Correlation Matrices In Randomly Connected Dynamical Systems Show Small-World-Like Properties

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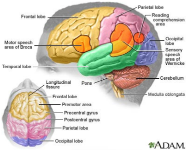

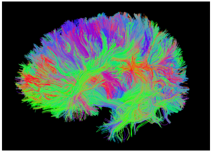

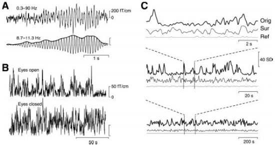

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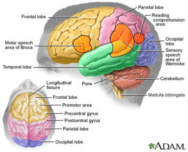

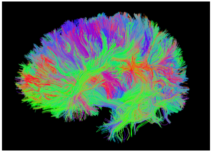

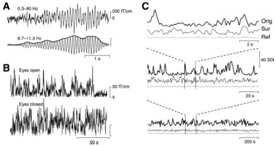

- other examples: financial time series, climate dynamics networks,...

Problems with quantifying FC

- data are usually noisy – preprocessing
- "all-to-all" connectivity? – too many signals!
  - need to reduce dimensionality (domain knowledge, clustering methods, PCA/ICA)
- choice of dependence measure
  - Pearson's correlation coefficient $\rho_{X,Y} = \frac{\mathrm{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{E[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X \sigma_Y}$
  - ordinal correlation measures (Spearman correlation, Kendal's tau,...)
  - Information-theoretical measures: mutual information

$$I(X;Y) = \sum_{y \in Y} \sum_{x \in X} p(x,y) \log \left(\frac{p(x,y)}{p(x)\,p(y)} \right)$$

[Hlinka et al., 2011, Neuroimage; Hartman et al, 2011, Chaos]

- "all-to-all" connectivity? – too many FC coefficients!
  - need to summarize the FC structure (graph theoretical methods)

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Graph-theoretical analysis

- FC transformed into a graph
- We study the graph properties
  - local (role of specific nodes)
    - degree ('degree centrality')
    - clustering
  - global 'type I'/mezoscale (depending on the whole graph)
    - path length
    - betweeness centrality
  - global 'type II' (characterizing the whole graph by one number)
    - density
    - average clustering
    - mean path length
    - small-world property
- characterize or compare graphs wrt these measures

Small-world property

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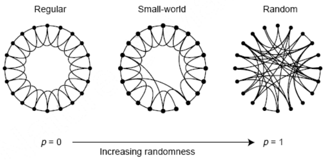
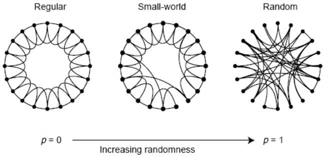


[Watts and Strogatz, 1998]

Small-world property



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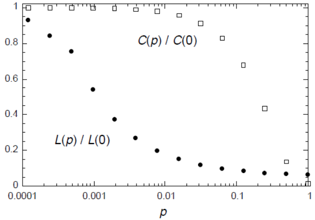

Graph: $G = (V, E)$; V set of nodes; $V = 1, \dots, n$;$E \subset V^2$ set of edges. $d_{i,j}$ shortest path between i a j.
Reprezentation by matrix A: $A_{i,j} = 1 \Leftrightarrow (i, j) \in E$; $k_i = \sum_j A_{i,j}$ degree.

$L = \frac{1}{n \cdot (n-1)} \cdot \sum_{i,j} d_{i,j}$ $C = \frac{1}{n} \sum_{i \in V} c_i$; $\quad c_i = \frac{\sum_{j,\ell} A_{i,j} A_{j,\ell} A_{\ell,i}}{k_i (k_i - 1)}$

small-world index ([Humphries, 2008]): $\sigma = \frac{\gamma}{\lambda} \gg 1$; $\lambda = \frac{L}{L_{rand}} \gtrsim 1$, $\gamma = \frac{C}{C_{rand}} \gg 1$

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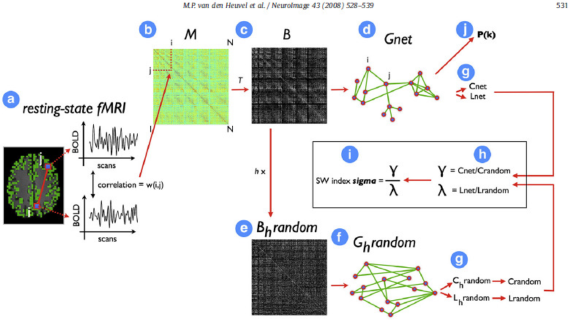
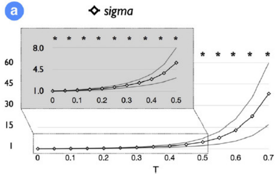
The brain correlation matrix is a small world:





Why is this interesting?

The brain is a small world...

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and randomly connected system also...

$$X_t = AX_{t-1} + e_t$$

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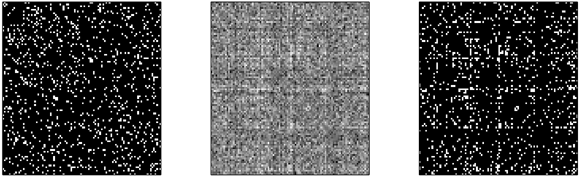

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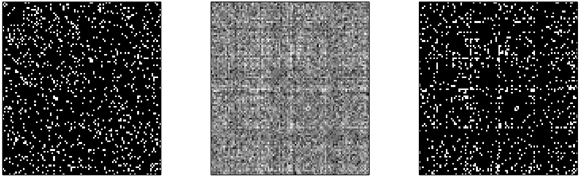

$L_S = 2.157, L_F = 2.308, C_S = 0.1081, C_F = 0.2355, \lambda = 1.07, \gamma = 2.1778, \sigma = 2.0353$.

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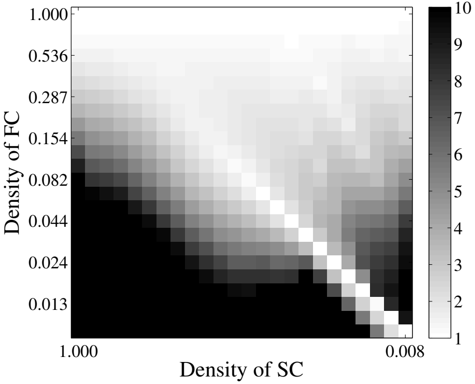
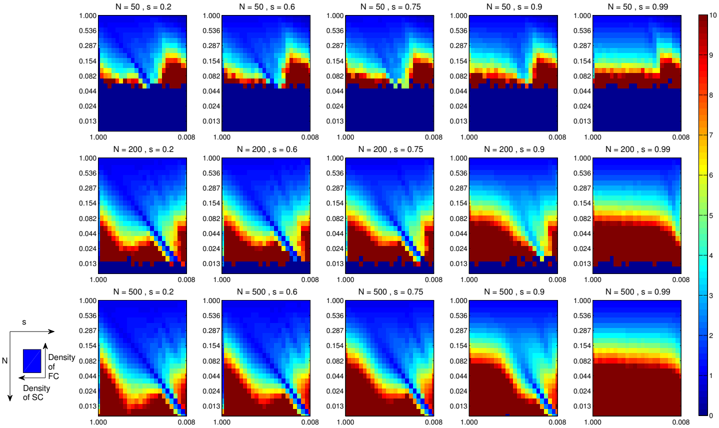


Detailed results

$\sigma \gg 1$, but depends on many parameters:



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Thank you for your attention!

Reference:

Hlinka, J., Hartman, D., Palus, M. *Small-world topology of functional connectivity in randomly connected dynamical system*, 2012, Chaos, 22, 033107

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