ and Y be as in (ii) of Definition 1.1. Fix any $x \in Y$ and any $r > 0$. Consider any $x_1, x_2 \in B(x,r)$. Find $\gamma > 0$ so small that $2\gamma < r$ and that $B(x, r-2\gamma) \ni x_1, x_2$. Find $n \in \mathbb{N}$ so big that $\operatorname{dist}(x, Y_n) < \gamma$; find then $y \in Y_n$ such that the distance between x and y is less than γ. Thus $x_1, x_2 \in B(y, r-\gamma)$. As $Y_n \in \mathcal{R}$ and $y \in Y_n$, we have that

$$\begin{aligned}|f(x_1) - f(x_2)| \leq\ & \operatorname{diam} f(B(y, r-\gamma)) \\ &= \operatorname{diam} f(B(y, r-\gamma) \cap Y_n) \leq f(B(x,r) \cap Y).\end{aligned}$$

Therefore $\operatorname{diam} f(B(x,r)) = \operatorname{diam} f(B(x,r) \cap Y)$, and so $Y \in \mathcal{R}$.

Thus, if we have two or even countably many functions, then by taking the intersection of the corresponding rich families, we immediately get a rich family that insures separable reduction of continuity simultaneously for all the functions.

This explains substantial advantage of dealing with rich families when we need separable reduction. Typically a cofinal family can do the job for an individual object or property. But when we need simultaneous reduction of a number of them and have a rich family for each one, then all we have to do is to take the intersection of all these rich families instead of devising a special proof in case our families are only cofinal.

2. Primal representation of Fréchet subdifferentiability

Here we wish to construct rich families for separable reduction of various properties associated with Fréchet subdifferentiability. In pursuing this goal we shall follow the traditional approach going back to [Preiss84], [Fabian-Zhivkov85] (see also [Fabian89], [Penot10], [Ioffe11], and [Fabian-Ioffe13]), whose first step is "primal" (not involving anything associated with the dual space) characterization of the desired property.

Let $(X, \|\cdot\|)$ be a Banach space, let $f : X \longrightarrow (-\infty, +\infty]$ be a proper function, and let $x \in X$ be an element of its domain, i.e., $f(x) < +\infty$. We say that f *Fréchet subdifferentiable* at x if there are an element x^* of the dual space X^* and a function $o : [0, +\infty) \longrightarrow [0, +\infty)$ such that $\frac{0(t)}{t} \to 0$ as $t \downarrow 0$ and

$$f(x+h) - f(x) - \langle x^*, h\rangle > -o(\|h\|)$$

holds for every non-zero $h \in X$. The (possibly empty) set of all x^*'s for which the above holds with a suitable function $o(\cdot)$ is called the *Fréchet subdifferential* of f at x.

Let $k \in \mathbb{N}$, let $X, X_1, \dots, X_k$ be (rather) non-separable Banach spaces, and let $A_i : X_i \to X,\ i = 1, \dots, k$, be bounded linear operators. The statement below is a slight extension of [Fabian-Ioffe13, Lemma 2.1].

Proposition 2.1. *Let $c \geq 0,\ \varepsilon_1 > 0,\ \dots, \varepsilon_k > 0,\ \rho_1 \geq 0,\ \dots,\ \rho_k \geq 0$ be given constants and let $\varphi : X \longrightarrow (-\infty, +\infty]$ be a convex function, with $\varphi(0) < +\infty$. Then the following two assertions are equivalent:*

(i) There exist $\varepsilon_i' \in (0, \varepsilon_i),\ i = 1, \dots, k$, and $(w_1, \dots w_k) \in S_{X_1} \times \cdots \times S_{X_k}$ such that

$$\varphi(x) \geq \varphi(0) - c\big\|x - \textstyle\sum A_i x_i\big\| - \sum \varepsilon_i' \|x_i\| - \sum \rho_i \|x_i - w_i\| + \sum \rho_i$$

holds for all $(x, x_1, \dots, x_k) \in X \times X_1 \times \cdots \times X_k$.

(ii) There exists $x^* \in \partial\varphi(0)$ *such that* $\|x^*\| \le c$ *and* $\big|\|A_i^* x^*\| - \rho_i\big| < \varepsilon_i$ *for every* $i = 1, \dots, k$.

Proof. (Above and below, $\sum$ means $\sum_{i=1}^k$.) Assume (ii) holds. Find $\varepsilon_i' \in (0, \varepsilon_i)$ so that $\big|\|A_i^* x^*\| - \rho_i\big| < \varepsilon_i'$ for every $i = 1, \dots, k$. For each i find a norm attaining $w_i^* \in X_i^*$ such that $\|w_i^*\| = \rho_i$ and $\|A_i^* x^* - w_i^*\| < \varepsilon_i'$. Take finally a $w_i \in S_{X_i}$ so that $\|w_i^*\| = \langle w_i^*, w_i\rangle$. Then for all $(x, , x_1, \dots, x_k) \in X \times X_1 \times \cdots \times X_k$ we have

$$\begin{aligned}\varphi(x) &\ge \varphi(0) + \langle x^*, x\rangle = \varphi(0) + \big\langle x^*, x - \textstyle\sum A_i x_i\big\rangle \\ &\quad + \textstyle\sum \big\langle A_i^* x^* - w_i^*, x_i\big\rangle + \sum\big\langle w_i^*, x_i - w_i\big\rangle + \sum\langle w_i^*, w_i\rangle \\ &\ge \varphi(0) - c\Big\|x - \textstyle\sum A_i x_i\Big\| - \sum \varepsilon_i'\|x_i\| - \sum\rho_i\|x_i - w_i\| + \sum\rho_i.\end{aligned}$$

Assume that (i) holds. Set

$$\psi(x, x_1, \dots, x_k) := \varphi(x) + c\big\|x - \textstyle\sum A_i x_i\big\| + \sum\varepsilon_i'\|x_i\| + \sum\rho_i\|x_i - w_i\| - \sum\rho_i.$$

Then

$$\psi(x, x_1, \dots, x_k) \ge \varphi(0) = \psi(0, 0, \dots, 0)$$

for all $x \in X$ and for all $(x_1, \dots, x_k) \in X_1 \times \cdots \times X_k$. Thus, by Moreau-Rockafellar theorem [Phelps93, page 47], there are $x^* \in \partial\varphi(0)$, $\xi \in cB_{X^*}$, and further, for $i = 1, \dots, k$, there are $\xi_i \in \varepsilon_i' B_{X_i^*}$ and $w_i^* \in X_i^*$, with $\langle w_i^*, w_i\rangle = \|w_i^*\| = \rho_i$, such that

$$(0, 0, \dots, 0) = (x^*, 0, \dots, 0) + (\xi, -A_1^*\xi, \dots, -A_i^*\xi) + (0, \xi_1, \dots, \xi_k) + (0, w_1^*, \dots, w_k^*).$$

Hence, $0 = x^* + \xi$ and $A_i^*\xi = \xi_i + w_i^*$ for $i = 1, \dots, k$. Therefore, $\|x^*\| \le c$ and

$$\big|\|A_i^*\xi\| - \rho_i\big| = \big|\|A_i^*\xi\| - \|w_i^*\|\big| \le \big\|A_i^*\xi - w_i^*\big\| = \|\xi_i\| \le \varepsilon_i' < \varepsilon_i$$

for every $i = 1, \dots, k$. □

The proposition above gives us the key instrument for finding the necessary primal characterization of Fréchet subdifferentiability and several associated properties.

Let us call *data* any triple $d = (c, \varepsilon, \rho)$ such that $c \ge 0$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k) \in (0, +\infty)^k$, and $\rho = (\rho_1, \dots, \rho_k) \in [0, +\infty)^k$. To begin with, we define for any given data d and any $w = (w_1, \dots, w_k) \in S_{X_1} \times \cdots \times S_{X_k}$ the function

$$p_{d,w}(h, x_1, \dots, x_k) := c\big\|h - \textstyle\sum A_i x_i\big\| + \sum\varepsilon_i\|x_i\| + \sum\rho_i\|x_i - w_i\| - \sum\rho_i,$$

where $(h, x_1, \dots, x_k) \in X \times X_1 \times \cdots \times X_k$ are the arguments of the function and and d and w are parameters changing within the indicated limits. For any fixed d and w this is a convex continuous function, equal to zero at $(0, 0, \dots, 0)$. Moreover for $u = (u_1, \dots, u_k) \in S_{X_1} \times \cdots \times S_{X_k}$ we have

$$p_{d,w}(h, x_1, \dots, x_k) - p_{d,u}(h, x_1, \dots, x_k) \le \textstyle\sum\rho_i\|w_i - u_i\|. \tag{2.1}$$

We need more notation for the statements below. Namely, we denote

- by Δ the collection of all sequences $\delta = (\delta_n) \in (0, +\infty)^\omega$ such that $\delta_1 \ge \delta_2 \ge \cdots$;

- by Λ the collection of all sequences $\lambda = (\lambda_n) \in [0,+\infty)^\omega$ such that $\sum_{n=1}^\infty \lambda_n = 1$ and $\{n \in \mathbb{N} :\ \lambda_n > 0\}$ is a finite set;
- by Υ the collection of all $(\nu_n) \in \mathbb{N}^\omega$ such that $\{n \in \mathbb{N} :\ \nu_n \neq 1\}$ is a finite set;
- and, given $\nu = (\nu_n) \in \Upsilon$ and $\delta = (\delta_n) \in \Delta$, we denote by $\mathcal{H}(\nu,\delta)$ the collection of all sequences (h_n) in X such that $\|h_n\| < \delta_{\nu_n}$ for every $n \in \mathbb{N}$.

The next proposition offers the desired primal characterization. It translates the non-emptiness of Fréchet subdifferential (even a subtler fact) completely into terms of the space X. The proof of the proposition repeats word for word the proof of [Fabian-Ioffe13, Lemma 2.2] if we replace the reference to [Fabian-Ioffe13, Lemma 2.1] by the reference to Proposition 2.1, so we omit it.

Proposition 2.2. *Consider a proper function $f : X \longrightarrow (-\infty,+\infty]$ and fix $x \in X$ such that $f(x) < +\infty$. Then, given data $d = (c,\varepsilon,\rho)$, the following two assertions are equivalent:*

(i) There exist $\varepsilon_i' \in (0,\varepsilon_i)$, $i = 1,\dots,k$, $w := (w_1,\dots,w_k) \in S_{X_1} \times \ldots \times S_{X_k}$, and a sequence $\delta := (\delta_1,\delta_2,\dots) \in \Delta \cap \mathbb{Q}^\omega$ such that for $d' := (c,\varepsilon',\rho)$ the inequality

$$\sum_{n=1}^\infty \lambda_n\Big(f(x+h_n) + \frac{1}{\nu_n}\|h_n\|\Big) + p_{d',w}\Big(\sum_{n=1}^\infty \lambda_n h_n, x_1,\dots,x_k\Big) \geq f(x) \tag{2.2}$$

holds whenever $x_1,\dots,x_k \in X_1 \times \cdots \times X_k$, $(\lambda_n) \in \Lambda$, $\nu \in \Upsilon$, and $(h_n) \in \mathcal{H}(\nu,\delta)$.

(ii) There exists $x^* \in \partial_F f(x)$ such that $\|x^*\| \leq c$ and $\big|\|A_i^* x^*\| - \rho_i\big| < \varepsilon_i$ for every $i = 1,\dots,k$.

Our aim is to find a rich family that could be used for a separable reduction of (ii). It is the first property (i) of the proposition that equips us with a suitable instrument for constructing such family.

3. Rich families associated with (*i*) in Proposition 2.2

Let $k, X, X_1,\dots,X_k, A_1,\dots A_k$ have the same meaning as before. By a *block* we understand any product $Y \times Y_1 \times \cdots \times Y_k$ where $Y, Y_1,\dots,Y_k$ are subspaces of $X, X_1,\dots,X_k$, respectively. Any $\mathcal{F} \subset \mathcal{S}(X \times X_1 \times \cdots \times X_k)$ whose elements are blocks is called a *block-family.* For every block $\mathcal{Y} := Y \times Y_1 \times \cdots \times Y_k$, every $x \in Y$, every $\lambda = (\lambda_n) \in \Lambda$, every $\nu = (\nu_n) \in \mathbb{N}^\omega$, every $\delta = (\delta_n) \in \Delta$, every data $d = (c,\varepsilon,\rho) \in [0,+\infty) \times (0,+\infty)^k \times [0,+\infty)^k$ and every $w \in S_{X_1} \times \cdots \times S_{X_k}$ we denote by $I(x,\lambda,\nu,\delta,d,w,\mathcal{Y})$ the following (possibly infinite) quantity

$$\begin{aligned}\inf\Big\{&\sum_{n=1}^\infty \lambda_n\Big(f(x+h_n)+\frac{1}{\nu_n}\|h_n\|\Big) + p_{d,w}\Big(\sum_{n=1}^\infty \lambda_n h_n, x_1,\dots,x_k\Big) :\\ &(h_n) \in \mathcal{H}(\nu,\delta) \cap Y^\omega,\ (x_1,\dots,x_k) \in Y_1 \times \cdots \times Y_k\Big\}.\end{aligned} \tag{3.1}$$

If $Y = X$ and $Y_i = X_i$ for all $i = 1,\dots,k$, we write just $I(x,\lambda,\nu,\delta,d,w)$. With this notation, (2.2) reads as $I(x,\lambda,\nu,\delta,d,w) \geq f(x)$.

For $\lambda = (\lambda_n) \in \Lambda,\ \nu = (\nu_n) \in \mathbb{N}^\omega$, and $d = (c, \varepsilon, \rho) \in [0, +\infty) \times (0, +\infty)^k \times [0, +\infty)^k$ we define the block-family $\mathcal{R}_{\lambda,\nu,d}$ as that consisting of all blocks $\mathcal{Y} := Y \times Y_1 \times \cdots \times Y_k \in \mathcal{S}(X \times X_1 \times \cdots \times X_k)$ such that

$$A_1(Y_1) \subset Y,\ \ldots,\ A_k(Y_k) \subset Y \tag{3.2}$$

and that for all $x \in Y,\ \delta \in \Delta \cap \mathbb{Q}^\omega$, and $w \in S_{Y_1} \times \cdots \times S_{Y_k}$

$$I(x, \lambda, \nu, \delta, d, w) = I(x, \lambda, \nu, \delta, d, w, \mathcal{Y}). \tag{3.3}$$

Proposition 3.1. *For any $\lambda = (\lambda_n) \in \Lambda,\ \nu = (\nu_n) \in \mathbb{N}^\omega$ and $d = (c, \varepsilon, \rho) \in [0, +\infty) \times (0, +\infty)^k \times [0, +\infty)^k$, the family $\mathcal{R}_{\lambda,\nu,d}$ defined above is rich.*

Proof. Fix any λ, ν and d as above and put, for simplicity, $\mathcal{R} := \mathcal{R}_{\lambda,\nu,d}$. We re-denote $I(x, \lambda, \nu, \delta, d, w)$ and $I(x, \lambda, \nu, \delta, d, w, \mathcal{Y})$, respectively, by $I(x, \delta, w)$ and $I(x, \delta, w, \mathcal{Y})$. Now, for every $x \in X$, every $w = (w_1, \ldots, w_k) \in S_{X_1} \times \ldots \times S_{X_k}$, every $\delta \in \Delta$, and for every $m, n \in \mathbb{N}$ we find vectors $v_1(x, \delta, w, m) \in X_1, \ldots, v_k(x, \delta, w, m) \in X_k$, and vectors $g_n(x, \delta, w, m) \in X$, with $\|g_n(x, \delta, w, m)\| < \delta_{\nu_n}$, such that

(3.4)
$$\begin{aligned} I(x, \delta, w) + \frac{1}{m} \geq & \sum_{n=1}^\infty \lambda_n \Big(f\big(x + g_n(x, \delta, w, m)\big) + \frac{1}{\nu_n} \|g_n(x, \delta, w, m)\| \Big) \\ & + p_{d,w} \Big(\sum_{n=1}^\infty \lambda_n g_n(x, \delta, w, m), v_1(x, \delta, w, m), ..., v_k(x, \delta, w, m) \Big) \end{aligned}$$

if $I(x, \delta, w) > -\infty$, and

$$\begin{aligned} -m > & \sum_{n=1}^\infty \lambda_n \Big(f\big(x + g_n(x, \delta, w, m)\big) + \frac{1}{\nu_n} \|g_n(x, \delta, w, m)\| \Big) \\ & + p_{d,w} \Big(\sum_{n=1}^\infty \lambda_n g_n(x, \delta, w, m), v_1(x, \delta, w, m), \ldots, v_k(x, \delta, w, m) \Big) \end{aligned} \tag{3.5}$$

if $I(x, \delta, w) = -\infty$. Here, we choose the vectors $v_i(x, \delta, w, m)$ and $g_n(x, \delta, w, m)$ in such a way that $v_i(x, \delta, w, m) = v_i(x, \delta', w, m)$ and $g_n(x, \delta, w, m) = g_n(x, \delta', w, m)$ whenever $\delta, \delta' \in \Delta$ and $\delta_{\nu_j} = \delta'_{\nu_j}$ for every $j \in \mathbb{N}$ such that $\lambda_j > 0$. By this we guarantee that for every $x \in X$, every $w \in S_{X_1} \times \cdots \times S_{X_k}$, and every $m \in \mathbb{N}$ the set

$$\{v_i(x, \delta, w, m) :\ i = 1, \ldots, k,\ \delta \in \Delta \cap \mathbb{Q}^\omega\} \bigcup \{g_n(x, \delta, w, m) :\ n \in \mathbb{N},\ \delta \in \Delta \cap \mathbb{Q}^\omega\}$$

is countable.

We first show that $\mathcal{R}$ is cofinal in $\mathcal{S}(X \times X_1 \times \cdots \times X_k)$. To begin with, fix any $Z \in \mathcal{S}(X)$ and any $Z_i \in \mathcal{S}(X_i),\ i = 1, \ldots, k$. Choose countable dense subsets C_0 in Z, C_0^1 in $Z_1, \ldots,$ and C_0^k in Z_k. Assume further that for some $m \in \mathbb{N}$ we have already constructed countable sets $C_0 \subset C_1 \subset \cdots \subset C_{m-1} \subset X$ and $C_0^i \subset C_1^i \subset \cdots \subset C_{m-1}^i \subset S_{X_i},\ i = 1, \ldots, k$. Define then C_m as the $\mathbb{Q}$-linear span of the union of C_{m-1}, $A_i(C_{m-1}^i),\ i = 1, \ldots, k$, and the set

$$\big\{g_n(x,\delta,w,m):\ n\in\mathbb{N},\ x\in C_{m-1},\ \delta\in\Delta\cap\mathbb{Q}^\omega,\ w\in C^1_{m-1}\times\cdots\times C^k_{m-1}\big\}$$

Likewise, for any $i=1,\dots,k$ define the set C^i_m as the $\mathbb{Q}$-linear span of the union of C^i_{m-1} and

$$\{v_i(x,\delta,w,m):\ i=1,\dots,k,\ x\in C_{m-1},\ \delta\in\Delta\cap\mathbb{Q}^\omega,\ w\in C^1_{m-1}\times\cdots\times C^k_{m-1}\}.$$

augmented with normalized versions of its elements (that is, vectors of the form $\xi/\|\xi\|$). Clearly, all these sets are still countable.

Set $Y:=\overline{C_0\cup C_1\cup\cdots}$ and $Y_i:=\overline{C^i_0\cup C^i_1\cup\cdots}$ for every $i=1,\dots,k$. Clearly, these are closed separable subspaces and $\mathcal{Y}:=Y\times Y_1\times\cdots\times Y_k\supset Z\times Z_1\times\cdots\times Z_k$ We have to show that $\mathcal{Y}$ belongs to $\mathcal{R}$, that is, that (3.2) and (3.3) hold. The verification of (3.2) is easy. As regards (3.3), fix any $x\in Y$, $\delta\in\Delta\cap\mathbb{Q}^\omega$, and $w=(w_1,\dots,w_k)\in S_{Y_1}\times\cdots\times S_{Y_k}$. Clearly, it is enough to prove that $I(x,\delta,w)\ge I(x,\delta,w,\mathcal{Y})$. Uniform continuity of the assignment $u\mapsto p_{d,u}(\cdots)$ (see(2.1)) allows us to assume that w_i belongs to $C^i_0\cup C^i_1\cup\cdots$ for every $i=1,\dots,k$. Now, consider any $(h_n)\in\mathcal{H}(\nu,\delta)$ and any $(x_1,\dots,x_k)\in X_1\times\cdots\times X_k$. Put $N:=\{n\in\mathbb{N}:\ \lambda_n>0\}$; this is a finite set. Take an arbitrary $r\in\mathbb{Q}_+$ so small that $\|h_n\|<\delta_{\nu_n}-2r$ for every $n\in N$. Find then $\delta'=(\delta'_n)\in\Delta\cap\mathbb{Q}^\omega$ such that $\delta'_n\le\delta_n$ for every $n\in\mathbb{N}$ and $\delta'_n=\delta_n-r$ if $n\in N$. Find $m\in\mathbb{N}$ so big that $w_1\in C^1_{m-1},\dots,\ w_k\in C^k_{m-1}$, and that $\operatorname{dist}(x,C_{m-1})<r$; pick then $y_m\in C_{m-1}$ such that $\|x-y_m\|<r$.

We are now ready to estimate

$$\begin{aligned}
(3.6)\quad &\sum_{n=1}^\infty\lambda_n\Big(f\big(y_m+g_n(y_m,\delta',w,m)\big)+\frac{1}{\nu_n}\|g_n(y_m,\delta',w,m)\|\Big)\\
&+p_{d,w}\Big(\sum_{n=1}^\infty\lambda_n g_n(y_m,\delta',w,m),v_1(y_m,\delta',w,m),\dots,v_k(y_m,\delta',w,m)\Big)\\
\ge\ &\sum_{n=1}^\infty\lambda_n\Big(f\big(x+(y_m-x+g_n(y_m,\delta',w,m))\big)+\frac{1}{\nu_n}\|y_m-x+g_n(y_m,\delta',w,m)\|\Big)\\
&+p_{d,w}\Big(\sum_{n=1}^\infty\lambda_n\big(y_m-x+g_n(y_m,\delta',w,m)\big),v_1(y_m,\delta',w,m),\dots,v_k(y_m,\delta',w,m)\Big)\\
&-r-cr\ge I(x,\delta,w,\mathcal{Y})-r-cr,
\end{aligned}$$

the last inequality being true because $v_i(y_m,\delta',w,m)\in C_m\subset Y$ and $\big(y_m-x+g_n(y_m,\delta',w,m):\ n\in\mathbb{N}\big)\in\mathcal{H}(\nu,\delta)\cap Y^\omega$.

If $I(y_m,\delta',w)=-\infty$ for infinitely many $m\in\mathbb{N}$, then (3.5) and (3.6) imply together that $-m>I(x,\delta,w,\mathcal{Y})-r-cr$ for all such m; hence $I(x,\delta,w,\mathcal{Y})=-\infty$, and thus $I(x,\delta,w)\ge-\infty=I(x,\delta,w,\mathcal{Y})$.

Assume now that $I(y_m,\delta',w)>-\infty$ for all sufficiently large $m\in\mathbb{N}$. Fix one such m, big enough to guarantee that $m>\frac{1}{r}$. Define $h'_n:=h_n+x-y_m$ if $n\in N$, and

$h_n' := 0$ if $n \in \mathbb{N} \setminus N$. Then $(h_n') \in \mathcal{H}(\nu, \delta')$ and we can estimate

$$
\begin{aligned}
(3.7)\qquad & \sum_{n=1}^{\infty} \lambda_n \Big(f(x+h_n) + \frac{1}{\nu_n}\|h_n\| \Big) + p_{d,w}\Big(\sum_{n=1}^{\infty} \lambda_n h_n, x_1, \dots, x_k \Big) \\
= \ & \sum_{n=1}^{\infty} \lambda_n \Big(f(y_m+h_n') + \frac{1}{\nu_n}\|h_n\| \Big) + p_{d,w}\Big(\sum_{n=1}^{\infty} \lambda_n h_n, x_1, \dots, x_k \Big) \\
\geq \ & \sum_{n=1}^{\infty} \lambda_n \Big(f(y_m+h_n') + \frac{1}{\nu_n}\|h_n'\| \Big) + p_{d,w}\Big(\sum_{n=1}^{\infty} \lambda_n h_n', x_1, \dots, x_k \Big) - r - cr \\
\geq \ & I(y_m, \delta', w) - r - cr \\
\geq \ & \sum_{n=1}^{\infty} \lambda_n \Big(f\big(y_m + g_n(y_m, \delta', w, m)\big) + \frac{1}{\nu_n}\|g_n(y_m, \delta', w, m)\| \Big) \\
& + p_{d,w}\Big(\sum_{n=1}^{\infty} \lambda_n g_n(y_m, \delta', w, m), v_1(y_m, \delta', w, m), \dots, v_k(y_m, \delta', w, m) \Big) \\
& - \frac{1}{m} - r - cr \geq I(x, \delta, w, \mathcal{Y}) - 3r - 2cr,
\end{aligned}
$$

by (3.4) and (3.6). Since $r \in \mathbb{Q}_+$ could be arbitrarily small, this proves that $I(x, \delta, w) \geq I(x, \delta, w, \mathcal{Y})$. Therefore $\mathcal{Y} \in \mathcal{R}$ and so the property (i) from Definition 1.1 is verified.

To prove that $\mathcal{R}$ is σ-complete, we have to somewhat elaborate on the above constructions. Let $\mathcal{Y}_1 = Y_1 \times Y_1^1 \times \cdots \times Y_1^k$, $\mathcal{Y}_2 = Y_2 \times Y_2^1 \times \cdots \times Y_2^k$, $\dots$, be an increasing sequence of elements of $\mathcal{R}$. Put $\mathcal{Y} := Y \times Y^1 \times \cdots \times Y^k$ where

$$
Y := \overline{Y_1 \cup Y_2 \cup \cdots}, \quad Y^1 := \overline{Y_1^1 \cup Y_2^1 \cup \cdots}, \quad \dots, \quad Y^k := \overline{Y_1^k \cup Y_2^k \cup \cdots}.
$$

We have to show that $\mathcal{Y}$ belongs to $\mathcal{R}$. This means to verify (3.2) and (3.3).

The proof of (3.2) is straightforward. As regards (3.3), fix some $x \in Y$, $\delta \in \Delta \cap \mathbb{Q}^\omega$ and $w = (w_1, \dots, w_k) \in S_{Y^1} \times \cdots \times S_{Y^k}$. We have to prove that $I(x, \delta, w) \geq I(x, \delta, w, \mathcal{Y})$. Because the assignment $u \mapsto p_{d,u}(\cdots)$ is uniformly continuous, we may and do assume that $w \in S_{Y_j^1} \times \cdots \times S_{Y_j^k}$ for some $j \in \mathbb{N}$. Now, take any $(h_n) \in \mathcal{H}(\nu, \delta)$ and any $(x_1, \dots, x_k) \in X_1 \times \cdots \times X_k$. Let again $N := \{n \in \mathbb{N} : \ \lambda_n > 0\}$; this is a finite set. Take an arbitrary $r \in \mathbb{Q}_+$ so small that $\|h_n\| < \delta_{\nu_n} - 2r$ for every $n \in N$. Find then $\delta' = (\delta_n') \in \Delta \cap \mathbb{Q}^\omega$ such that $\delta_n' \leq \delta_n$ for every $n \in \mathbb{N}$ and $\delta_n' = \delta_n - r$ if $n \in N$. Take $m \in \mathbb{N}$ so big that $m > j$ and $\operatorname{dist}(x, Y_m) < r$; pick then $y_m \in Y_m$ so that $\|x - y_m\| < r$. Define $h_n' := h_n + x - y_m$ if $n \in N$, and $h_n' := 0$ if $n \in \mathbb{N} \setminus N$. Then $(h_n') \in \mathcal{H}(\nu, \delta')$ and from the first half of (3.7) (valid also now) we have

$$
\begin{aligned}
(3.8)\qquad r + cr \ + \ & \sum_{n=1}^{\infty} \lambda_n \Big(f(x+h_n) + \frac{1}{\nu_n}\|h_n\| \Big) + p_{d,w}\Big(\sum_{n=1}^{\infty} \lambda_n h_n, x_1, \dots, x_k \Big) \\
\geq \ & I(y_m, \delta', w) = I(y_m, \delta', w, \mathcal{Y}_m) \geq I(y_m, \delta', w, \mathcal{Y})
\end{aligned}
$$

since $y_m \in Y^m$, $\mathcal{Y}_m \in \mathcal{R}$, and $\mathcal{Y}_m \subset \mathcal{Y}$.

Now, consider any $(k_n) \in \mathcal{H}(\nu, \delta') \cap Y^\omega$ and any $(x'_1, \ldots, x'_k) \in Y_1 \times \cdots \times Y_k$. Set $k'_n := k_n + y_m - x$ if $n \in N$, and $k'_n := 0$ if $n \in \mathbb{N} \setminus N$. Then $(k'_n) \in \mathcal{H}(\nu, \delta) \cap Y^\omega$ and we can estimate

$$\begin{aligned}
& \sum_{n=1}^{\infty} \lambda_n \Big(f(y_m + k_n) + \frac{1}{\nu_n} \|k_n\| \Big) + p_{d,w} \Big(\sum_{n=1}^{\infty} \lambda_n k_n, x'_1, \ldots, x'_k \Big) \\
= \; & \sum_{n=1}^{\infty} \lambda_n \Big(f(x + k'_n) + \frac{1}{\nu_n} \|k_n\| \Big) + p_{d,w} \Big(\sum_{n=1}^{\infty} \lambda_n k_n, x'_1, \ldots, x'_k \Big) \\
\geq \; & \sum_{n=1}^{\infty} \lambda_n \Big(f(x + k'_n) + \frac{1}{\nu_n} \|k'_n\| \Big) + p_{d,w} \Big(\sum_{n=1}^{\infty} \lambda_n k'_n, x'_1, \ldots, x'_k \Big) - r - cr \\
\geq \; & I(x, \delta, w, \mathcal{Y}) - r - cr.
\end{aligned}$$

Hence

$$I(y_m, \delta', w, \mathcal{Y}) \geq I(x, \delta, w, \mathcal{Y}) - r - cr. \tag{3.9}$$

Therefore, combining (3.8) and (3.9) and recalling that $r \in \mathbb{Q}_+$ was arbitrarily small, we conclude that $I(x, \delta, w) \geq I(x, \delta, w, \mathcal{Y})$. This verifies (3.3) for our $\mathcal{Y}$ and hence guarantees that $\mathcal{Y} \in \mathcal{R}$. We proved that $\mathcal{R}$ is σ-complete. $\square$

Remark 3.2. There are other rich families associated with (i). For instance we can drop condition (3.2) in the definition of C_m. But the family so obtained cannot be used for separable reduction of (ii) in Proposition 2.2.

4. Main result

We can now state and prove the main result of the paper.

Theorem 4.1. *Let $k \in \mathbb{N}$, let $X, X_1, \ldots, X_k$ be Banach spaces, let $A_i : X_i \to X$, $i = 1, \ldots, k$, be bounded linear operators, and let f be a proper extended real-valued function on X. Let finally $c \geq 0$, $\varepsilon_1 > 0, \ldots, \varepsilon_k > 0$, $\rho_1 \geq 0, \ldots, \rho_k \geq 0$ be given constants. Then there exists a rich block-family $\mathcal{R} \subset \mathcal{S}(X \times X_1 \times \cdots \times X_k)$ such that for every $Y \times Y_1 \times \cdots \times Y_k \in \mathcal{R}$ we have $A_1(Y_1) \subset Y, \ldots, A_k(Y_k) \subset Y$, and for every $x \in Y$ the following holds:*
There is an $x^* \in \partial_F f(x)$ such that $\|x^*\| \leq c$ and $\big| \|A_i^* x^*\| - \rho_i \big| < \varepsilon_i$, $i = 1, \ldots, k$, whenever there is a $y^* \in \partial_F (f|_Y)(x)$ such that $\|y^*\| \leq c$ and $\big| \|(A_i|_{Y_i})^* y^*\| - \rho_i \big| < \varepsilon_i$, $i = 1, \ldots, k$.

Proof. Put $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_k)$, $\rho := (\rho_1, \ldots, \rho_k)$, and $d := (c, \varepsilon, \rho)$. For every $\lambda \in \Lambda \cap \mathbb{Q}^\omega$ and every $\nu \in \Upsilon$ let $\mathcal{R}_{\lambda,\nu,d}$ be the corresponding rich family from Proposition 3.1. As there are countably many such λ and ν, the intersection $\mathcal{R}$ of all such families over λ and ν is also a rich family by Proposition 1.2. This is precisely the family we need.

Indeed, take any $\mathcal{Y} := Y \times Y_1 \times \cdots \times Y_k \in \mathcal{R}$. Take any $x \in Y$ and assume that there is $y^* \in \partial_F(f|_Y)(x)$, with $\|y^*\| \le c$ and $\big|\|(A_i|_{Y_i})^* y^*\| - \rho_i\big| < \varepsilon_i$, $i = 1, \ldots, k$. By Proposition 2.2, there are $\varepsilon_i' \in (0, \varepsilon_i) \cap \mathbb{Q}$, $i = 1, \ldots, k$, $w \in S_{Y_1} \times \cdots \times S_{Y_k}$ and $\delta \in \Delta \cap \mathbb{Q}^\omega$ such that, when putting $d' := (c, (\varepsilon_1', \ldots, \varepsilon_k'), \rho)$, we have $I(x, \lambda, \nu, \delta, d', w, \mathcal{Y}) \ge f(x)$ for every $\lambda \in \Lambda$ and for every $\nu \in \Upsilon$. But then, by the definition of our $\mathcal{R}$ and by (3.3), we have that $I(x, \lambda, \nu, \delta, d', w) \ge f(x)$ for every $\lambda \in \Lambda \cap \mathbb{Q}^\omega$ and $\nu \in \Upsilon$. Applying again Proposition 2.2, we conclude that there exists $x^* \in \partial_F f(x)$, with $\|x^*\| \le c$ and $\big|\|(A_i|_{Y_i})^* x^*\| - \rho_i\big| < \varepsilon_i$ for every $i = 1, \ldots, k$. $\square$

All the results to follow are consequences of the theorem above.

Corollary 4.2 (Preiss-Zajíček; see [Lindenstrauss-Preiss-Tišer12])**.** *Let X be a Banach space and f an extended real-valued function on X. Then there is a rich family $\mathcal{R} \subset \mathcal{S}(X)$ such that for any $Y \in \mathcal{R}$, any $x \in Y$, and any $c \ge 0$ the following two properties are equivalent:*

(a) f is Fréchet differentiable at x and $\|f'(x)\| \le c$;

(b) $f|_Y$ is Fréchet differentiable at x and $\|(f|_Y)'(x)\| \le c$.

Proof. Applying the theorem to f and $X_1 := \cdots := X_k := \{0\}$, we immediately get a rich family $\mathcal{R}_+ \subset \mathcal{S}(X)$ such that for any $Y \in \mathcal{R}_+$ and any $x \in Y$ we can be sure that $\partial_F f(x)$ contains an element with norm not greater than c if the same is true for $\partial_F(f|_Y)(x)$. Likewise, applying the theorem, we find a rich family $\mathcal{R}_- \subset \mathcal{S}(X)$ with similar properties for the function $-f$. It remains to set $\mathcal{R} := \mathcal{R}_+ \cap \mathcal{R}_-$ and apply Proposition 1.2 taking into account that f is Fréchet differentiable at x if (and only if) both $\partial_F f(x)$ and $\partial_F(-f)(x)$ are nonempty. This proves that (b) $\Rightarrow$ (a). The opposite implication is trivial. $\square$

Theorem 4.1 is suitable for separable reductions of various statements on Fréchet subdifferential of one function. As a very particular case of it we get the existence of a rich family of separable subspaces that guarantees separable reduction of the non-emptiness of Fréchet subdifferential. But Theorem 4.1 allows to say more.

Corollary 4.3. *Given a Banach space X, a proper function $f : X \longrightarrow (-\infty, +\infty]$, and constants $0 \le \delta < c$, then there exists a rich family $\mathcal{R} \subset \mathcal{S}(X)$ such that $\delta < \|x^*\| < c$ for some $x^* \in \partial_F f(x)$ whenever $Y \in \mathcal{R}$, $x \in Y$, and $\delta < \|y^*\| < c$ for some $y^* \in \partial_F(f|_Y)f(x)$.*

Proof. Let $k := 1$, $X_1 := X$, and let A_1 be the identity operator on X. For every $\varepsilon_1 > 0$, $\rho_1 > 0$ (and our given c) let $\mathcal{R}_{\varepsilon_1, \rho_1}$ be the corresponding rich block-family in $\mathcal{S}(X \times X)$ found in Theorem 4.1. Put $\mathcal{R}_0 := \bigcap \big\{\mathcal{R}_{\varepsilon_1, \rho_1} :\ \varepsilon_1, \rho_1 \in \mathbb{Q}_+\big\}$; this is again a rich block-family in $\mathcal{S}(X \times X)$ by Proposition 1.2. Put $\mathcal{R}_1 := \{Y \times Y :\ Y \in \mathcal{S}(X)\}$; clearly this is a rich family in $\mathcal{S}(X \times X)$. Put $\mathcal{R}_2 := \mathcal{R}_0 \cap \mathcal{R}_1$; this is a rich family by Proposition 1.2. Define finally $\mathcal{R} := \{Y \in \mathcal{S}(X) :\ Y \times Y \in \mathcal{R}_2\}$; it is easy to show that this is a rich family in $\mathcal{S}(X)$.

It remains to verify that this $\mathcal{R}$ "works". So take any Y in it, any $x \in Y$, and assume there is $y^* \in \partial_F(f|_Y)(x)$ satisfying that $\delta < \|y^*\| < c$. Find $\varepsilon, \rho \in \mathbb{Q}_+$

such that $\delta < \rho - \varepsilon < \|y^*\| < \rho + \varepsilon < c$. By Theorem 4.1, as $Y \in \mathcal{R}_{\varepsilon,\rho}$, there is $x^* \in \partial_F f(x)$ such that $\|x^*\| \le c$ and $\big|\|x^*\| - \rho\big| < \varepsilon$. It then follows that $\delta < \rho - \varepsilon < \|x^*\| < \rho + \varepsilon < c$. □

If the f is an indicator function of a closed subset Ω of X, then we get separable reduction (via a rich family) of non-zeroness of the Fréchet normal cone of Ω.

We can make a one step further and apply Theorem 4.1 to get the existence of rich families for separable reduction of Fréchet subdifferentiability of composite functions obtained by means of one or another functional operation with various quantitative requirements on elements of Fréchet subdifferentials. The following umbrella theorem is a gateway to many results of this sort.

Theorem 4.4. *Let $m \in \mathbb{N}$, let $Z, Z_1, \dots, Z_m$ be Banach spaces, and let constants $c \ge 0,\ \gamma > 0,\ \varepsilon_i > 0,\ \rho_i \ge 0$, proper functions $f_i : Z_i \longrightarrow (-\infty, +\infty]$, and linear bounded operators $\Lambda_i : Z \to Z_i,\ i = 1, \dots, m$, be given. Then there exists a rich block-family $\mathcal{R} \subset \mathcal{S}(Z \times Z_1 \times \cdots \times Z_m)$ such that for every $V \times V_1 \times \cdots \times V_m \in \mathcal{R}$ we have $\Lambda_1(V) \subset V_1, \dots,\ \Lambda_m(V) \subset V_m$, and for every $(z_1, \dots, z_m) \in V_1 \times \cdots \times V_m$, the following holds:*
There are $z_1^* \in \partial_F f_1(z_1), \dots,\ z_m^* \in \partial_F f_m(z_m)$ such that

$$\textstyle\sum_{i=1}^m \|z_i^*\| \le c, \quad \|\sum_{i=1}^m \Lambda_i^* z_i^*\| < \gamma, \quad \big|\|\Lambda_i^* z_i^*\| - \rho_i\big| < \varepsilon_i,\ \ i = 1, \dots, m,$$

whenever there are $v_1^* \in \partial_F(f_1|_{V_1})(z_1), \dots,\ v_m^* \in \partial_F(f_m|_{V_m})(z_m)$ such that

$$\textstyle\sum_{i=1}^m \|v_i^*\| \le c, \quad \|\sum_{i=1}^m (\Lambda_i|_V)^* v_i^*\| < \gamma, \quad \big|\|(\Lambda_i|_V)^* v_i^*\| - \rho_i\big| < \varepsilon_i,\ \ i = 1, \dots, m.$$

Proof. Set $X := Z_1 \times \cdots \times Z_m$, and endow it with the ℓ_∞-norm, so that for $x = (z_1, \dots, z_m) \in X$ and $x^* = (z_1^*, \dots, z_m^*) \in X^*$ we have $\|x\| = \max\{\|z_1\|, \dots, \|z_m\|\}$ and $\|x^*\| = \|z_1^*\| + \cdots + \|z_m^*\|$. For every subspace U of Z we denote $\Delta U := \{(z, \dots, z) :\ z \in U\}$. Set further $X_0 := \Delta Z,\ X_1 := Z, \dots,\ X_m := Z$, and define operators $A_i : X_i \to X,\ i = 0, 1, \dots, m$, as follows: $A_0(z, \dots, z) := (\Lambda_1 z, \dots, \Lambda_m z)$ and, for $i = 1, \dots, m$, $A_i(z) := (0, \dots, 0, \Lambda_i z, 0, \dots 0)$ with $\Lambda_i z$ at the i-th place. An elementary calculation reveals that for $z_1^* \in Z_1^*, \dots, z_m^* \in Z_m^*$ we have

(4.1)
$$\|A_0^*(z_1^*, \dots, z_m^*)\| = \|\Lambda_1^* z_1^* + \cdots + \Lambda_m^* z_m^*\|; \quad \|A_i^*(z_1^*, \dots, z_m^*)\| = \|\Lambda_i^* z_i^*\|,\ i = 1, \dots, m.$$

More generally, if $V \in \mathcal{S}(Z),\ V_i \in \mathcal{S}(Z_i)$, and $v_1^* \in V_1^*, \dots,\ v_m^* \in V_m^*$, we have

$$\big\|\big(A_0|_{\Delta V}\big)^*(v_1^*, \dots, v_m^*)\big\| = \big\|\big(\Lambda_1|_V\big)^* v_1^* + \cdots + \big(\Lambda_m|_V\big)^* v_m^*\big\| \tag{4.2}$$

$$\big\|\big(A_i|_{V_i}\big)^*(v_1^*, \dots, v_m^*)\big\| = \big\|\big(\Lambda_i|_V\big)^* v_i^*\big\|, \ \ i = 1, \dots, m. \tag{4.3}$$

Let now $f : X \longrightarrow (-\infty, +\infty]$ be defined by

$$f(z_1, \dots, z_m) = f_1(z_1) + \cdots + f_m(z_m), \quad (z_1, \dots, z_m) \in X.$$

Clearly, this is a proper function. Moreover, this is a "separable" function, i.e., the sum of functions depending on mutually different arguments; so

$$\partial_F f(z_1, \dots, z_m) = \partial_F f_1(z_1) \times \cdots \times \partial_F f_m(z_m). \tag{4.4}$$

Finally, we set $\varepsilon_0 = \gamma$, $\rho_0 = 0$.

Let $\mathcal{R}_0 \subset \mathcal{S}(X \times X_0 \times X_1 \times \cdots \times X_m)$ be the rich block-family found in Theorem 4.1 for our constants, $c,\ \varepsilon_i,\ \rho_i,\ i = 0, 1, \ldots, m$, and for our operators $A_0, A_1, \ldots, A_m$. Consider the block-family

$$\mathcal{R}_1 := \big\{V_1 \times \cdots \times V_m \times \Delta V \times V \times \cdots \times V :\ V_1 \in \mathcal{S}(Z_1), \ldots, V_m \in \mathcal{S}(Z_m),\ V \in \mathcal{S}(Z)\big\};$$

clearly, it is rich in $\mathcal{S}\big(X \times X_0 \times X_1 \times \cdots \times X_m\big)$. Put $\mathcal{R}_2 := \mathcal{R}_0 \cap \mathcal{R}_1$; it is also rich by Proposition 1.2. Finally, put

$$\mathcal{R} := \big\{V \times V_1 \times \cdots \times V_m :\ V_1 \times \cdots \times V_m \times \Delta V \times V \times \cdots \times V \in \mathcal{R}_2\big\};$$

this block-family is also rich, now in $\mathcal{S}\big(Z \times Z_1 \times \cdots \times Z_m\big)$.

We shall show that $\mathcal{R}$ has the desired properties. So, fix any $V \times V_1 \times \cdots \times V_m \in \mathcal{R}$. Then $V_1 \times \cdots \times V_m \times \Delta V \times V \times \cdots \times V \in \mathcal{R}_0$. Now, apply Theorem 4.1 where we plug $k := m + 1,\ Y := V_1 \times \cdots \times V_m,\ Y_0 := \Delta V,\ Y_1 := V,\ \ldots,\ Y_m := V$, and get that $A_0(\Delta V) \subset V_1 \times \cdots \times V_m,\ A_1(V) \subset V_1 \times \cdots \times V_m, \ldots,\ A_m(V) \subset V_1 \times \cdots \times V_m$. Thus, using the definition of A_i's, we get that $\Lambda_1(V) \subset V_1, \ldots,\ \Lambda_m(V) \subset V_m$.

Take now any $x = (z_1, \ldots, z_m) \in V_1 \times \cdots \times V_m$. Then the statement: "there is an $x^* \in \partial_F f(x)$ such that $\|x^*\| \le c$ and $\big|\|A_i^* x^*\| - \rho_i\big| < \varepsilon_i$ for $i = 0, 1, \ldots, m$" means, by (4.1), that $x^* = (z_1^*, \ldots, z_m^*)$ for some $z_i^* \in \partial_F f_i(z_i)$ and $\|z_1^*\| + \cdots + \|z_m^*\| \le c$, $\|\Lambda_1^* z_1^* + \cdots + \Lambda_m^* z_m^*\| < \varepsilon_0 = \gamma$ and $\big|\|\Lambda_i^* z_i^*\| - \rho_i\big| < \varepsilon_i,\ i = 1, \ldots, m$.

Likewise, the statement: "there is $v^* \in \partial_F\big(f|_{V_1 \times \cdots \times V_m}\big)(x)$ such that $\|v^*\| \le c$ and $\big|\|\big(A_i|_V\big)^* v^*\| - \rho_i\big| < \varepsilon_i$ for $i = 0, \ldots, m$" means by (4.2) and (4.3), that $v^* = (v_1^*, \ldots, v_m^*)$ for some $v_i^* \in \partial_F\big(f_i|_{V_i}\big)(z_i)$, $\|v_1^*\| + \cdots + \|v_m^*\| \le c$, $\|(\Lambda_1|_V)^* v_1^* + \cdots + (\Lambda_m|_V)^* v_m^*\| < \varepsilon_0 = \gamma$ and $\big|\|(\Lambda_i|_V)^* v_i^*\| - \rho_i\big| < \varepsilon_i,\ i = 1, \ldots, m$.

As, by Theorem 4.1, the first statement holds at $x = (z_1, \ldots, z_m) \in V_1 \times \cdots \times V_m$ if the second statement holds at the point, this completes the proof. $\square$

As consequences of Theorem 4.4, we can get quantitative versions of separable reductions (via suitable rich families) for a fuzzy calculus and an extremal principle for Fréchet subdifferentials and Fréchet normal cones, respectively. In the following corollaries we consider (as simple but basic examples) the operations of composition with a linear operator and sum of functions.

Corollary 4.5. *Let X and Y be Banach spaces, let f be a proper function on Y, let $A : X \to Y$ be a bounded linear operator, and let $x^* \in X^*$. Given an $\varepsilon > 0$ and $c > \|x^*\|$, then there exists a rich family $\mathcal{R} \subset \mathcal{S}(X \times Y)$ such that for every $U \times V \in \mathcal{R}$ we have $A(U) \subset V$ and for every $y \in V$ the following holds:*
There is $y^* \in \partial_F f(y)$ such that $\|y^*\| + \|x^*\| \le c$ and $\|A^* y^* - x^*\| < \varepsilon$ whenever there is $v^* \in \partial_F(f|_V)(y)$ such that $\|v^*\| + \big\|x^*|_U\big\| \le c$ and $\big\|(A|_U)^* v^* - x^*|_U\big\| < \varepsilon$.

Proof. Applying Theorem 4.4 to $m := 2$, $\gamma := \varepsilon$, to any $\varepsilon_1 > 0, \varepsilon_2 > 0, \rho_1 > 0, \rho_2 > 0$, and to $Z := X,\ Z_1 := Y,\ Z_2 := X,\ f_1 := f,\ f_2 := -x^*,\ A_1 := A$, to A_2 being the identity operator on Z_2, we get a rich block-family $\mathcal{R}_{\varepsilon_1, \varepsilon_2, \rho_1, \rho_2} \subset \mathcal{S}(X \times Y \times X)$. Put then $\mathcal{R}_0 := \bigcap \big\{\mathcal{R}_{\varepsilon_1, \varepsilon_2, \rho_1, \rho_2} :\ \varepsilon_1, \varepsilon_2, \rho_1, \rho_2 \in \mathbb{Q}_+\big\}$; this is a rich block-family. Further put $\mathcal{R}_1 := \{U \times V \times U : U \in \mathcal{S}(X),\ V \in \mathcal{S}(Y)\}$ and then $\mathcal{R}_2 := \mathcal{R}_0 \cap \mathcal{R}_1$; we again got a rich block-family. Finally, define $\mathcal{R} := \{U \times V : U \times V \times U \in \mathcal{R}_2\}$; it is easy to check that this is also a rich block-family. Now, the verification that our $\mathcal{R}$ has the desired properties is routine. $\square$

As a consequence, we get the fuzzy chain rule for the Fréchet subdifferential of the composition with a linear operator.

Theorem 4.6. *In addition to the assumptions of Corollary 4.5, suppose that Y is an Asplund space, f is function on Y Lipschitzian in a vicinity of $\overline{y} := A\overline{x}$, let $x^* \in \partial_F(f \circ A)(\overline{x})$, and let $\varepsilon > 0$ be given. Then there are $y \in Y$ and $y^* \in \partial_F f(y)$ such that $\|y - A\overline{x}\| < \varepsilon$ and $\|A^*y^* - x^*\| < \varepsilon$.*

The novelty of this result, compared with the known fuzzy chain rules for compositions, is that X is no longer assumed to be Asplund. The (small) price we pay is the necessity to assume that f is Lipschitz near $\overline{y}$. It is not clear to us whether this assumption is essential or is connected only with the techniques used in the proof below.

Proof. For $x \in X$ set $g(x) = (f \circ A)(x)$; then $g(x) \geq g(\overline{x}) + \langle x^*, x - \overline{x}\rangle - r(\|x - \overline{x}\|$, where $r : [0, +\infty) \longrightarrow [(0, +\infty)$ is a function such that $\frac{r(t)}{t} \to 0$ as $t \downarrow 0$. We can moreover assume that $r(t) > 0$ for all $t > 0$, that r is convex, and that $\partial r(t) \to 0$ as $t \downarrow 0$. (For convexity see [Ioffe13]; the rest is easy to show.) Fix any $\delta > 0$ such that f is Lipschitz with constant ℓ on the ball $B(\overline{y}, \delta)$.

For every $n = 1, 2, \ldots$ consider the function

$$\varphi_n(u, y) = f(y) + n\|y - Au\|^2 + 2r(\|u - \overline{x}\|) - \langle x^*, u - \overline{x}\rangle, \quad (u, y) \in X \times Y.$$

Set $Q := B(\overline{x}, \delta) \times B(\overline{y}, \delta)$ and choose for every $n \in \mathbb{N}$ a pair $(u_n, v_n) \in Q$ such that

$$\varphi_n(u_n, v_n) \leq \inf_Q \varphi_n + n^{-2}. \tag{4.5}$$

We claim that $(u_n, v_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.

Indeed, we notice first that, $\|v_n - Au_n\| \to 0$ as all terms of φ_n are bounded below on Q and $\inf_Q \varphi_n \leq \varphi_n(\overline{x}, \overline{y}) = f(\overline{y}) < +\infty$. Furthermore

$$\begin{array}{rcl} n^{-2} + f(\overline{y}) \geq \varphi(u_n, v_n) & \geq & f(v_n) + 2r(\|u_n - \overline{x}\|) - \langle x^*, u_n - \overline{x}\rangle \\ & \geq & f(Au_n) - \ell\|v_n - Au_n\| + 2r(\|u_n - \overline{x}\|) - \langle x^*, x_n - \overline{x}\rangle \\ & \geq & f(\overline{y}) + \langle x^*, u_n - \overline{x}\rangle - r(\|u_n - \overline{x}\|) - \ell\|v_n - Au_n\| \\ & & \qquad +2r(\|u_n - \overline{x}\|) - \langle x^*, u_n - \overline{x}\rangle \\ & = & f(\overline{y}) - \ell\|v_n - Au_n\| + r(\|u_n - \overline{x}\|) \end{array}$$

and we conclude that

$$0 \leq r(\|u_n - \overline{x}\|) \leq n^{-2} + \ell\|v_n - Ax_n\| \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Now, the properties of r force that $u_n \to \overline{x}$, and consequently $v_n \to \overline{y}$ as claimed.

We shall first verify that the result is true when Y is a separable space. In this case as Y is Asplund, there is a dense collection of Fréchet smooth equivalent norms in Y ([Deville-Godefroy-Zizler], pp. 43, 48). So we assume that Y is endowed with such a norm, say $\|\cdot\|_Y$.

For every $n \in \mathbb{N}$ so big that (u_n, v_n) lies in Q, the variational principle of Ekeland applied to φ_n restricted to Q yields a pair $(x_n, y_n) \in Q$ such that $\|x_n - u_n\| \le n^{-1}$, $\|y_n - v_n\| \le n^{-1}$ and that

$$\psi_n(x, y) := \varphi_n(x, y) + n^{-1}(\|x - x_n\| + \|y - y_n\|) \ge \psi_n(x_n, y_n) = \varphi_n(x_n, y_n)$$

for all $(x, y) \in Q$. From the above, we get that $\|x_n - \overline{x}\| \to 0$ and $\|y_n - \overline{y}\| \to 0$ as $n \to \infty$. Thus for all $n \in \mathbb{N}$ big enough the couple (x_n, y_n) lies in the interior of Q and we can subdifferentiate in Fréchet sense without any troubles. From now on, consider any such $n \in \mathbb{N}$.

Let v_n^* be the derivative of $n\|\cdot\|_Y^2$ at the point $y_n - Ax_n$, and let $\gamma_n(x, y)$ be the sum of all terms of ψ_n except $n\|y - Ax\|^2 - \langle x^*, x - \overline{x}\rangle$. As the latter is Fréchet differentiable, we have that $\partial_F\psi_n(x_n, y_n) = \partial_F\gamma_n(x_n, y_n) + (-A^*v_n^*, v_n^*) - (x^*, 0)$. On the other hand, γ_n is the sum of two functions, one $\lambda_n(y) = f(y) + n^{-1}\|y - y_n\|$, $y \in Y$, depending only on y and the other $\mu_n(x) = 2r(\|x - \overline{x}\|) + n^{-1}\|x - x_n\|$, $x \in X$, depending only on x. Therefore $\partial_F\gamma_n(x_n, y_n) = \partial_F\mu_n(x_n) \times \partial_F\lambda_n(y_n)$. The Fréchet subdifferentials of λ_n and μ_n are easy to estimate. Indeed, as Y is Asplund, the standard fuzzy sum rule (see Theorem 4.8 below) gives $\partial_F\lambda_n(y_n) \subset \partial_F f(\tilde{y}_n) + n^{-1}B_{Y^*} + n^{-1}B_{Y^*}$ with some $\tilde{y}_n \in Y$ satisfying $\|\tilde{y}_n - y_n\| < n^{-1}$. On the other hand, μ_n is the sum of two convex continuous functions. Therefore the Fréchet subdifferentials of μ_n and both its component functions coincide with their convex subdifferentials, so that $\partial_F\mu_n(x_n) \subset 2\partial r(\|x_n - \overline{x}\|)B_{X^*} + n^{-1}B_{X^*} =: \delta_n B_{X^*}$, where $\delta_n \to 0$ as $n \to \infty$.

Summarizing, we conclude that $0 \in \partial_n f(\tilde{y}_n) + v_n^* + 2n^{-1}B_{Y^*}$ and $0 \in -Av_n^* - x^* + \delta_n B_{X^*}$, where $\delta_n \to 0$ as $n \to \infty$. This means that there is a $y_n^* \in \partial_F f(\tilde{y}_n)$ such that $\|y_n^* + v_n^*\| \le 2n^{-1}$ and $\|x^* - A^*y_n^*\| \le \delta_n + 2n^{-1}\|A\| \to 0$. It remains to note that $\|\tilde{y}_n - Ax_n\| \to 0$ as $\tilde{y}_n \to \overline{y}$ and $x_n \to \overline{x}$. This completes the proof of the theorem for the case of separable Y.

Returning to the statement of the theorem, assume now that Y is a non-separable Asplund space. Let $\mathcal{R}_{mn} \subset \mathcal{S}(X \times Y)$ be a rich block-family satisfying Corollary 4.5 and corresponding to $c := m$ and $\varepsilon := n^{-1}$. Then $\mathcal{R} = \bigcap \mathcal{R}_{mn}$ is also a rich family by Proposition 1.2. Take a $(U, V) \in \mathcal{R}$ with $\overline{x} \in U$ and $\overline{y} \in V$. Clearly, $A|_U : U \to V$ and $x^*|_U \in \partial_F(f|_V \circ A|_U)(\overline{x})$.

Now, from the separable case already proved, we can find a $y \in V$ and a $v^* \in \partial_F f|_V(y)$ such that $\|y - \overline{y}\| < \varepsilon$ $\|x^*|_U - A|_U^* v^*\| < \varepsilon$. It remains to choose $m, n \in \mathbb{N}$ such that the two inequalities remain valid with ε replaced by n^{-1} and $\|v^*\| + \|x^*\| \le n$ and to apply Corollary 4.5 taking into account that $(U, V) \in \mathcal{R}_{mn}$. □

The second corollary of Theorem 4.4 is related to sums of functions.

Corollary 4.7. *Let Z be a Banach space, consider constants $c \ge 0$, $\varepsilon > 0$, $\rho_1 \ge 0, \ldots, \rho_m \ge 0$, and let proper functions $f_i : Z \longrightarrow (-\infty, +\infty]$, $i = 1, \ldots, m$, be*

given. Then there exists a rich family $\mathcal{R} \in \mathcal{S}(Z)$ such that for every $V \in \mathcal{R}$ and every $z_1, \ldots, z_m \in V$ the following holds:
There are $z_i^* \in \partial_F f_i(z_i),\ i = 1, \ldots, m$, such that

$$\|z_1^*\| + \cdots + \|z_m^*\| \le c, \quad \|z_1^* + \cdots + z_m^*\| < \varepsilon, \quad \big|\|z_i^*\| - \rho_i\big| < \varepsilon,\ i = 1, \ldots, m$$

whenever there are $v_i^* \in \partial_F(f_i|_V)(z_i),\ i = 1, \ldots, m$, such that

$$\|v_1^*\| + \cdots + \|v_m^*\| \le c, \quad \|v_1^* + \cdots + v_m^*\| < \varepsilon, \quad \big|\|v_i^*\| - \rho_i\big| < \varepsilon,\ i = 1, \ldots, m.$$

Proof. Apply Theorem 4.4, with $Z_1 := \cdots := Z_m := Z$, Λ_i being identities and $\gamma := \varepsilon_1 := \cdots := \varepsilon_m := \varepsilon$, and get a rich block-family $\mathcal{R}_0 \subset \mathcal{S}(Z^{m+1})$. Using a simple gymnastics like in the proof of Corollary 4.5, we produce a rich family $\mathcal{R}$ in $\mathcal{S}(Z)$ with the desired property. □

The corollary, in turn, provides a direct access to the fuzzy sum rule in Asplund spaces which, in the simplest form, is stated as follows.

Theorem 4.8. *Let X be an Asplund space, and let f_1 and f_2 be two lower semicontinuous functions on X with one of them Lipschitz near a certain $x \in X$. If $x^* \in \partial_F(f_1 + f_2)(x)$, then for any $\varepsilon > 0$ there are $x_i \in X$ and $x_i^* \in \partial_F f_i(x_i)$, $i = 1, 2$, such that $\|x_i - x\| < \varepsilon$ and $\|x_1^* + x_2^* - x^*\| < \varepsilon$.*

Proof. The statement is true if X is a separable Asplund space. For the proof, first find an equivalent Fréchet smooth norm, see e.g. [Deville-Godefroy-Zizler, pages 48, 43], and then proceed as in [Ioffe83]. If X is non-separable, put together the validity of the separable case and Corollary 4.7 in a way similar to how in has been done in the proof of Theorem 4.6. □

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