Mediterr. J. Math. 13 (2016), 1885–1891
DOI 10.1007/s00009-015-0614-3
1660-5446/16/041885-7
published online July 26, 2015

Mediterranean Journal of Mathematics

A Note on Boundedness of the Hardy–Littlewood Maximal Operator on Morrey Spaces

A. Gogatishvili and R. Ch. Mustafayev

Abstract. In this paper we prove that the Hardy–Littlewood maximal operator is bounded on Morrey spaces $\mathcal{M}_{1,\lambda}(\mathbb{R}^n)$, $0 \le \lambda < n$ for radial, non-increasing functions on $\mathbb{R}^n$.

Mathematics Subject Classification. Primary 42B25; Secondary 42B35.

Keywords. Morrey spaces, maximal operator.

1. Introduction

Morrey spaces $\mathcal{M}_{p,\lambda} \equiv \mathcal{M}_{p,\lambda}(\mathbb{R}^n)$, were introduced by Morrey in [8] in order to study regularity questions which appear in the Calculus of Variations, and defined as follows: for $0 \le \lambda \le n$ and $1 \le p < \infty$,

$$\mathcal{M}_{p,\lambda} := \left\{ f \in L_p^{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{M}_{p,\lambda}} := \sup_{x \in \mathbb{R}^n,\, r>0} r^{\frac{\lambda-n}{p}} \|f\|_{L_p(B(x,r))} < \infty \right\},$$

where $B(x,r)$ is the open ball centered at x of radius r.

Note that $\mathcal{M}_{p,0}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$ and $\mathcal{M}_{p,n}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, when $1 \le p < \infty$.

These spaces describe local regularity more precisely than Lebesgue spaces and appeared to be quite useful in the study of the local behavior of solutions to partial differential equations, a priori estimates and other topics in PDE (cf. [4]).

The research of A. Gogatishvili was partly supported by the Grants P201-13-14743S of the Grant Agency of the Czech Republic and RVO: 67985840, by Shota Rustaveli National Science Foundation Grants No. 31/48 (Operators in some function spaces and their applications in Fourier Analysis) and No. DI/9/5-100/13 (Function spaces, weighted inequalities for integral operators and problems of summability of Fourier series). The research of A. Gogatishvili and R. Ch. Mustafayev was partly supported by the joint project between Academy of Sciences of Czech Republic and The Scientific and Technological Research Council of Turkey.

Birkhäuser

Given a locally integrable function f on $\mathbb{R}^n$ and $0 \le \alpha < n$, the fractional maximal function $M_\alpha f$ of f is defined by

$$M_\alpha f(x) := \sup_{Q \ni x} |Q|^{\frac{\alpha-n}{n}} \int_Q |f(y)| \, \mathrm{d}y, \quad (x \in \mathbb{R}^n),$$

where the supremum is taken over all cubes Q containing x. The operator $M_\alpha : f \to M_\alpha f$ is called the fractional maximal operator. $M := M_0$ is the classical Hardy–Littlewood maximal operator.

The study of maximal operators is one of the most important topics in harmonic analysis. These significant non-linear operators, whose behavior is very informative in particular in differentiation theory, provided the understanding and the inspiration for the development of the general class of singular and potential operators (see, for instance, [3, 5–7, 9–11]).

The boundedness of the Hardy–Littlewood maximal operator M in Morrey spaces $\mathcal{M}_{p,\lambda}$ was proved by Chiarenza and Frasca in [2]: It was shown that Mf is a.e. finite if $f \in \mathcal{M}_{p,\lambda}$ and an estimate

$$\|Mf\|_{\mathcal{M}_{p,\lambda}} \le c\|f\|_{\mathcal{M}_{p,\lambda}} \tag{1.1}$$

holds if $1 < p < \infty$ and $0 < \lambda < n$, and a weak type estimate replaces (1.1) for $p = 1$, that is, the inequality

$$t|\{Mf > t\} \cap B(x,r)| \le c r^{n-\lambda} \|f\|_{\mathcal{M}_{1,\lambda}} \tag{1.2}$$

holds with constant c independent of x, r, t and f.

In this paper we show that (1.1) is not true for $p = 1$. According to our example the right result is (1.2). If restricted to the cone of radial, non-increasing functions on $\mathbb{R}^n$, inequality (1.1) holds true for $p = 1$.

The paper is organized as follows. We start with notations and preliminary results in Sect. 2. In Sect. 3, we prove that the Hardy–Littlewood maximal operator M is bounded on $\mathcal{M}_{1,\lambda}$, $0 < \lambda < n$, for radial, non-increasing functions, and we give an example which shows that M is not bounded on $\mathcal{M}_{1,\lambda}$, $0 < \lambda < n$.

2. Notations and Preliminaries

Now we make some conventions. Throughout the paper, we always denote by c a positive constant, which is independent of main parameters, but it may vary from line to line. By $a \lesssim b$ we mean that $a \le cb$ with some positive constant c independent of appropriate quantities. If $a \lesssim b$ and $b \lesssim a$, we write $a \approx b$ and say that a and b are equivalent. Throughout this paper cubes will be assumed to have their sides parallel to the coordinate axes. For a measurable set E, χ_E denotes the characteristic function of E.

Let Ω be any measurable subset of $\mathbb{R}^n$, $n \ge 1$. Let $\mathfrak{M}(\Omega)$ denote the set of all measurable functions on Ω and $\mathfrak{M}_0(\Omega)$ the class of functions in $\mathfrak{M}(\Omega)$ that are finite a.e., while $\mathfrak{M}^{\downarrow}(0,\infty)$ ($\mathfrak{M}^{+,\downarrow}(0,\infty)$) is used to denote the subset of those functions which are non-increasing (non-increasing and non-negative) on $(0,\infty)$. Denote by $\mathfrak{M}^{\mathrm{rad},\downarrow} = \mathfrak{M}^{\mathrm{rad},\downarrow}(\mathbb{R}^n)$ the set of all measurable, radial, non-increasing functions on $\mathbb{R}^n$, that is,

$$\mathfrak{M}^{\mathrm{rad},\downarrow} := \{f \in \mathfrak{M}(\mathbb{R}^n) : f(x) = \varphi(|x|),\ x \in \mathbb{R}^n \text{ with } \varphi \in \mathfrak{M}^{\downarrow}(0,\infty)\}.$$

Recall that $Mf \approx Hf$, $f \in \mathfrak{M}^{\mathrm{rad},\downarrow}$, where

$$Hf(x) := \frac{1}{|B(0,|x|)|} \int_{B(0,|x|)} |f(y)|\,\mathrm{d}y$$

is n-dimensional Hardy operator. Obviously, $Hf \in \mathfrak{M}^{\mathrm{rad},\downarrow}$, when $f \in \mathfrak{M}^{\mathrm{rad},\downarrow}$.

For $p \in (0,\infty]$, we define the functional $\|\cdot\|_{p,\Omega}$ on $\mathfrak{M}(\Omega)$ by

$$\|f\|_{p,\Omega} := \begin{cases} \left(\int_\Omega |f(x)|^p\,\mathrm{d}x\right)^{1/p} & \text{if} \quad p < \infty, \\ \operatorname{ess\,sup}_\Omega |f(x)| & \text{if} \quad p = \infty. \end{cases}$$

The Lebesgue space $L_p(\Omega)$ is given by

$$L_p(\Omega) := \{f \in \mathfrak{M}(\Omega) : \|f\|_{p,\Omega} < \infty\}$$

and it is equipped with the quasi-norm $\|\cdot\|_{p,\Omega}$.

The non-increasing rearrangement (see, e.g., [1, p. 39]) of a function $f \in \mathfrak{M}_0(\mathbb{R}^n)$ is defined by

$$f^*(t) := \inf\{\lambda > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| \le t\} \quad (0 < t < \infty).$$

3. Boundedness of M on $\mathcal{M}_{1,\lambda}$ for Radial, Non-Increasing Functions

Recall that

$$\begin{aligned} M_\alpha f(x) &\approx \sup_{B \ni x} |B|^{\frac{\alpha-n}{n}} \int_B |f(y)|\,\mathrm{d}y \\ &\approx \sup_{r>0} |B(x,r)|^{\frac{\alpha-n}{n}} \int_{B(x,r)} |f(y)|\,\mathrm{d}y, \quad (x \in \mathbb{R}^n), \end{aligned}$$

where the supremum is taken over all balls B containing x.

In order to prove our main result we need the following auxiliary lemmas.

Lemma 3.1. *Assume that* $0 < \lambda < n$. *Let* $f \in \mathfrak{M}^{\mathrm{rad},\downarrow}(\mathbb{R}^n)$ *with* $f(x) = \varphi(|x|)$. *The equivalency*

$$\|f\|_{\mathcal{M}_{1,\lambda}} \approx \sup_{x>0} x^{\lambda-n} \int_0^x |\varphi(\rho)|\rho^{n-1}\,\mathrm{d}\rho$$

holds with positive constants independent of f.

Proof. Recall that

$$\|f\|_{\mathcal{M}_{1,\lambda}} \approx \sup_B |B|^{\frac{\lambda-n}{n}} \int_B |f| = \|M_\lambda f\|_\infty, \quad f \in \mathfrak{M}(\mathbb{R}^n).$$

Switching to polar coordinates, we have that

$$\begin{aligned} M_\lambda(f)(y) &\gtrsim |B(0,|y|)|^{\frac{\lambda-n}{n}} \int_{B(0,|y|)} |f(z)|\,\mathrm{d}z \\ &\approx |y|^{\lambda-n} \int_0^{|y|} |\varphi(\rho)|\rho^{n-1}\,\mathrm{d}\rho. \end{aligned}$$

Consequently,

$$\|f\|_{\mathcal{M}_{1,\lambda}} \gtrsim \operatorname*{ess\,sup}_{y\in\mathbb{R}^n} |y|^{\lambda-n} \int_0^{|y|} |\varphi(\rho)|\rho^{n-1}\,\mathrm{d}\rho = \sup_{x>0} x^{\lambda-n} \int_0^x |\varphi(\rho)|\rho^{n-1}\,\mathrm{d}\rho,$$

where $f(\cdot) = \varphi(|\cdot|)$.

On the other hand,

$$\|f\|_{\mathcal{M}_{1,\lambda}} \lesssim \sup_B |B|^{\frac{\lambda-n}{n}} \int_0^{|B|} f^*(t)\,\mathrm{d}t = \sup_B |B|^{\frac{\lambda-n}{n}} \int_0^{|B|} |\varphi(t^{\frac{1}{n}})|\,\mathrm{d}t \approx \sup_B |B|^{\frac{\lambda-n}{n}} \int_0^{|B|^{\frac{1}{n}}} |\varphi(\rho)|\rho^{n-1}\,\mathrm{d}\rho = \sup_{x>0} x^{\lambda-n} \int_0^x |\varphi(\rho)|\rho^{n-1}\,\mathrm{d}\rho,$$

where $f(\cdot) = \varphi(|\cdot|)$. □

Corollary 3.2. *Assume that* $0 < \lambda < n$. *Let* $f \in \mathfrak{M}^{\mathrm{rad},\downarrow}(\mathbb{R}^n)$ *with* $f(x) = \varphi(|x|)$. *The equivalency*

$$\|Mf\|_{\mathcal{M}_{1,\lambda}} \approx \sup_{x>0} x^{\lambda-n} \int_0^x \varphi(\rho)\rho^{n-1} \ln\left(\frac{x}{\rho}\right)\,\mathrm{d}\rho$$

holds with positive constants independent of f.

Proof. Let $f \in \mathfrak{M}^{\mathrm{rad},\downarrow}$ with $f(x) = \varphi(|x|)$. Since $Mf \approx Hf$ and $Hf \in \mathfrak{M}^{\mathrm{rad},\downarrow}$, by Lemma 3.1, switching to polar coordinates, using Fubini's Theorem, we have that

$$\|Mf\|_{\mathcal{M}_{1,\lambda}} \approx \sup_{x>0} x^{\lambda-n} \int_0^x \left(\frac{1}{|B(0,t)|}\int_{B(0,t)} |f(y)|\,dy\right) t^{n-1}\,\mathrm{d}t \approx \sup_{x>0} x^{\lambda-n} \int_0^x \frac{1}{t}\int_0^t \varphi(\rho)\rho^{n-1}\,d\rho\,\mathrm{d}t = \sup_{x>0} x^{\lambda-n} \int_0^x \varphi(\rho)\rho^{n-1} \ln\left(\frac{x}{\rho}\right)\,\mathrm{d}\rho.$$

□

Lemma 3.3. *Assume that* $0 < \lambda < n$. *Let* $f \in \mathfrak{M}^{\mathrm{rad},\downarrow}$ *with* $f(x) = \varphi(|x|)$. *The inequality*

$$\|Mf\|_{\mathcal{M}_{1,\lambda}} \lesssim \|f\|_{\mathcal{M}_{1,\lambda}}, \quad f \in \mathfrak{M}^{\mathrm{rad},\downarrow}$$

holds if and only if the inequality

$$\sup_{x>0} x^{\lambda-n} \int_0^x \varphi(\rho)\rho^{n-1} \ln\left(\frac{x}{\rho}\right)\,\mathrm{d}\rho \lesssim \sup_{x>0} x^{\lambda-n} \int_0^x \varphi(\rho)\rho^{n-1}\,\mathrm{d}\rho$$

holds for all $\varphi \in \mathfrak{M}^{+,\downarrow}(0,\infty)$.

Proof. The statement immediately follows from Lemma 3.1 and Corollary 3.2. □

Lemma 3.4. *Let* $0 < \lambda < n$. *Then the inequality*

$$\sup_{x>0} x^{\lambda-n} \int_0^x \varphi(\rho)\rho^{n-1} \ln\left(\frac{x}{\rho}\right) \mathrm{d}\rho \lesssim \sup_{x>0} x^{\lambda-n} \int_0^x \varphi(\rho)\rho^{n-1}\, \mathrm{d}\rho \qquad (3.1)$$

holds for all $\varphi \in \mathfrak{M}^{+,\downarrow}(0,\infty)$.

Proof. Indeed:

$$\begin{aligned}
&\sup_{x>0} x^{\lambda-n} \int_0^x \varphi(\rho)\rho^{n-1} \ln\left(\frac{x}{\rho}\right) \mathrm{d}\rho \\
&\quad = \sup_{x>0} x^{\lambda-n} \int_0^x \frac{1}{t} \int_0^t \varphi(\rho)\rho^{n-1}\, \mathrm{d}\rho\, \mathrm{d}t \\
&\quad = \sup_{x>0} x^{\lambda-n} \int_0^x t^{n-\lambda-1} t^{\lambda-n} \int_0^t \varphi(\rho)\rho^{n-1}\, \mathrm{d}\rho\, \mathrm{d}t \\
&\quad \le \sup_{t>0} t^{\lambda-n} \int_0^t \varphi(\rho)\rho^{n-1}\, \mathrm{d}\rho \cdot \left(\sup_{x>0} x^{\lambda-n} \int_0^x t^{n-\lambda-1}\, \mathrm{d}t \right) \\
&\quad \approx \sup_{t>0} t^{\lambda-n} \int_0^t \varphi(\rho)\rho^{n-1}\, \mathrm{d}\rho.
\end{aligned}$$

□

Now we are in position to prove our main result.

Theorem 3.5. *Assume that* $0 < \lambda < n$. *The inequality*

$$\|Mf\|_{\mathcal{M}_{1,\lambda}} \lesssim \|f\|_{\mathcal{M}_{1,\lambda}} \qquad (3.2)$$

holds for all $f \in \mathfrak{M}^{\mathrm{rad},\downarrow}$ *with a constant independent of* f.

Proof. The statement follows by Lemmas 3.3 and 3.4. □

Remark 3.6. Note that inequality (3.2) holds true when $\lambda = 0$, for $\mathcal{M}_{1,0}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$ and M is bounded on $L_\infty(\mathbb{R}^n)$.

Remark 3.7. It is obvious that the statement of Theorem 3.5 does not hold when $\lambda = n$, for in this case $\mathcal{M}_{1,n}(\mathbb{R}^n) = L_1(\mathbb{R}^n)$ and the inequality

$$\|Mf\|_{L_1(\mathbb{R}^n)} \lesssim \|f\|_{L_1(\mathbb{R}^n)}$$

is true only for $f = 0$ a.e., which follows from the fact that $Mf(x) \approx |x|^{-n}$ for $|x|$ large when $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Example. We show that M is not bounded on $\mathcal{M}_{1,\lambda}(\mathbb{R}^n)$, $0 < \lambda < n$. For simplicity let $n = 1$ and $\lambda = 1/2$. Consider the function

$$f(x) = \sum_{k=0}^{\infty} \chi_{[k^2,k^2+1]}(x), \ \ (x \in \mathbb{R}).$$

Then

$$\|f\|_{\mathcal{M}_{1,1/2}(\mathbb{R})} = \sup_I |I|^{-1/2} \int_I f \le \sup_{I:\,|I|\le 1} |I|^{-1/2} \int_I f + \sup_{I:\,|I|>1} |I|^{-1/2} \int_I f,$$

where the supremum is taken over all open intervals $I \subset \mathbb{R}$. It is easy to see that

$$\sup_{I:\,|I|\le 1} |I|^{-1/2} \int_I f \le \sup_{I:\,|I|\le 1} |I|^{1/2} \le 1.$$

Note that

$$\begin{aligned} \sup_{I:\,|I|>1} |I|^{-1/2} \int_I f &= \sup_{m\in\mathbb{N}} \sup_{I:\,m<|I|\le m+1} |I|^{-1/2} \int_I f \\ &= \sup_{m\in\mathbb{N}} \sup_{I:\,m<|I|\le m+1} |I|^{-1/2} \int_I \left(\sum_{k=0}^{\infty} \chi_{[k^2,k^2+1]}(x) \right) \mathrm{d}x \\ &= \sup_{m\in\mathbb{N}} \sup_{I:\,m<|I|\le m+1} |I|^{-1/2} \left| I \cap \bigcup_{k=0}^{\infty} [k^2, k^2+1] \right|. \end{aligned}$$

Since

$$\left| I \cap \bigcup_{k=0}^{\infty} [k^2, k^2+1] \right| \le \left| [0, m+1] \cap \bigcup_{k=0}^{\infty} [k^2, k^2+1] \right|$$

for any interval I such that $m < |I| \le m+1$, we obtain that

$$\begin{aligned} \sup_{I:\,|I|>1} |I|^{-1/2} \int_I f &\lesssim \sup_{m\in\mathbb{N}} m^{-1/2} \left| [0, m+1] \cap \bigcup_{k=0}^{\infty} [k^2, k^2+1] \right| \\ &\lesssim \sup_{m\in\mathbb{N}} m^{-1/2} m^{1/2} = 1. \end{aligned}$$

Consequently, we arrive at

$$\|f\|_{\mathcal{M}_{1,1/2}(\mathbb{R})} \lesssim 2.$$

On the other hand, since

$$Mf \ge \sum_{k=0}^{\infty} \left(\chi_{[k^2,k^2+1]} + \frac{\chi_{[k^2+1,k^2+k+1]}}{x-k^2} + \frac{\chi_{[k^2+k+1,(k+1)^2]}}{(k+1)^2+1-x} \right),$$

we have that

$$\begin{aligned} \|Mf\|_{\mathcal{M}_{1,1/2}(\mathbb{R})} &\ge \sup_{k\in\mathbb{N}} k^{-1} \int_0^{k^2} Mf \ge \sup_{k\in\mathbb{N}} k^{-1} \sum_{i=1}^{k-1} \int_{i^2}^{(i+1)^2} Mf \\ &\ge \sup_{k\in\mathbb{N}} k^{-1} \sum_{j=1}^{k-1} \ln j \gtrsim \sup_{k\in\mathbb{N}} \ln k = \infty. \end{aligned}$$

References

[1] Bennett, C., Sharpley, R., Interpolation of operators. In: Pure and Applied Mathematics, vol. 129. Academic Press, Boston (1988)

[2] Chiarenza, F., Frasca, M.: Morrey spaces and Hardy–Littlewood maximal function. Rend. Mat. Appl. (7) **7**(3–4), 273–279 (1987)

[3] Garcia-Cuerva, J., Rubio de Francia, J.L.: Weighted norm inequalities and related topics. In: North-Holland Mathematics Studies, vol. 116. North-Holland Publishing Co., Amsterdam (1985) (Notas de Matemática [Mathematical Notes], 104)

[4] Gilbarg, D., Trudinger, N.S.: Elliptic partial differential equations of second order, 2nd edn. Springer, Berlin (1983)

[5] Grafakos, L.: Classical Fourier analysis, 2nd edn. In: Graduate Texts in Mathematics, vol. 249. Springer, New York (2008)

[6] Grafakos, L.: Modern Fourier analysis, 2nd edn. In: Graduate Texts in Mathematics, vol. 250. Springer, New York (2009)

[7] de Guzmán, M.: Differentiation of integrals in R^n. In: Lecture Notes in Mathematics, vol. 481. Springer, Berlin (1975) (with appendices by Antonio Córdoba, and Robert Fefferman, and two by Roberto Moriyón)

[8] Morrey, C.B.: On the solutions of quasi-linear elliptic partial differential equations. Trans. Am. Math. Soc. **43**(1), 126–166 (1938). doi:10.2307/1989904

[9] Stein, E.M.: Singular integrals and differentiability properties of functions. In: Princeton Mathematical Series, vol. 30. Princeton University Press, Princeton (1970)

[10] Stein, E.M.: Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. In: Princeton Mathematical Series, vol. 43. Princeton University Press, Princeton (1993) (with the assistance of Timothy S. Murphy; Monographs in Harmonic Analysis, III)

[11] Torchinsky, A.: Real-variable methods in harmonic analysis. In: Pure and Applied Mathematics, vol. 123. Academic Press, Orlando (1986)

A. Gogatishvili
Institute of Mathematics of the Academy of Sciences of the Czech Republic
Źitna 25
115 67 Prague 1
Czech Republic
e-mail: gogatish@math.cas.cz

R. Ch. Mustafayev
Department of Mathematics
Faculty of Science and Arts
Kirikkale University
71450 Yahsihan
Kirikkale
Turkey
e-mail: rzamustafayev@gmail.com

Received: January 20, 2015.
Revised: July 8, 2015.
Accepted: July 12, 2015.