A direct approach to membrane reinforced bodies

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Abstract The paper deals with membrane reinforced bodies with the membrane treated as a two dimensional surface with concentrated material properties. The membrane response is linearized so that it depends linearly on the surface strain tensor. The response of the matrix is treated separately in three cases: (a) as a nonlinear material, (b) as a linear material and finally (c) as a notension material. For the general nonlinear material, the principle of minimum energy and complementary energy are proved. For the linearly elastic matrix the surface Korn inequality is used to prove the existence of equilibrium state under general loads. Finally, for the notension material a theorem stating that the total energy of the system is bounded from below on the space of admissible displacements if and only if the loads are equilibrated by a statically admissible stress that is negative semidefinite. An example presenting an admissible stress solution is given for a rectangular panel with membrane occupying the main diagonal plane.

Key Words Membranes in the bulk matrix, equilibrium of forces, no–tension body

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1. Introduction

The present paper outlines an approach to membrane-reinforced bodies in which the membrane is treated as an ideal two dimensional surface with concentrated material properties. We refer to [12] and [13] for details.

We consider static situations. The system of forces is described by the bulk stress tensor describing the situation in the matrix and of the surface stress tensor describing the situation on the membranes. The interaction of the matrix with the membranes is implicitly contained in the virtual power principle which postulates effectively the equilibrium. A localization procedure leads to the strong form of the equilibrium equations which contain some new forms, see Equation (3) (below).

The constitutive theory starts with large deformation approach. However, within the present approach, which is ultimately oriented on masonry (no–tension) materials, we soon move to the small strain for both the bulk and membrane response. Thus for the bulk we use the familiar small strain tensor but the response is assumed linear only in Section 4, where we treat membrane in a linearly elastic matrix. The rest deals with a nonlinear response of the matrix. As for the membrane, we linearize the response in the reference configuration to deal with the linearly elastic membrane; the surface strain tensor plays essential role.

In Section 3 we deal with a nonlinear matrix reinforced by linearly elastic membrane. We introduce the equilibrium states and state, for a material with the convex energy function, the principle of minimum potential energy at equilibrium. Under appropriate invertibility condition on the stress strain relation we prove the principle of minimum complementary energy.

In Section 4 we deal with the linearly elastic matrix with the linearly elastic reinforcement. The main goal is to prove the existence of the equilibrium solution for arbitrary loads. For this we need an appropriate version of the surface Korn inequality which we take from Ciarlet [2, Theorem 2.7-1]. The form of the inequality motivates the definition of the space of admissible displacements which is essentially $W^{1,2}$ for the bulk and $W^{1,2}$ for the tangential part of the displacement of $\mathcal{M}$ and which is L^2 for the normal component of the displacement. The existence of the equilibrium displacement is then proved in [13] by an application of the Lax Milgram lemma.

Section 5 deals with the matrix made of no–tension material. The existence theory is beyond the scope of the present paper. Instead, we treat simpler topics like the principle of minimum complementary energy which does not follow from the aforementioned proof as the stress strain relation is not invertible. The main result is the necessary and sufficient condition for the boundedness of the total energy from below: this occurs if and only if the loads can be balanced by a square integrable stressfield consisting of the negative semidefinite bulk stress and square integrable surface stress in the membrane. In [10] it is argued that the collapse of the masonry body occurs exactly at the point of the loading process at which the total energy ceases to be bounded from below. Here a simple considerations show that the reinforcement always improves (or at least does not worsen) the situation, the collapse of the body with the reinforcement occurs later in the loading process.

Throughout we use the conventions for vectors and second order tensors identical with those in [5]. Thus Lin denotes the set of all second order tensors on $\mathbb{R}^3$, i.e., linear transformations from $\mathbb{R}^3$ into itself, Sym is the subspace of symmetric tensors, Skw is the subspace of skew (antisymmetric) tensors, Sym_+ the set of all positive semidefinite elements of Sym; additionally, Sym_- is the set of all negative semidefinite elements of Sym . The scalar product of $\mathbf{A}, \mathbf{B} \in \mathrm{Lin}$ is defined by $\mathbf{A} \cdot \mathbf{B} = \mathrm{tr}(\mathbf{A}\mathbf{B}^{\mathrm{T}})$ and $|\cdot|$ denotes the associated euclidean norm on Lin . We denote by $\mathbf{1} \in \mathrm{Lin}$ the unit tensor. If $\mathbf{A}, \mathbf{B} \in \mathrm{Sym}$, we write $\mathbf{A} \le \mathbf{B}$ if $\mathbf{B} - \mathbf{A} \in \mathrm{Sym}_+$.

2. Equilibrium of forces

To describe the forces in the reinforced body, we consider the stress system which consists of the bulk stress in the matrix and the surface stress in the membrane. The equilibrium is postulated in the form of the virtual power principle which allows us to effectively introduce the force interactions between the matrix and the membrane. Next we postulate the constitutive equations of the given system. Here the main measures of deformation are the bulk and surface deformation gradients. We treat the basic properties of the constitutive equations like the principle of objectivity and the symmetry group. Then in the last section we introduce the linearization of the membrane response whose results will be used throughout the paper. The correct form of the linearization has been given in the paper by Gurtin & Murdoch [6].

2.1. **The system of forces.** We identify the body with its reference configuration Ω which is a bounded open subset of $\mathbb{R}^3$ with sufficiently smooth boundary $\partial\Omega$. We assume that within the bulk body Ω there is a collection of nonintersecting surfaces whose union we denote by $\mathcal{M}$, which represents the membranes in the body with different material properties. We denote by $\partial\mathcal{M}$ the collection of boundaries of the membranes and consider the general situation when part of $\partial\mathcal{M}$ is contained in $\partial\Omega$ and part in Ω itself. We denote the general material point in Ω by $\mathbf{x}$ and below we postulate different properties for $\mathbf{x} \in \Omega \setminus \mathcal{M}$ and for $\mathbf{x} \in \mathcal{M}$. We assume that $\mathcal{M}$ is a 2 dimensional manifold so that the tangent cone reduces to a 2 dimensional tangent space $\mathrm{Tan}(\mathcal{M}, \mathbf{x})$ for every point $\mathbf{x} \in \mathcal{M}$. We call a relative normal to $\partial\mathcal{M}$ at $\mathbf{a} \in \mathcal{M}$ the normal to $\partial\mathcal{M}$ which lies in the tangent space to $\mathcal{M}$ at $\mathbf{a}$. We use the same terminology for relative normals to the boundary of a subregion of $\mathcal{M}$.

A system of forces for the body with membranes consists of the bulk stress tensor $\mathbf{T}$, the bulk body force $\mathbf{b}$, the surface stress tensor $\mathbb{T}$ on membranes, and the surface body force $\mathbb{b}$. Here $\mathbf{T}$ and $\mathbf{b}$ are defined on Ω with values in the set Sym of symmetric second order tensors, and $\mathbb{R}^3$, respectively. For every $\mathbf{x} \in \mathcal{M}$, $\mathbb{T}(\mathbf{x})$ is an element of Sym which is superficial in that $\mathbb{T}\mathbb{m} = \mathbf{0}$ where $\mathbb{m}$ is the unit normal to $\mathcal{M}$ and finally $\mathbb{b}$ is defined on $\mathcal{M}$ with values in $\mathbb{R}^3$.

The system of forces is in internal equilibrium and in equilibrium with the environment if the principle of virtual power holds:

$$\int_\Omega \mathbf{T}\cdot\nabla\mathbf{v} - \mathbf{b}\cdot\mathbf{v}\, d\mathcal{L}^3 + \int_\mathcal{M} \mathbb{T}\cdot\nabla\mathbf{v} - \mathbb{b}\cdot\mathbf{v}\, d\mathcal{H}^1 = \int_\mathcal{S} \mathbf{s}\cdot\mathbf{v}\, d\mathcal{H}^2 + \int_{\partial\mathcal{M}\cap\mathcal{S}} \mathbb{s}\cdot\mathbf{v}\, d\mathcal{H}^1 \quad (1)$$

for every virtual velocity field $\mathbf{v}$ on $\mathrm{cl}\,\Omega$ such that $\mathbf{v} = \mathbf{0}$ on $\mathcal{D}$.

To state the strong form of the balance of forces, we introduce the following notation:

$$[\mathbf{T}]_\mathcal{M}\mathbb{m}(\mathbf{x}) := \lim_{\rho\to 0} \big(\mathbf{T}(\mathbf{x} + \rho\mathbb{m}(\mathbf{x}))\mathbb{m}(\mathbf{x}) - \mathbf{T}(\mathbf{x} - \rho\mathbb{m}(\mathbf{x}))\mathbb{m}(\mathbf{x})\big)$$

for each $\mathbf{x} \in \mathcal{M}$ and

$$\langle\mathbf{T}\rangle_\mathcal{M}(\mathbf{a}) := \lim_{\rho\to 0} \rho \int_{\{\mathbf{e}\in\mathrm{Norm}(\partial\mathcal{M},\mathbf{a}):\mathbf{e}\cdot\mathbb{t}(\mathbf{a})\geq 0, |\mathbf{e}|=1\}} \mathbf{T}(\mathbf{a} + \rho\mathbf{e})\mathbf{e}\, d\mathcal{H}^1(\mathbf{e}) \quad (2)$$

for each $\mathbf{a} \in \partial\mathcal{M} \cap \Omega$. The existence of the limits is an assumption. The integration range in (2) is the unit hemicircle normal to $\partial\mathcal{M}$ at $\mathbf{a}$ and the integration variable is $\mathbf{e}$. We obtain the following system of equilibrium equations:

$$\left.\begin{array}{lll} \operatorname{div}\mathbf{T} + \mathbf{b} = \mathbf{0} & \mathcal{L}^3 \text{ a.e. on } & \Omega \\ \mathbb{div}\,\mathbb{T} + [\mathbf{T}]_{\mathcal{M}}\mathbb{n} + \mathbb{b} = \mathbf{0} & \mathcal{H}^2 \text{ a.e. on } & \mathcal{M} \\ \mathbb{T}\mathbb{t} - \langle\mathbf{T}\rangle_{\mathcal{M}} = \mathbf{0} & \mathcal{H}^1 \text{ a.e. on } & \partial\mathcal{M}\cap\Omega \\ \mathbf{T}\mathbf{n} = \mathbf{s} & \mathcal{H}^2 \text{ a.e. on } & \mathcal{S} \\ \mathbb{T}\mathbb{t} = \mathbb{s} & \mathcal{H}^1 \text{ a.e. on } & \partial\mathcal{M}\cap\mathcal{S} \end{array}\right\} \qquad (3)$$

where $\mathbb{t}$ is the relative normal to $\partial\mathcal{M}$ and $\mathbf{n}$ is the normal to $\partial\Omega$.

2.2. **Constitutive equations.** The bulk response is determined by the response functions

$$\hat{f} : \mathrm{Lin}_+ \to \mathbb{R}, \quad \hat{\mathbf{T}} : \mathrm{Lin}_+ \to \mathrm{Lin} \qquad (4)$$

giving the referential volume density of stored energy and stress in terms of the deformation gradient $\mathbf{F}$ where Lin_+ is the set of second order tensors with positive determinant. The surface response is determined by giving, for every $\mathbf{x} \in \mathcal{S}$, the response functions

$$\hat{\mathbf{f}}_{\mathbf{x}} : \mathrm{Lin}_+^{\mathbf{x}} \to \mathbb{R}, \quad \hat{\mathbb{T}}_{\mathbf{x}} : \mathrm{Lin}_+^{\mathbf{x}} \to \mathrm{Lin}^{\mathbf{x}}, \qquad (5)$$

delivering the referential surface density of stored energy and the surface stress in terms of the surface deformation gradient. Here $\mathrm{Lin}^{\mathbf{x}}$ is the set of all $\mathbf{A} \in \mathrm{Lin}$ such that $\mathbf{A}\mathbb{P}(\mathbf{x}) = \mathbf{A}$ where $\mathbb{P}(\mathbf{x})$ is the projection from $\mathbb{R}^3$ onto the tangent space of $\mathcal{M}$ at $\mathbf{x} \in \mathcal{M}$. We have attached the subscript $\mathbf{x}$ to the response functions since the domain of $\hat{\mathbf{f}}$, $\hat{\mathbb{T}}$ is different for every $\mathbf{x} \in \mathcal{S}$; however, for reasons of notational simplicity we often omit the subscript and write $\hat{\mathbf{f}}$ and $\hat{\mathbb{T}}$ in place of $\hat{\mathbf{f}}_{\mathbf{x}}$ and $\hat{\mathbb{T}}_{\mathbf{x}}$. The stress relations read (are postulated here)

$$\hat{\mathbf{T}} = \mathrm{D}\,\hat{f}, \quad \hat{\mathbb{T}}_{\mathbf{x}} = \mathrm{D}\,\hat{\mathbf{f}}_{\mathbf{x}}$$

where D denotes the differentiation of a function with respect to its argument, which is the deformation gradient and the surface deformation gradient, respectively, keeping $\mathbf{x}$ fixed. The constitutive equations then say that the stress corresponding to the deformation $\mathbf{y} : \Omega \to \mathbb{R}^3$ is given by

$$\left.\begin{array}{rcll} \mathbf{T}(\mathbf{x}) & = & \hat{\mathbf{T}}(\mathbf{F}(\mathbf{x})), & \mathbf{x} \in \Omega \setminus \mathcal{S}, \\ \mathbb{T}(\mathbf{x}) & = & \hat{\mathbb{T}}_{\mathbf{x}}(\mathbb{F}(\mathbf{x})), & \mathbf{x} \in \mathcal{S}. \end{array}\right\} \qquad (6)$$

Here $\mathbf{F}$ and $\mathbb{F}$ are the bulk and surface deformation gradients, given by

$$\begin{array}{rcll} \mathbf{F} & = & \nabla\mathbf{y}, & \text{on } \Omega \setminus \mathcal{M}, \\ \mathbb{F} & = & \mathbb{\nabla}\mathbb{y} & \text{on } \mathcal{M}, \end{array}$$

where $\mathbb{y}$ is the restriction of $\mathbf{y}$ to $\mathcal{M}$ and $\mathbb{\nabla}$ is the surface gradient on $\mathcal{M}$, in the specific form defined in [15, Appendix A and B]. The definition makes the surface gradient at $\mathbf{x} \in \mathcal{M}$ of a function h defined on $\mathcal{M}$ and with values in a finite dimensional vectorspace V a linear transformation from $\mathbb{R}^3$ to V and not just the linear transformation from the tangent space $\mathrm{Tan}(\mathcal{M}, \mathbf{x})$ at $\mathbf{x}$ to V. This differs from the definition employed in [6], which is just the restriction of the present ∇h to $\mathrm{Tan}(\mathcal{M}, \mathbf{x})$. Thus in particular $\mathbb{F}(\mathbf{x})$ is an element of Lin, in fact of $\mathrm{Lin}^{\mathbf{x}}$, for any $\mathbf{x} \in \mathcal{M}$.

We define the bulk and surface energy f and $\mathbf{f}$ corresponding to the deformation $\mathbf{y} : \Omega \to \mathbb{R}^3$ by

$$\left.\begin{array}{rcll} f(\mathbf{x}) &=& \hat f(\mathbf{F}(\mathbf{x})), & \mathbf{x} \in \Omega \setminus \mathcal{S}, \\ \mathbf{f}(\mathbf{x}) &=& \hat{\mathbf{f}}_{\mathbf{x}}(\mathbb{F}(\mathbf{x})), & \mathbf{x} \in \mathcal{S}. \end{array}\right\}$$

2.3. **Linearization of the membrane response.** Since we deal with small deformations here, we use the bulk and surface small strain tensors to be introduced here.

Define the displacement $\mathbf{u}$ corresponding to the deformation process $\mathbf{y}$ by $\mathbf{u}(\mathbf{x}) = \mathbf{y}(\mathbf{x}) - \mathbf{x}$ for $\mathbf{x} \in \Omega$, and let $\mathbb{u}$ be the restriction of $\mathbf{u}$ to $\mathcal{M}$. Define the bulk and surface small strain tensors by

$$\left.\begin{array}{ll} \mathbf{E} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}) & \text{in } \Omega \setminus \mathcal{M}, \\ \mathbb{E}_{\mathbf{x}} = \frac{1}{2}\mathbb{P}(\nabla\mathbb{u} + \nabla\mathbb{u}^{\mathrm{T}})\mathbb{P} & \text{on } \mathcal{M}; \end{array}\right\} \tag{7}$$

We have

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{F} + \mathbf{F}^{\mathrm{T}} - 2\mathbf{1}), \quad \mathbb{E} = \tfrac{1}{2}\mathbb{P}(\mathbb{F} + \mathbb{F}^{\mathrm{T}} - 2\mathbb{P})\mathbb{P},$$
$$\mathbb{E} = \mathbb{P}\mathbf{E}\mathbb{P}.$$

Proposition 2.1. *Consider the response specified by the response functions* $\hat f, \hat{\mathbf{f}}$ *as in* $(4)_1$ *and* $(5)_1$ *with the stress response given by* $(4)_2$ *and* $(5)_2$*; assume that the response is objective and that the reference configuration is stress free, i.e.,*

$$\hat{\mathbf{T}}(\mathbf{1}) = \mathbf{0} \quad \text{and} \quad \hat{\mathbb{T}}_{\mathbf{x}}(\mathbb{P}(\mathbf{x})) = \mathbf{0} \quad \text{for every} \quad \mathbf{x} \in \mathcal{M}.$$

Define the bulk and surface elasticity tensors C *and* $\mathfrak{C}_{\mathbf{x}}$ *by*

$$\mathsf{C} = \mathrm{D}\,\hat{\mathbf{T}}(\mathbf{1}) = \mathrm{D}^2\,\hat f(\mathbf{1}),$$

$$\mathfrak{C}_{\mathbf{x}} = \mathrm{D}\,\hat{\mathbb{T}}_{\mathbf{x}}(\mathbb{P}(\mathbf{x})) = \mathrm{D}^2\,\hat{\mathbf{f}}_{\mathbf{x}}(\mathbb{P}(\mathbf{x})), \quad \mathbf{x} \in \mathcal{M}.$$

Let $\mathbf{x} \in \mathcal{M}$ *be fixed and write* $\mathfrak{C} \equiv \mathfrak{C}_{\mathbf{x}}$. *Then*

(i) C *and* $\mathfrak{C}$ *have major symmetry;*

(ii) C *and* $\mathfrak{C}$ *map* Lin *into* Sym *and* Skw *into* $\{\mathbf{0}\}$;

(iii) $\mathfrak{C}$ *is superficial in the sense that* $\mathfrak{C}[\mathbf{A}] = \mathfrak{C}[\mathbb{P}\mathbf{A}\mathbb{P}] = \mathbb{P}\mathfrak{C}[\mathbf{A}]\mathbb{P}$ *for each* $\mathbf{A} \in \mathrm{Lin}$.

Hence

$$\mathsf{C}[\mathbf{H}] = \mathsf{C}[\mathbf{E}], \quad \mathfrak{C}[\mathbb{H}] = \mathfrak{C}[\mathbb{E}]$$

for each $\mathbf{H}, \mathbb{H} \in \mathrm{Lin}$ *where*

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{H} + \mathbf{H}^{\mathrm{T}}), \quad \mathbb{E} = \tfrac{1}{2}\mathbb{P}(\mathbb{H} + \mathbb{H}^{\mathrm{T}})\mathbb{P}.$$

We denoted the value of the linear transformations C and $\mathfrak{C}_{\mathbf{x}}$ on their respective arguments by $\mathsf{C}[\mathbf{E}]$ and $\mathfrak{C}_{\mathbf{x}}[\mathbb{E}]$ to emphasize that they are fourth order tensors. However, we often simplify the notation and write $\mathsf{C}\mathbf{E}$ and $\mathfrak{C}_{\mathbf{x}}\mathbb{E}$. We also often omit the subscript $\mathbf{x}$ and write $\mathfrak{C}$ for $\mathfrak{C}_{\mathbf{x}}$ for notational convenience.

Let $\mathrm{Sym}^{\mathbf{x}} = \mathrm{Sym} \cap \mathrm{Lin}^{\mathbf{x}}$. Assume that we are given for every $\mathbf{x} \in \mathcal{M}$ a fourth order tensor $\mathfrak{C}_{\mathbf{x}}$ such that Assertions (i)–(iii) of Proposition 2.1 hold.

If $\mathbf{u}:\Omega\times\mathbb{R}\to\mathbb{R}^3$ is a displacement field, we define the superficial stress in the linear response by

$$\mathbb{T}(\mathbf{x})=\mathfrak{C}_{\mathbf{x}}\mathbb{E}(\mathbf{x})\qquad \mathbf{x}\in\mathcal{M} \tag{8}$$

in place of $(6)_2$, where $\mathbb{E}$ is given by (7). In the same situation, we define the linearized free energy $\mathbf{f}$ by

$$\mathbf{f}(\mathbf{x})=\tfrac{1}{2}\mathbb{E}(\mathbf{x})\cdot\mathfrak{C}_{\mathbf{x}}\mathbb{E}(\mathbf{x}),\qquad \mathbf{x}\in\mathcal{M}. \tag{9}$$

If the membrane is isotropic then (8) read

$$\mathbb{T}=\mathfrak{l}(\operatorname{tr}\mathbb{E})\mathbb{P}+2\mathfrak{m}\mathbb{E};$$

with $\mathfrak{l}$ and $\mathfrak{m}$ the Lamé coefficients.

About the bulk response we assume generally that the free energy is expressed as a function of the small strain tensor $\mathbf{E}$, by a possibly nonlinear function, i.e.,

$$f(\mathbf{x})=\hat{f}(\hat{\mathbf{E}}(\mathbf{x})),\quad \mathbf{x}\in\Omega\setminus\mathcal{M}, \tag{10}$$

where $\hat{f}$ is a given response function and

$$\hat{\mathbf{E}}(\mathbf{u})=\tfrac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^{\mathrm{T}})$$

with

$$\mathbf{u}(\mathbf{x})=\mathbf{y}(\mathbf{x})-\mathbf{x}$$

the displacement. In section 3 we asume that $\hat{f}$ is a general nonliner (convex) function of quadratic growth. In section 4 we shall assume that $\hat{f}$ is quadratic in $\mathbf{E}$, (linear elastic material), and in section 5 (below) we shall deal with $\hat{f}$ the response function of a no–tension material, in which case $\hat{f}$ is genuinely nonlinear, but does not satisfy all the hypotheses of section 3. The stress relation gives the symmetric stress $\hat{\mathbf{T}}$ as the derivative of $\hat{f}$,

$$\hat{\mathbf{T}}=\mathrm{D}\,\hat{f}.$$

This gives the constitutive equation

$$\mathbf{T}=\hat{\mathbf{T}}(\hat{\mathbf{E}}(\mathbf{u})). \tag{11}$$

3. Admissible displacements. Energy

In this section we assume that the symmetric bulk stress is a generally nonlinear function of the small strain tensor with a convex store energy function of quadratic growth while the membrane response is linear as outline in Subsection 2.3. We define the space of admissible displacements in (12) below which is based on the Sobolev spaces of square integrable displacements with the square integrable gradients both in the bulk and on the membrane. We consider the total energy and under the invertibility assumption on the stress strain relation also the complementary energy. The minima of these corresponding to the equilibrium states are proved.

3.1. **Basic hypotheses. Admissible displacements.** Throughout the section V is a finite dimensional real vectorspace. Let $\mathcal{M}$ be a surface of dimension 2 in $\mathbb{R}^3$. Throughout this section we assume that $\mathcal{M}$ is of class ≥ 1. We denote by $W^{1,2}(\mathcal{M}, V)$ the set of all $\mathbf{u} : \mathcal{M} \to V$ such that both $|\mathbf{u}|$ and its derivative are square integrable on $\mathcal{M}$. Thus each $\mathbf{u} \in W^{1,2}(\mathcal{M}, V)$ has a well defined weak surface derivative $\mathbb{\nabla}\mathbf{u} \in L^2(\mathcal{M}, \mathrm{Lin}(\mathbb{R}^3, V))$. We define a norm $|\cdot|_{W^{1,2}(\mathcal{M},V)}$ on $W^{1,2}(\mathcal{M}, V)$ by

$$|\mathbf{u}|^2_{W^{1,2}(\mathcal{M},V)} = |\mathbf{u}|^2_{L^2(\mathcal{M},V)} + |\mathbb{\nabla}\mathbf{u}|^2_{L^2(\mathcal{M},\mathrm{Lin}(\mathbb{R}^3,V))}$$

for every $\mathbf{u} \in W^{1,2}(\mathcal{M}, V)$. It is easy to see that $W^{1,2}(\mathcal{M}, V)$ is a reflexive Banach space.

We assume that we are given the objects $\hat{f}$, $\mathbf{f}$, $\mathcal{D}$, $\mathcal{S}$, $\mathbf{b}$, $\mathbb{b}$, $\mathbf{s}$, and $\mathbb{s}$ as in the preceding sections. About these objects, and about the objects derived thereof we stipulate the following hypotheses:

H1 The function $\hat{f}$ is continuously differentiable and convex and the function $\hat{\mathbf{T}}$ satisfies the growth condition

$$|\hat{\mathbf{T}}(\mathbf{E})| \leq c(1 + |\mathbf{E}|)$$

for each $\mathbf{E} \in \mathrm{Sym}$ and some c.

H2 For $\mathcal{H}^2$ a.e. $\mathbf{x} \in \mathcal{M}$ the tensor $\mathfrak{C}_\mathbf{x}$ satisfies

$$\mathfrak{C}_\mathbf{x}\mathbf{E} = \mathfrak{C}_\mathbf{x}(\mathbb{P}(\mathbf{x})\mathbf{E}\mathbb{P}(\mathbf{x}))$$

for every $\mathbf{E} \in \mathrm{Sym}$,

$$\mathbb{E}_1 \cdot \mathfrak{C}_\mathbf{x}\mathbb{E}_2 = \mathfrak{C}_\mathbf{x}\mathbb{E}_1 \cdot \mathbb{E}_2$$

for every $\mathbb{E}_1$, $\mathbb{E}_2 \in \mathrm{Sym}$ such that

$$\mathbb{E}_i = \mathbb{P}(\mathbf{x})\mathbb{E}_i\mathbb{P}(\mathbf{x}), \quad i = 1, 2,$$

and

$$d|\mathbb{E}|^2 \geq \mathbb{E} \cdot \mathfrak{C}_\mathbf{x}\mathbb{E} \geq c|\mathbb{E}|^2$$

for each $\mathbb{E} \in \mathrm{Sym}$ such that

$$\mathbb{E} = \mathbb{P}(\mathbf{x})\mathbb{E}\mathbb{P}(\mathbf{x}),$$

where $c > 0$, and d are constants independent of $\mathbf{x}$ and $\mathbb{E}$.

H3 The map $\mathbf{x} \mapsto \mathfrak{C}_\mathbf{x}$ is $\mathcal{H}^2$ measurable on $\mathcal{M}$.

H4 We have $\mathbf{b} \in L^2(\Omega, \mathbb{R}^3), \mathbb{b} \in L^2(\mathcal{M}, \mathbb{R}^3), \mathbf{s} \in L^2(\mathcal{S}, \mathbb{R}^3), \mathbb{s} \in L^2(\mathcal{S} \cap \partial\mathcal{M}, \mathbb{R}^3)$. Moreover, $\mathbb{P}(\mathbf{x})\mathbb{s}(\mathbf{x}) = \mathbb{s}(\mathbf{x})$ for $\mathcal{H}^1$ a.e. $\mathbf{x} \in \mathcal{S} \cap \partial\mathcal{M}$.

H5 $\mathcal{H}^2(\mathcal{D}) > 0$.

H6 $\mathcal{M}$ is an oriented admissible surface with Lipschitz boundary; moreover, $\mathcal{M}$ is a union of finitely many connected components whose closures are pairwise disjoint.

The last condition in H6 is a mild condition guaranteeing that every $\mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^3)$ has a well defined trace $\tau\mathbf{u}$ on $\mathcal{M}$ which is an element of $L^2(\mathcal{M}, \mathbb{R}^3)$. We often write $\mathbf{u}$ for $\tau\mathbf{u}$ when there is no danger of confusion. We put

$$H := \{\mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^3) : \mathbb{u} := \tau\mathbf{u},\ \mathbb{u}^{\parallel} := \mathbb{P}\mathbb{u} \in W^{1,2}(\mathcal{M}, \mathbb{R}^3)\} \tag{12}$$

and

$$|\mathbf{u}|_H = |\mathbf{u}|_{W^{1,2}(\Omega,\mathbb{R}^3)} + |\mathbb{P}\mathbb{u}|_{W^{1,2}(\mathcal{M},\mathbb{R}^3)}.$$

It is easy to see that under $|\cdot|_H$, the space H is a Hilbert space. We furthermore put

$$U = \{\mathbf{u} \in H : \mathbf{u} = \mathbf{0} \text{ on } \mathcal{D}\}.$$

We call the elements $\mathbf{u} \in U$ the kinematically admissible displacements.

3.2. **Energy and complementary energy.** We define the total energy of the displacement $\mathbf{u} \in U$ by

$$\mathsf{F}(\mathbf{u}) = \mathsf{E}(\mathbf{u}) - \mathsf{W}(\mathbf{u}), \tag{13}$$

where E is the internal energy, given by

$$\mathsf{E}(\mathbf{u}) := \int_\Omega f \, d\mathcal{L}^3 + \int_\mathcal{M} \mathbf{f} \, d\mathcal{H}^2 \tag{14}$$

where f and $\mathbf{f}$ are given by the constitutive equations (10) and (9). Furthermore, the potential energy W of the loads is

$$\mathsf{W}(\mathbf{u}) := \int_\Omega \mathbf{b} \cdot \mathbf{u} \, d\mathcal{L}^3 + \int_\mathcal{M} \mathbb{b} \cdot \mathbb{u} \, d\mathcal{H}^2 + \int_\mathcal{S} \mathbf{s} \cdot \mathbf{u} \, d\mathcal{H}^2 + \int_{\mathcal{S} \cap \partial\mathcal{M}} \mathbb{s} \cdot \beta \mathbb{u}^\parallel \, d\mathcal{H}^1 \tag{15}$$

where it will be recalled that $\mathbb{u}^\parallel = \mathbb{P}\mathbb{u}$ and where we denote by $\beta\mathbb{u}^\parallel$ the trace of $\mathbb{u}^\parallel$ on $\partial\mathcal{M}$, see [12, Appendix C]. We here note that $\mathbb{u}^\parallel$ is an element of $W^{1,2}(\mathcal{M}, \mathbb{R}^3)$, whereas $\mathbb{u}$ is generally not in $W^{1,2}(\mathcal{M}, \mathbb{R}^3)$. This is reflected in the definitions of the spaces H and U.

For the given objects as above the equilibrium boundary value problem seeks a displacement $\mathbf{u} \in U$ which satisfies the constitutive equations (11) and (8) and the equilibrium equations (3) in the weak form (1).

Proposition 3.1. *Assume H1–H6. The displacement* $\mathbf{u} \in U$ *solves the equilibrium problem* (1) *if and only if the first variation of the total energy* F *vanishes. Moreover, any solution* $\mathbf{u}$ *of the equilibrium problem is also a minimizer of the total energy among all kinematically admissible displacements, i.e.,*

$$\mathsf{F}(\mathbf{v}) \ge \mathsf{F}(\mathbf{u})$$

for all $\mathbf{v}$ *satisfying* $\mathbf{v} = \mathbf{0}$ *on* $\mathcal{D}$.

This is standard in the absence of the inner membrane. The case with the membrane is treated in a way similar to [6, Proof of Theorem 9.4].

Assume further the following.

H7 The equation

$$\hat{\mathbf{T}}(\mathbf{E}) = \mathbf{T}$$

has exactly one solution $\mathbf{E}$ for every $\mathbf{T} \in \mathrm{Sym}$.

We let $\tilde{\mathbf{E}} : \mathrm{Sym} \to \mathrm{Sym}$ denote the inverse of $\hat{\mathbf{T}}$, i.e., $\tilde{\mathbf{E}}$ satisfies

$$\hat{\mathbf{T}}(\tilde{\mathbf{E}}(\mathbf{T})) = \mathbf{T}$$

for every $\mathbf{T} \in \mathrm{Sym}$. Define the density of the complementary energy $\tilde{f}$ as the Legendre transformation of $\hat{f}$, i.e., by

$$\tilde{f}(\mathbf{T}) = \tilde{\mathbf{E}}(\mathbf{T}) \cdot \mathbf{T} - \hat{f}(\tilde{\mathbf{E}}(\mathbf{T}))$$

for any $\mathbf{T} \in \mathrm{Sym}$. Then under H1 the energy $\tilde{f}$ is a convex function of $\mathbf{T}$ and

$$\tilde{\mathbf{E}} = D\tilde{f}.$$

Assuming H2, we denote, for every $\mathbf{x} \in \mathcal{M}$, by $\mathfrak{C}_{\mathbf{x}}^{-1} : \mathrm{Sym} \to \mathrm{Sym}$ the psudoinverse of $\mathfrak{C}_{\mathbf{x}}$, i.e., a linear transformation such that (a) $\mathfrak{C}[\mathbb{H}] = \mathbb{P}\mathfrak{C}[\mathbb{P}\mathbb{H}]$ for every $\mathbf{H} \in \mathrm{Sym}$ and (b)

$$\mathfrak{C}_{\mathbf{x}}^{-1}[\mathfrak{C}_{\mathbf{x}}[\mathbb{E}]] = \mathbb{E}$$

for all $\mathbb{E} \in \mathrm{Sym}$ such that $\mathbb{P}(\mathbf{x})\mathbb{E}\mathbb{P}(\mathbf{x}) = \mathbb{E}$.

Let $(\mathbf{T}, \mathbb{T})$ be a pair of functions with $\mathbf{T} \in L^2(\Omega, \mathrm{Sym})$ and $\mathbb{T} \in L^2(\mathcal{M}, \mathrm{Sym})$ such that $\mathbb{T}\mathbb{m} = \mathbf{0}$ everywhere on $\mathcal{M}$. We say that $(\mathbf{T}, \mathbb{T})$ is a statically admissible stressfield if it satisfies (1) for every virtual velocity field $\mathbf{v} \in U$. For every statically admissible stressfield $(\mathbf{T}, \mathbb{T})$, define the complementary energy $\mathsf{G}(\mathbf{T}, \mathbb{T})$ by

$$\mathsf{G}(\mathbf{T}, \mathbb{T}) = \int_\Omega \tilde{f}(\mathbf{T})\, d\mathcal{L}^n + \tfrac{1}{2} \int_{\mathcal{M}} \mathfrak{C}^{-1}\mathbb{T} \cdot \mathbb{T}\, d\mathcal{H}^{n-1}.$$

Note that if $\mathbf{u}$ is the solution of the equilibrium equations then the pair $(\mathbf{T}, \mathbb{T})$ of functions given by $\mathbf{T} = \hat{\mathbf{T}}(\hat{\mathbf{E}}(\mathbf{u}))$ and $\mathbb{T} = \mathfrak{C}\hat{\mathbb{E}}(\mathbb{u})$ is a statically admissible stressfield. We have the following principle of minimum complementary energy:

Proposition 3.2. *Under H1–H7, the stressfield* $(\mathbf{T}, \mathbb{T})$ *corresponding to an equilibrium solution gives the minimum complementary energy among all statically admissible stresses, i.e.,*

$$\mathsf{G}(\mathbf{T}, \mathbb{T}) \le \mathsf{G}(\mathbf{S}, \mathbb{S})$$

for all statically admissible stressfields $(\mathbf{S}, \mathbb{S})$.

4. Linearly elastic matrix

In this section we consider the linear response both in the matrix and in the membrane. Under the positive definiteness of the bulk and membrane tensor of elasticities we prove the existence of the equilibrium state of minimum energy. The proof of the positive definite character of the total energy, which is the main step to the proof of existence of the equilibrium state, requires the bulk and membrane Korn's inequalities; of these the membrane Korn inequality is less known.

4.1. **Linearization of the bulk response.** In this section it is assumed that not only the surface response, but also the bulk response, is linear.

Assume that we are given a fourth order tensor C such that Assertions (i)–(iii) of Proposition 2.1 hold. If $\mathbf{u} : \Omega \times \mathbb{R} \to \mathbb{R}^n$ is a displacement field, we define the bulk stresses in the linear response by

$$\mathbf{T}(\mathbf{x}) = \mathsf{C}\mathbf{E}(\mathbf{x}), \qquad \mathbf{x} \in \Omega \setminus \mathcal{M},$$

in place of $(6)_1$, where $\mathbf{E}$ is given by $(7)_1$. In the same situation, we define the linearized free energy f by

$$f(\mathbf{x}) = \tfrac{1}{2}\mathbf{E}(x) \cdot \mathsf{C}\mathbf{E}(\mathbf{x}).$$

4.2. **Korns inequality for membranes.** The proof of the existence of the solution to the equilibrium problem requires the coercivity of the total energy. The Korn inequalities for the membrane and for the matrix are required. These reads as follows.

Theorem 4.1. *If $\mathcal{M}$ is an admissible surface then there exists a $c > 0$ such that*

$$c|\nabla \mathbb{u}^{\parallel}|_{L^2(\mathcal{M},\mathbb{R}^3)} \le |\hat{\mathbb{E}}(\mathbb{u})|_{L^2(\mathcal{M},\mathrm{Lin})} + |\mathbb{u}|_{L^2(\mathcal{M},\mathbb{R}^3)}$$

for each $\mathbb{u} \in S$.

This is essentially the second inequality of Korn's type "without boundary conditions" on a general surface, [2, Theorem 2.7-1] according to the terminology of [2].

Recall also the classical Korn inequality for the bulk strain tensor.

Theorem 4.2. *Let Ω be a bounded open subset of $\mathbb{R}^3$ with Lipschitz boundary and let $\mathcal{D} \subset \partial\Omega$ be a $\mathcal{H}^2$ measurable set with $\mathcal{H}^2(\mathcal{D}) > 0$. Then there exists a constant $c > 0$ such that*

$$c|\nabla \mathbf{u}|_{L^2(\Omega,\mathrm{Lin})} \le |\hat{\mathbf{E}}(\mathbf{u})|_{L^2(\Omega,\mathrm{Sym})}$$

for all $\mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^3)$ such that $\mathbf{u} = \mathbf{0}$ on $\mathcal{D}$ in the sense of trace, where

$$\hat{\mathbf{E}}(\mathbf{u}) = \tfrac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}})$$

is the small strain tensor.

4.3. **Existence of solutions for membranes in a linearly elastic matrix.** Assume that we are given a bulk elasticity tensor $\mathsf{C} \in \mathrm{Lin}(\mathrm{Sym}, \mathrm{Sym})$ and for each $\mathbf{x} \in \mathcal{M}$ the surface elasticity tensor $\mathfrak{C}_{\mathbf{x}} \in \mathrm{Lin}(\mathrm{Sym}, \mathrm{Sym})$, and the associated objects, with $f(\mathbf{E}) = \frac{1}{2}\mathbf{E} \cdot \mathsf{C}\mathbf{E}$ for each $\mathbf{E} \in \mathrm{Sym}$. The proof of the existence of the solution to the equilibrium problem is based on appropriate conditions of the positivity of the tensors of elastic constants. When combined with the Korn inequalities, it leads to the coercivity of the total energy mentioned above. We recall the positivity of the membrane elasticity tensor embodied in Hypothesis H2. For the bulk elasticity tensor we assume the following.

H8 The tensor C satisfies

$$\mathbf{E}_1 \cdot \mathsf{C}\mathbf{E}_2 = \mathsf{C}\mathbf{E}_1 \cdot \mathbf{E}_2$$

for all $\mathbf{E}_1$, $\mathbf{E}_2 \in \mathrm{Sym}$ and there exist constant $c > 0$ such that

$$\mathbf{E} \cdot \mathsf{C}\mathbf{E} \ge c|\mathbf{E}|^2$$

for each $\mathbf{E} \in \mathrm{Sym}$.

It is immediate that H8 implies H1 and H7.

Recall the definition

$$U = \{\mathbf{u} \in H : \mathbf{u} = \mathbf{0} \text{ on } \mathcal{D}\}. \tag{16}$$

We define the potential energy F of the displacement $\mathbf{u} \in U$ by (13). Here the internal energy E is given by (14) where $\mathbb{u} = \tau\mathbf{u}$ is the trace of $\mathbf{u}$ on $\mathcal{M}$ and $f, \mathbf{f}$ are given by

$$\left.\begin{array}{rcll} f(\mathbf{x}) & = & \frac{1}{2}\hat{\mathbf{E}}(\mathbf{u})(x) \cdot \mathsf{C}\hat{\mathbf{E}}(\mathbf{u})(\mathbf{x}), & \mathbf{x} \in \Omega \setminus \mathcal{M}, \\ \mathbf{f}(\mathbf{x}) & = & \frac{1}{2}\hat{\mathbb{E}}(\mathbb{u})(\mathbf{x}) \cdot \mathfrak{C}_{\mathbf{x}}\hat{\mathbb{E}}(\mathbb{u})(\mathbf{x}), & \mathbf{x} \in \mathcal{M}. \end{array}\right\} \tag{17}$$

Theorem 4.3. *Assume that Hypotheses H8 and H2–H6 hold. Then F has a unique minimum relative to U at some point $\mathbf{u} \in U$.*

5. Matrix made of no–tension material

In this section we assume that the matrix response is that of a no–tension material, i.e., an elastic material with the stress constrained to be negative semidefinite for all values of strain. Constitutive equations of the no–tension material are introduced. Then the principle of minimum complementary energy is stated: this version of the principle is different from that stated in Proposition 3.2 as the invertibility hypothesis H7 is not satisfied by the no–tension material. The existence theory for the no tension matrix is out of the scope of the present paper. We recall that without the reinforcement the proof requires the introduction of the space of displacements of bounded deformation [1], [4]. We prove a simpler result saying that the total energy is bounded from below if and only if the loads can be equilibrated by a stressfield that is square integrable and negative semidefinite in its bulk part.

5.1. **Constitutive equations of no--tension materials.** In this section we assume that the bulk body is made of a no–tension material to be introduced below while we continue to assume that the membrane response is linear in the sense of Subsection 2.3. The purpose of the present section is to introduce the response functions of no–tension materials. The stress $\mathbf{T}$ depends on the small deformation tensor $\mathbf{E} = \hat{\mathbf{E}}(\mathbf{u})$,

$$\mathbf{T} = \hat{\mathbf{T}}(\mathbf{E}),$$

where $\hat{\mathbf{T}}$ is given by the constitutive equation of a masonry material defined in (18) (below).

Proposition 5.1. *Assume H8. If* $\mathbf{E} \in \mathrm{Sym}$, *there exists a unique triplet* $(\mathbf{T}, \mathbf{E}^{\mathrm{e}}, \mathbf{E}^{\mathrm{f}})$ *of elements of* Sym *such that*

$$\left.\begin{array}{c} \mathbf{E} = \mathbf{E}^{\mathrm{e}} + \mathbf{E}^{\mathrm{f}}, \\ \mathbf{T} = \mathsf{C}\mathbf{E}^{\mathrm{e}}, \\ \mathbf{T} \in \mathrm{Sym}_{-}, \quad \mathbf{E}^{\mathrm{f}} \in \mathrm{Sym}_{+}, \\ \mathbf{T} \cdot \mathbf{E}^{\mathrm{f}} = 0. \end{array}\right\}$$

We refer to [1], [4] and [3] for various forms of the above statement and the proof.

We define the elastic stress $\hat{\mathbf{T}} : \mathrm{Sym} \to \mathrm{Sym}$ and stored energy $\hat{f} : \mathrm{Sym} \to \mathbb{R}$ of a masonry material by

$$\hat{\mathbf{T}}(\mathbf{E}) = \mathbf{T}, \quad \hat{f}(\mathbf{E}) = \tfrac{1}{2}\hat{\mathbf{T}}(\mathbf{E}) \cdot \mathbf{E} \tag{18}$$

for any $\mathbf{E} \in \mathrm{Sym}$ where $(\mathbf{T}, \mathbf{E}^{\mathrm{e}}, \mathbf{E}^{\mathrm{f}})$ is the triplet associated with $\mathbf{E}$ as in Proposition 5.1; $\mathbf{E}^{\mathrm{e}}$ and $\mathbf{E}^{\mathrm{f}}$ are called the elastic and fracture parts of the deformation $\mathbf{E}$. The explicit form of the response function $\hat{\mathbf{T}}$ and its further analysis have been given in [7], [8], [9] in dimensions 2 and 3, respectively, in case C is isotropic.

5.2. **Equilibrium displacements.** We assume the partition of $\partial\Omega$ into the two complementary sets $\mathcal{D}$ and $\mathcal{S}$ and assume that we are given the forces $\mathbf{b}$, $\mathbb{b}$, $\mathbf{s}$, $\mathbb{s}$ as in Subsection 2.1. We assume that $\mathcal{M}$ is of class 1 and well placed in Ω. This allows us to define the space U as in (16), and the total energy

$\mathsf{F} : U \to \mathbb{R}$ by (13). Here, for the given displacement $\mathbf{u} \in U$, the internal energy $\mathsf{E}(\mathbf{u})$ is given by (14) where $\mathbb{u} = \tau\mathbf{u}$ is the trace of $\mathbf{u}$ on $\mathcal{M}$ and where f is given by the constitutive equation

$$f(\mathbf{x}) = \hat{f}(\mathbf{E}(\mathbf{x}))$$

with $\mathbf{E}$ the small strain tensor of $\mathbf{u}$ and $\hat{f}$ the energy function of a no–tension material, and $\mathbf{f}$ is given by $(17)_2$. We note that $\hat{f}$ satisfies H1 but not H7. The potential energy W of the loads is given by (15) where it will be recalled that $\mathbb{u}^{\|} = \mathbb{P}\mathbb{u}$ and where we denote by $\beta\mathbb{u}^{\|}$ the trace of $\mathbb{u}^{\|}$ on $\partial\mathcal{M}$.

The notion of the equilibrium state is that defined in section 3 and Proposition 3.1 about the variation of energy and minimum energy holds true for no–tension body, i.e., $\mathbf{u}$ is an equilibrium state if and only if the fields $\mathbf{T} = \hat{\mathbf{T}} \circ \hat{\mathbf{E}}(\mathbf{u})$ and $\mathbb{T} = \mathfrak{C}[\hat{\mathbb{E}}(\mathbf{u})]$ satisfy

$$\int_\Omega \mathbf{T} \cdot \hat{\mathbf{E}}(\mathbf{v})\, d\mathcal{L}^3 + \int_{\mathcal{M}} \mathbb{T} \cdot \hat{\mathbb{E}}(\mathbf{v})\, d\mathcal{H}^2 = \mathsf{W}(\mathbf{v})$$

for every $\mathbf{v} \in U$ where U is given by (16) and W is given by (15).

Let $(\mathbf{T}, \mathbb{T})$ be a pair of functions with $\mathbf{T} \in L^2(\Omega, \mathrm{Sym})$ and $\mathbb{T} \in L^2(\mathcal{M}, \mathrm{Sym})$ such that we have $\mathbb{T}\mathbb{m} = \mathbf{0}$ over $\mathcal{M}$. We say that the pair $(\mathbf{T}, \mathbb{T})$ is doubly admissible if it is statically admissible in the sense of definition is section 3 and moreover $\mathbf{T} \le \mathbf{0}$ for almost every point of Ω. For each stressfield $(\mathbf{T}, \mathbb{T})$ which is doubly admissible we define the complementary energy $\mathsf{G}(\mathbf{T}, \mathbb{T})$ by

$$\mathsf{G}(\mathbf{T}, \mathbb{T}) = \tfrac{1}{2} \int_\Omega \mathbb{C}^{-1}\mathbf{T} \cdot \mathbf{T}\, d\mathcal{L}^n + \tfrac{1}{2} \int_{\mathcal{M}} \mathfrak{C}^{-1}\mathbb{T} \cdot \mathbb{T}\, d\mathcal{H}^{n-1}$$

where $\mathfrak{C}^{-1}$ is the pseudoinverse of $\mathfrak{C}$ defined in Subsection 2.3. We have the following principle of minimum complementary energy:

Proposition 5.2. *Let H2–H6 and H8 hold. Let $\mathbf{u}$ be an equilibrium state of the system and define the pair $(\mathbf{T}, \mathbb{T})$ by*

$$\mathbf{T} = \hat{\mathbf{T}}(\hat{\mathbf{E}}(\mathbf{u})), \quad \mathbb{T} = \mathfrak{C}\hat{\mathbb{E}}(\mathbb{u}).$$

Then the pair $(\mathbf{T}, \mathbb{T})$ has the minimum complementary energy among all doubly admissible stressfields equilibrating the loads, i.e.,

$$\mathsf{G}(\mathbf{T}, \mathbb{T}) \le \mathsf{G}(\mathbf{S}, \mathbb{S})$$

for any doubly admissible equilibrating stressfield $(\mathbf{S}, \mathbb{S})$.

This is identical in form with Proposition 3.2; however, that proposition does not apply as Hypothesis H7 is not satisfied. We refer to [1] and [4] for the proof for a no–tension body without the reinforcement.

We note that the existence theory of the equilibrium states based on the minimization of the total energy in U does not work as the total energy functions is not generally coercive. Even to obtain the weaker property than the coercivity, one has to require that the loads be compatible in the sense of Subsection 5.3 (below). The existence theory in the absence of the membrane has to be build in the space $BD(\mathbb{R}^3)$ of the displacements of bounded deformation [1] and [4], but even in this case the loads have to be safe in the sense that they can be equilibrated by an uniformly negative stressfield. In the presence of the membrane, the condition easily generalizes but unfortunately it does not lead to the coercivity of energy as the trace on $\mathcal{M}$ of displacements from $BD(\mathbb{R}^3)$ is not a continuous map.

5.3. **Lower bound on energy.** According to the limit analysis proposed in [10] the collapse occurs for the given loads if and only if the total energy is not bounded from below. Furthermore [10] shows that the total energy is bounded from below if and only if the loads are compatible in the sense of the existence of an admissible equilibrating stressfield. We here extend this equivalence to the case of the presence of membranes.

Assume that the partition $\mathcal{D}$ and $\mathcal{S}$ of $\partial\Omega$ is in Subsection 2.1 is given and let $\mathbf{b}, \mathbf{s}, \mathbb{b}, \mathbb{s}$ be the loads as in that section. We say that the loads are compatible if there exists a doubly admissible stressfield equilibrating them.

Proposition 5.3. *Assume that H2–H6 and H8 hold. Then the total energy functional* F *is bounded from below if and only if the loads are compatible.*

5.4. **Fiber reinforced panel.** In this subsection we demonstrate the construction of the admissible stressfield equilibrating the loads considered in Subsection 5.3.

Let us consider the square panel $\Omega = (0,1)\times(0,1)$ with the origin of the coordinate system $\mathbf{o}$ in the upper right corner of the panel, the x axis pointing to the left and the y axis pointing downwards. Let us denote by $\mathbf{i}$ and $\mathbf{j}$ the unit vectors corresponding to the x and y axis, respectively. The panel, made of masonry-like material, is fixed at its base $(0,1)\times\{1\}$ and, in absence of gravity, undergoes a vertical load $p > 0$ and a tangential load $q < 0$ that are uniformly distributed on its top $(0,1)\times\{0\}$. The panel is reinforced by an elastic fiber that is applied on the diagonal $\{\mathbf{r} = (x,y) \in \Omega : x = y\}$ and it is fixed at the point $(1,1)$. The fiber divides the panel into the triangular regions

$$\Omega_+ = \{\mathbf{r} = (x,y) \in \Omega : x > y\} \quad \text{and} \quad \Omega_- = \{\mathbf{r} = (x,y) \in \Omega : x < y\}.$$

We want to construct a negative semidefinite and equilibrated stress field $\mathbf{T}$ that is defined in $\Omega_+ \cup \Omega_-$. Note that in the absence of the fibre there is no such a stressfield. For Ω_+ we take

$$\mathbf{T}_+ = \begin{cases} -q^2/p\,\mathbf{i}\otimes\mathbf{i} - q(\mathbf{i}\otimes\mathbf{j} + \mathbf{j}\otimes\mathbf{i}) - p\mathbf{j}\otimes\mathbf{j} & \text{if} \quad \mathbf{r} \in \Omega_+^1, \\ \mathbf{0} & \text{if} \quad \mathbf{r} \in \Omega_+ \setminus \Omega_+^1 \end{cases} \tag{19}$$

where $\Omega_+^1 = \{\mathbf{r} \in \Omega_+ : x < 1 + qy/p\}$ is the triangular region delimited by the top of the panel, the fiber and the isostatic line of equation $y = p(x-1)/q$ that starts at the upper left corner of the panel $\mathbf{q} \equiv (1,0)$ and meets the fiber at the point $\mathbf{r}_0 \equiv (p/(p-q), p/(p-q))$. For Ω_- we are looking for a negative semidefinite stress field

$$\mathbf{T}_- = \begin{cases} \sigma\kappa^2\mathbf{i}\otimes\mathbf{i} + \sigma\kappa(\mathbf{i}\otimes\mathbf{j} + \mathbf{j}\otimes\mathbf{i}) + \sigma\mathbf{j}\otimes\mathbf{j} & \text{if} \quad \mathbf{r} \in \Omega_-^1, \\ \mathbf{0} & \text{if} \quad \mathbf{r} \in \Omega_- \setminus \Omega_-^1, \end{cases} \tag{20}$$

with $\sigma < 0$. Here κ is the cotangent of the angle between the active isostatic lines and the x axis and $\Omega_-^1 \subset \Omega_-$ is a region that is delimited by the fiber for $0 < x < p/(p-q)$ and will be specified below.

Let us denote by $s = s(x)$ the normal force in the fiber and by

$$\delta_{11}\mathbf{i}\otimes\mathbf{i} + \delta_{12}(\mathbf{i}\otimes\mathbf{j}+\mathbf{j}\otimes\mathbf{i}) + \delta_{22}\mathbf{j}\otimes\mathbf{j}$$

the jump of the stress field $[\mathbf{T}] = \mathbf{T}_+ - \mathbf{T}_-$ across the fiber. From [11, p. 518], as in this case $\omega(x) = x$ and $J = \sqrt{2}$, s has to satisfy the jump conditions

$$\left.\begin{aligned} s'/\sqrt{2} &= -\delta_{11} + \delta_{12}, \\ s'/\sqrt{2} &= \delta_{22} - \delta_{12}, \end{aligned}\right\} \tag{21}$$

where s' is the derivative of s with respect to x. Let us denote by σ_0 the value of σ in $(20)_1$ for $x = y$. In view of (19) and (20) we have

$$\left.\begin{aligned} \delta_{11} &= -q^2/p - \sigma_0\kappa^2, \\ \delta_{12} &= -q - \sigma_0\kappa, \\ \delta_{22} &= -p - \sigma_0 \end{aligned}\right\} \tag{22}$$

for $0 < x < p/(p-q)$ and

$$\delta_{11} = \delta_{12} = \delta_{22} = 0$$

for $p/(p-q) < x < 1$. By subtracting $(21)_2$ from $(21)_1$ we obtain $\delta_{11} + \delta_{22} - 2\delta_{12} = 0$, which, in view of (22), implies

$$\sigma_0 = -\frac{(p-q)^2}{p(1-\kappa)^2}. \tag{23}$$

Then, from $(21)_2$ and (23) we obtain

$$s'/\sqrt{2} = -p + q - \sigma_0(1-\kappa) \tag{24}$$

for $0 \leq x \leq p/(p-q)$ and $s' = 0$, for $p/(p-q) < x < 1$. From (24) we deduce that in order to have $s' \geq 0$ we need $q/p < \kappa < 1$.

For region Ω_- we will construct two different kinds of stress fields. Firstly, we consider a stress field that is constant in Ω_-^1. In this case Ω_-^1 is the region that is delimited by the fiber, the base of the panel and the two isostatic lines

$$y = \frac{1}{\kappa}\left(x - \frac{p}{p-q}\right) + \frac{p}{p-q} \quad \text{and} \quad y = x/\kappa$$

starting from $\mathbf{r}_0$ and the origin $\mathbf{o}$, respectively (Fig. 1). If we denote by $\mathbf{m}$ and $\mathbf{n}$ the points where these isostatic lines meet the base of the panel, Ω_-^1 is the quadrilateral region $\mathbf{mnor}_0$. Moreover, in order to avoid that the isostatic lines meet the right hand side of the panel we require $0 \leq \kappa < 1$. Thus, we find

$$\mathbf{T} = \begin{cases} -q^2/p\mathbf{i}\otimes\mathbf{i} - q(\mathbf{i}\otimes\mathbf{j}+\mathbf{j}\otimes\mathbf{i}) - p\mathbf{j}\otimes\mathbf{j} & \text{if} \quad \mathbf{r}\in\Omega_+^1, \\ -\dfrac{(p-q)^2}{p(1-\kappa)^2}(\kappa^2\mathbf{i}\otimes\mathbf{i} + \kappa(\mathbf{i}\otimes\mathbf{j}+\mathbf{j}\otimes\mathbf{i}) + \mathbf{j}\otimes\mathbf{j}) & \text{if} \quad \mathbf{r}\in\Omega_-^1, \\ \mathbf{0} & \text{otherwise.} \end{cases} \tag{25}$$

From (23) and (24) we deduce that the normal force in the fiber is

$$s(x) = \begin{cases} \frac{\sqrt{2}(p-q)(\kappa p - q)}{p(1-\kappa)}x & if \quad 0 \leq x \leq p/(p-q), \\ \frac{\sqrt{2}(\kappa p - q)}{(1-\kappa)} & if \quad p/(p-q) < x \leq 1 \end{cases} \tag{26}$$

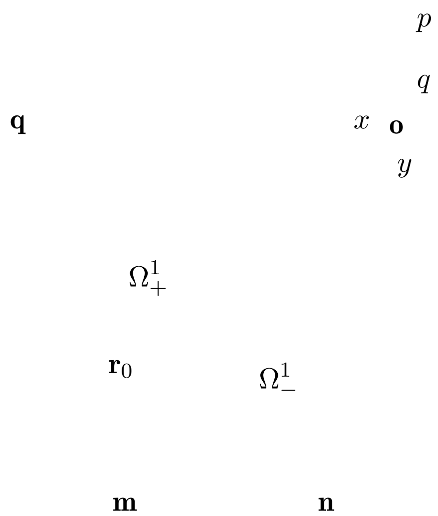

FIGURE 1.

and then the reaction at the fixed end of the fiber is

$$\frac{(\kappa p - q)}{(1-\kappa)}(\mathbf{i}, \mathbf{j}).$$

The other stress field that we consider for region Ω_- is different from zero in the wedge

$$\Omega_-^1 = \{\mathbf{r} = (x, y) \in \Omega_- : p(y-1)/q < x < 0\}$$

having its apex at the lower right corner of the panel $\mathbf{a} \equiv (0, 1)$ and delimited by the isostatic curve joining the points $\mathbf{a}$ and $\mathbf{r}_0$, the right lateral side of the panel and the fiber (Fig. 2). We suppose that in the wedge Ω_-^1 all isostatic curves intersect at $\mathbf{a}$, so that their equation is $y - 1 = x/\kappa$ and then

$$\kappa = \frac{x}{(y-1)} \tag{27}$$

with $p/q < \kappa < 0$. Moreover, from (23) and (27) we deduce

$$\sigma_0 = -\frac{(p-q)^2}{p}(x-1)^2.$$

We assume that in Ω_-^1 the stress field $\mathbf{T}_-$ has the same form as in $(20)_1$. Then, in view of (27) σ has to satisfy the equilibrium equation [14]

$$\frac{x}{y-1}\sigma_{,x} + \sigma_{,y} = -\frac{1}{y-1}\sigma \tag{28}$$

with the condition $\sigma = \sigma_0$ for $x = y$. The linear PDE (28) can be explicitly solved to obtain

$$\sigma = -\frac{(p-q)^2}{p}\frac{(y-1)^2}{(x-y+1)^3}. \tag{29}$$

Finally we have

$$\mathbf{T}_- = \begin{cases} \sigma\kappa^2 \mathbf{i}\otimes\mathbf{i} + \sigma\kappa(\mathbf{i}\otimes\mathbf{j} + \mathbf{j}\otimes\mathbf{i}) + \sigma\mathbf{j}\otimes\mathbf{j} & if \quad \mathbf{r} \in \Omega_-^1, \\ \mathbf{0} & if \quad \mathbf{r} \in \Omega_- \setminus \Omega_-^1, \end{cases} \tag{30}$$

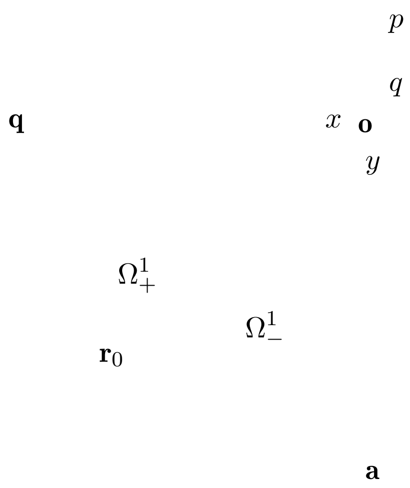

FIGURE 2.

with κ and σ given by (27) and (29), respectively. It is an easy matter to verify that in this case $\mathbf{T}_-$ is an integrable but not a square integrable function. For the fiber, from $(21)_2$ we deduce

$$s'/\sqrt{2} = q - p + \frac{(p-q)^2}{p(1-\kappa)} \tag{31}$$

for $0 \leq x \leq p/(p-q)$ and $s' = 0$ for $p/(p-q) < x < 1$. In order to have $s' > 0$ we need $-q > p$ by (31) and then stress field (30) can be used only when this condition is satisfied. In this case from (31) we deduce

$$s(x) = \begin{cases} -\frac{\sqrt{2}q(p-q)}{p}x - \frac{\sqrt{2}(p-q)^2}{2p}x^2 & if \quad 0 \leq x \leq p/(p-q), \\ -\sqrt{2}(\frac{1}{2}p + q) & if \quad p/(p-q) < x \leq 1. \end{cases}$$

The density of the complementary energy in regions Ω_+ and Ω_- is given by

$$e_c = \frac{1}{4\mu}\mathbf{T}_0 \cdot \mathbf{T}_0 + \frac{1}{8(\mu+\lambda)}\,\mathrm{tr}(\mathbf{T})^2 \tag{32}$$

where λ and μ are the Lamé constants of the masonry. The density of the complementary energy of the fiber is

$$e_{cf} = \frac{1}{\alpha}s^2 \tag{33}$$

where α is the extensional rigidity of the fiber. For the stress field defined in (25) we want to study the behavior of the complementary energy as a function of κ, and because the density in the region Ω^1_+ is a constant, we limit ourselves to considering the complementary energies of region Ω^1_- and the fiber. For region Ω^1_- we have

$$\mathrm{tr}(\mathbf{T}) = -\frac{(p-q)^2}{p(1-\kappa)^2}(1+\kappa^2),$$

$$\mathbf{T}_0 = -\frac{(p-q)^2}{p(1-\kappa)^2}\Big(\tfrac{1}{2}(\kappa^2-1)\mathbf{i}\otimes\mathbf{i} + \kappa(\mathbf{i}\otimes\mathbf{j} + \mathbf{j}\otimes\mathbf{i}) + \tfrac{1}{2}(1-\kappa^2)\mathbf{j}\otimes\mathbf{j}\Big)$$

by (30) and then, from (32) we obtain

$$e_c = \frac{2\mu+\lambda}{8\mu(\mu+\lambda)} \frac{(p-q)^4}{p^2} \frac{(1+\kappa^2)^2}{(1-\kappa)^4}. \tag{34}$$

Because the area of region Ω_-^1 is $p(1-\kappa)(p-2q)/2(p-q)^2$, $\lambda = \nu E/(1+\nu)(1-2\nu)$ and $\mu = E/2(1+\nu)$, where E and ν are the Young modulus and the Poisson ratio of the masonry, respectively, from (34) we obtain the complementary energy of region Ω_-,

$$\mathcal{E}_c(\Omega_-) = \frac{1-\nu^2}{4E} \frac{(p-q)^2(p-2q)}{p} \frac{(1+\kappa^2)^2}{(1-\kappa)^3}.$$

Moreover, from (33) and (26) we deduce the complementary energy of the fiber

$$\mathcal{E}_{cf} = \frac{1}{2\alpha} \int_0^{p/(p-q)} \left(\frac{\sqrt{2}(p-q)(\kappa p - q)}{p(1-\kappa)} \right)^2 x^2 dx + \frac{-q}{2\alpha(p-q)} \left(\frac{\sqrt{2}(\kappa p - q)}{1-\kappa} \right)^2 =$$

$$\frac{1}{3\alpha} \frac{(p-3q)}{(p-q)} \left(\frac{\kappa p - q}{1-\kappa} \right)^2.$$

As for $0 \leq \kappa < 1$ the derivatives with respect to κ of both $\mathcal{E}_c(\Omega_-)$ and $\mathcal{E}_{cf}$ are positive functions we conclude that the minimum of the total complementary energy $\mathcal{E}_c(\Omega) + \mathcal{E}_{cf}$ is attained for $\kappa = 0$. Moreover, we note that $d\sigma/d\kappa = 2(p-q)^2/\ p(1-\kappa)^3 > 0$, by (23) and $ds'/d\kappa = \sqrt{2}(p-q)^2/p(1-\kappa)^2 > 0$ by (24). Then, for $\kappa = 0$ we have the minimum compressive stress in Ω_- and the minimum value of the net shear stress acting on the fiber. Finally, we note that the complementary energy of region Ω_- corresponding to the stress field defined in (30) is equal to $+\infty$, because in this case $\mathbf{T}_-$ is not a square integrable function.

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