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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS TO A SYSTEM OF SCHRÖDINGER EQUATIONS

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Abstract. This paper is concerned with the behaviour of solutions to a system of coupled Schrödinger equations (1.1) which has applications in many physical problems, especially in nonlinear optics. In particular, when the solution exists globally, we obtain the growth of the solutions in the energy space. Finally, some conditions are also obtained in order to have blow-up in this space.

1. Introduction

In this work, we consider the following initial value problem (IVP) for two coupled nonlinear Schrödinger equations (NLS):

$$\begin{cases} iu_t + \Delta u + (\alpha|u|^{2p} + \beta|u|^q|v|^{q+2})u = 0, \\ iv_t + \Delta v + (\alpha|v|^{2p} + \beta|v|^q|u|^{q+2})v = 0, \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^n$, α, $\beta \in \mathbb{R}$, $p > 0$ and $q > 0$.

For β a real positive constant, $\alpha = 1$ and $q = p - 1$, the system (1.1) leads to the following model

$$\begin{cases} iu_t + \Delta u + (|u|^{2p} + \beta|u|^{p-1}|v|^{p+1})u = 0, \\ iv_t + \Delta v + (|v|^{2p} + \beta|v|^{p-1}|u|^{p+1})v = 0, \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x). \end{cases} \tag{1.2}$$

This kind of problem arises as a model for propagation of polarized laser beams in birefringent Kerr medium in nonlinear optics (see, for example, [4, 16, 24, 27, 35, 36] and the references therein for a complete discussion about the physical standpoint of the problem). The two functions u and

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v are the components of the slowly varying envelope of the electrical field, t is the distance in the direction of propagation, x are orthogonal variables and Δ is the diffraction operator. The case $n = 1$ corresponds to propagation in a planar geometry, the case $n = 2$ describes the propagation in a bulk medium and the case $n = 3$ represents the propagation of pulses in a bulk medium with time dispersion. The focusing nonlinear terms in (1.2) describes the dependence of the refraction index of material on the electric field intensity and the birefringence effects. The parameter $\beta > 0$ has to be interpreted as the birefringence intensity and describes the coupling between the two components of the electric-field envelope.

If α, β are real constants and $u = v$, the system (1.1) reduces to the nonlinear Schrödinger with double power nonlinearity.

$$\begin{cases} iu_t + \Delta u + (\alpha |u|^{2p} + \beta |u|^{2(q+1)})u = 0, \\ u(x,0) = u_0(x), \end{cases} . \tag{1.3}$$

Special case of (1.3) is the cubic-quintic nonlinear Schrödinger equation ($p = q = 1$)

$$iu_t + \Delta u + (\alpha |u|^2 + \beta |u|^4)u = 0. \tag{1.4}$$

This equation arises in a number of independent physics field: nuclear hydrodynamic with Skyrme [20], the optical pulse propagations in dielectrical media of non-Kerr type [23]. Also, it is used to describe the boson gas with two and three body interaction [2, 3].

The equation (1.3) is just one of many models of Schrödinger equations. Many of different aspects of this model were investigated by various techniques by any authors [10, 14, 18, 17, 19, 28, 21, 33] and references therein. In [33] was consider

$$\begin{cases} iu_t + \Delta u + (\alpha |u|^{p_1} + \beta |u|^{p_2})u = 0, \\ u(x,0) = u_0(x), \end{cases} . \tag{1.5}$$

with $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, $n \geq 3$ and $0 < p_1 < p_2 \leq \frac{4}{n-2}$ and they proved local and global well-posedness, they also addresses issues related to finite time blow-up, assymptotic behaviour and scattering in the energy space $H^1(\mathbb{R}^n)$.

The system (1.1), admits the mass and the energy conservation in the spaces $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ respectively. More precisely, the mass (L^2 norm):

$$M(u(t), v(t)) := \|u(t)\|^2_{L^2(\mathbb{R}^n)} + \|v(t)\|^2_{L^2(\mathbb{R}^n)} = M(u_0, v_0), \tag{1.6}$$

and the energy:

$$\begin{aligned} E(t) := E(u(t), v(t)) &:= \|\nabla u(t)\|^2_{L^2(\mathbb{R}^n)} + \|\nabla v(t)\|^2_{L^2(\mathbb{R}^n)} - \mathcal{X}(t) \\ &= E(0) := E(u_0, v_0), \end{aligned} \tag{1.7}$$

are conserved by the flow of (1.1), where

$$\mathcal{X}(t) = \frac{\alpha}{p+1}\left[\|u(t)\|^{2p+2}_{L^{2p+2}(\mathbb{R}^n)} + \|v(t)\|^{2p+2}_{L^{2p+2}(\mathbb{R}^n)}\right] + \frac{2\beta}{q+2}\|u(t)\,v(t)\|^{q+2}_{L^{q+2}(\mathbb{R}^n)}. \tag{1.8}$$

For some remarks on proofs of conservation laws for nonlinear Schrödinger equations, we refer to [29].

Well-posedness issues and the blow-up phenomenon for the IVP (1.1) has been studied in the literature, see for example in [11, 13, 16, 26, 27, 30, 35] and references therein. The system (1.2) has scaling, this is if u and v are two solutions from (1.2) and $\lambda > 0$ then

$$\eta(x,t) = \lambda^{1/p}u(\lambda x, \lambda^2 t), \quad \omega(x,t) = \lambda^{1/p}v(\lambda x, \lambda^2 t), \tag{1.9}$$

also are two solutions of (1.2). Hence, putting

$$p = \frac{2}{n-2s_0},$$

the Sobolev space $\dot{H}^{s_0}$ is invariant under the scaling (1.9). In what follows we list some important results that are relevant in our work.

1) *Local existence:*

Under assumptions $s \geq \max\{s_0, 0\}$ *and* $p > [s]/2$ if $p \notin \mathbb{Z}$

the solution of the Cauchy problem (1.2), exists locally in time.

2) *Global existence.* Assuming

i) $0 < p < 2/n$

the solution of the Cauchy problem (1.2), exists globally in time (see [16], see also Theorem 1.2 and Section 4 in this work).

3) When $p \geq 2/n$, the solution of the Cauchy problem (1.2), *blows-up in a finite time* for some initial data, especially for a class of sufficiently large data (see [13, 16, 26, 30] and Theorem 1.3 in this work). On the other hand, the solution of the Cauchy problem (1.2) *exists globally for other initial data*, especially for a class of sufficiently small data (see [11, 16, 27]).

In [35], Xiaoguang et al. obtained a sharp threshold of blow-up solution for (1.2). To study the blow-up threshold, the following stationary system

$$\begin{cases} \Delta Q - \frac{(2-n)p+2}{2}Q + (|Q|^{2p} + \beta|Q|^{p-1}|R|^{p+1})Q = 0, \\ \Delta R - \frac{(2-n)p+2}{2}R + (|R|^{2p} + \beta|R|^{p-1}|Q|^{p+1})R = 0, \end{cases} \tag{1.10}$$

associated with (1.2) was considered.

Let, $s_c = n/2 - 1/p$, $\sigma_{p,n,\beta} := (\frac{pn}{2})^{1/4(1-1/p)}\sqrt{\|Q\|^2_{L^2(\mathbb{R}^n)} + \|R\|^2_{L^2(\mathbb{R}^n)}}$,

$$\Gamma(u,v) := E^{s_c}(u,v)M^{1-s_c}(u,v),$$

and

$$\vartheta(u,v) := (\|\nabla u\|^2_{L^2(\mathbb{R}^n)} + \|\nabla v\|^2_{L^2(\mathbb{R}^n)})^{s_c/2}(\|u\|^2_{L^2(\mathbb{R}^n)} + \|v\|^2_{L^2(\mathbb{R}^n)})^{(1-s_c)/2}.$$

The following is the result proved in Xiaoguang et al. [35].

Theorem 1.1 ([35]). *Let $2/n \le p < A_n$, where $A_n = \infty$ if $n = 1, 2$, $A_n = 2/(n-2)$ if $n \ge 3$ and $(|x|u_0, |x|v_0) \in L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Assume that*

$$\Gamma(u_0, v_0) < \Gamma(Q, R) \equiv \frac{{s_c}^{s_c}}{n}(\sigma_{p,n,\beta})^2,$$

then the following two conclusions are valid.
1) If $\vartheta(u_0, v_0) < \vartheta(Q, R)$, then the solution of the Cauchy problem (1.2) exists globally in time.
2) If $\vartheta(u_0, v_0) > \vartheta(Q, R)$, then the solution of the Cauchy problem (1.2) blows-up in finite time.

In [7], they considered the initial value problem (IVP) associated to the coupled system of supercritical nonlinear Schrödinger equations

$$\begin{cases} iu_t + \Delta u + \theta_1(\omega t)(|u|^{2p} + \beta|u|^{p-1}|v|^{p+1})u = 0, \\ iv_t + \Delta v + \theta_2(\omega t)(|v|^{2p} + \beta|v|^{p-1}|u|^{p+1})v = 0, \end{cases} \tag{1.11}$$

where θ_1 and θ_2 are periodic functions. They proved that, for given initial data $\varphi, \psi \in H^1(\mathbb{R}^n)$, as $|\omega| \to \infty$, the solution (u_ω, v_ω) of the IVP (1.11) converges to the solution (U, V) of the IVP associated to

$$\begin{cases} iU_t + \Delta U + I(\theta_1)(|U|^{2p} + \beta|U|^{p-1}|V|^{p+1})U = 0, \\ iV_t + \Delta V + I(\theta_2)(|V|^{2p} + \beta|V|^{p-1}|U|^{p+1})V = 0, \end{cases} \tag{1.12}$$

with the same initial data, where $I(g)$ is the average of the periodic function g. Moreover, if the solution (U, V) is global and bounded, then they also proved that the solution (u_ω, v_ω) is also global provided $|\omega| \gg 1$.

Our main result characterize the asymptotic properties of solutions of (1.1) and gives the growth of the Sobolev norm in H^1

Theorem 1.2. *Let $u_0, v_0 \in L^2(|x|^2dx) \cap H^1(\mathbb{R}^n)$ and $u(t), v(t)$ be solutions of (1.1) with $t \ge 1$, we have*

1) *If $0 < p \le \frac{2}{n}$ and $p \ge q+1$ if $\beta > 0$ or $p \le q+1$ if $\beta < 0$ then*

$$E(0) - \frac{b_0}{4\,t^{np}} \le \int \left(|\nabla u(x,t)|^2 + |\nabla v(x,t)|^2\right) dx.$$

And if moreover $\mathfrak{X} \le 0$ (see (1.8), e.g., $\alpha \le 0$ and $\beta \le 0$), we also have

$$\|\nabla u(t)\|_{L^2_x(\mathbb{R}^n)} + \|\nabla v(t)\|_{L^2_x(\mathbb{R}^n)} \le \min\left\{\left(c_0 + \frac{2b_0^{1/2}}{np}\right) - \frac{b_0^{1/2}(2-np)}{np}t^{-np/2}, \quad E(0)\right\}, \tag{1.13}$$

and

$$\|x\,u(t)\|_{L^2_x} + \|x\,v(t)\|_{L^2_x} \le 2t\left(c_0 + \frac{2b_0^{1/2}}{np}\right) + \frac{4b_0^{1/2}(np-1)}{np}t^{1-np/2}, \tag{1.14}$$

and

$$\lim_{t\to+\infty} \int \left(|\nabla u(x,t)|^2 + |\nabla v(x,t)|^2\right) dx = E(0), \tag{1.15}$$

where $b_0 := b_0(n,p)$ *and* $c_0 = c_0(u_0, v_0)$ *are defined in* (5.7) *and* (5.16) *respectively.*

2) *If* $0 < q \leq \frac{2}{n} - 1$ *and* $p \leq q+1$ *if* $\alpha > 0$ *or* $p \geq q+1$ *if* $\alpha < 0$ *then*

$$E(0) - \frac{b_1}{4\, t^{n(q+1)}} \leq \int \left(|\nabla u(x,t)|^2 + |\nabla v(x,t)|^2\right) dx.$$

And if moreover $\mathcal{X} \leq 0$ *(e.g.,* $\alpha \leq 0$ *and* $\beta \leq 0$*), we also have*

$$\|\nabla u(t)\|_{L^2_x(\mathbb{R}^n)} + \|\nabla v(t)\|_{L^2_x(\mathbb{R}^n)} \leq \min\left\{\left(c_0 + \frac{2b_1^{1/2}}{n(q+1)}\right) - \frac{b_1^{1/2}(2-n(q+1))}{n(q+1)} t^{-n(q+1)/2}, \quad E(0)\right\},$$

and

$$\|x\, u(t)\|_{L^2_x} + \|x\, v(t)\|_{L^2_x} \leq 2t\left(c_0 + \frac{2b_1^{1/2}}{n(q+1)}\right) + \frac{4b_1^{1/2}(n(q+1)-1)}{n(q+1)} t^{1-n(q+1)/2}, \tag{1.16}$$

and

$$\lim_{t\to+\infty} \int \left(|\nabla u(x,t)|^2 + |\nabla v(x,t)|^2\right) dx = E(0),$$

where $b_1 := b_1(n,q) \geq 0$ *and* $c_0 = c_0(u_0,v_0) \geq 0$ *are defined in* (5.20) *and* (5.16) *respectively.*

Remark. i) *The restriction* $t \geq 1$ *in Theorem 1.2 can be replaced by* $t \geq c_0$*, where* $c_0 > 0$ *is any arbitrarily small constant.*

ii) *Observe also that using interpolation*

$$\|u\|_{H^\theta} \leq \|u\|_{L^2}^{1-\theta} \|u\|_{H^1}^{\theta}, \quad \theta \in [0,1],$$

the theorem above also gives the growth of the Sobolev norm in $H^\theta(\mathbb{R}^n)$*,* $\theta \in [0,1]$*.*

The growth of Sobolev norms, in the Schrödinger equation was studied by J. Bourgain in [6]*. See also* [31]*,* [9] *and references there.*

iii) *If* $np = 2$ *and* $n(q+1) = 2$ *then*

$$\frac{\partial}{\partial t}\left[\int \left(|J(u)|^2 + |J(v)|^2\right)\, dx - t f(t)\right] = 0$$

(see equality (5.1)*) and therefore if* $\alpha < 0$*,* $\beta < 0$ *and* $u_0, v_0 \in L^2(|x|^2 dx)$ *then*

$$\|v\|_{L^{2p+2}}^{2p+2} + \|u\|_{L^{2p+2}}^{2p+2} \leq \frac{(p+1)(\|xu_0\|_{L^2}^2 + \|xv_0\|_{L^2}^2)}{4|\alpha| t^2}$$

and

$$\|uv\|_{L^{q+2}}^{q+2} \leq \frac{(q+2)(\|xu_0\|_{L^2}^2 + \|xv_0\|_{L^2}^2)}{8|\beta| t^2}$$

And our blow-up result is

Theorem 1.3. *Let* $u_0, v_0 \in L^2(|x|^2 dx) \cap H^1(\mathbb{R}^n)$ *and* $u(t), v(t)$ *be solutions of* (1.1)*, we have*

1) *If* $np \geq 2$ *and* $p \leq q+1$ *if* $\beta > 0$ *or* $p \geq q+1$ *if* $\beta < 0$*, then there exists* $0 < T^* < \infty$ *such that*

$$\lim_{t\to T^*} \|\nabla u(t)\|_{L^2} = \infty, \qquad \lim_{t\to T^*} \|\nabla v(t)\|_{L^2} = \infty,$$

in the following cases

(1)

$$E(0)=0 \quad and \quad Im\int (\overline{u}_0 x\cdot\nabla u_0+\overline{v}_0 x\cdot\nabla v_0)\;dx<0,$$

(2)

$$E(0)<0,$$

(3)

$$E(0)>0,$$

and

$$\left(Im\int (\overline{u}_0 x\cdot\nabla u_0+\overline{v}_0 x\cdot\nabla v_0)\;dx\right)^2>\frac{npE(0)}{2}\int |x|^2(|u_0|^2+|v_0|^2)\,dx.$$

2) *If* $n(q+1)\geq 2$ *and* $p\geq q+1$ *if* $\alpha>0$ *or* $p\leq q+1$ *if* $\alpha<0$, *then there exists* $0<T^*<\infty$ *such that*

$$\lim_{t\to T^*}\|\nabla u(t)\|_{L^2}=\infty, \qquad \lim_{t\to T^*}\|\nabla v(t)\|_{L^2}=\infty,$$

in the following cases:

(1)

$$E(0)=0 \quad and \quad Im\int (\overline{u}_0 x\cdot\nabla u_0+\overline{v}_0 x\cdot\nabla v_0)\;dx<0,$$

(2)

$$E(0)<0,$$

(3)

$$E(0)>0,$$

and

$$\left(Im\int (\overline{u}_0 x\cdot\nabla u_0+\overline{v}_0 x\cdot\nabla v_0)\;dx\right)^2>\frac{n(q+1)E(0)}{2}\int |x|^2(|u_0|^2+|v_0|^2)dx.$$

Remark. *If*

$$\lim_{t\to T^*}\|\nabla u(t)\|_{L^2}=\infty, \qquad \lim_{t\to T^*}\|\nabla v(t)\|_{L^2}=\infty,$$

then by the energy conservation (1.7) *we have that also* $\lim_{t\to T^*}\mathcal{X}(t)=\infty$, *and this limit implies*

$$\lim_{t\to T^*}\|u(t)\|_{L^\infty}=\infty, \qquad \lim_{t\to T^*}\|v(t)\|_{L^\infty}=\infty.$$

ii)

2. Notation

Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we denote the partial derivative of u with relation to spatial variable x_j as u_{x_j}, $\partial_{x_j} u$ or $\frac{\partial u}{\partial x_j}$, similarly we denote the partial derivative of u with relation to time variable $t \in \mathbb{R}$ as u_t, $\partial_t u$ or $\frac{\partial u}{\partial t}$. All the integrals in our work are defined in $\mathbb{R}^n$, in this way

$$\int f(x)\, dx := \int_{\mathbb{R}^n} f(x)\, dx.$$

If $f(x)$, $x \in \mathbb{R}^n$ is a function, the laplacian of f is denoted by

$$\Delta f(x) = \sum_{j=1}^{n} \partial_{x_j}^2 f(x), \quad x = (x_1, \cdots, x_n).$$

The gradient of f is denoted by

$$\nabla f(x) = (\partial_{x_1} f, \cdots, \partial_{x_n} f).$$

The product of two vectors $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ $y = (y_1, \dots, y_n) \in \mathbb{C}^n$ is denoted by

$$x \cdot y = \sum_{j=1}^{n} x_j y_j,$$

and this manner $|x|^2 = x \cdot \overline{x}$.

3. Preliminary Results

In this section we present important results that will be useful in the following sections.

Lemma 3.1. (*Gronwall Inequality*) *Let u and β be continuous and α and δ Riemann integrable functions on $J = [a, b]$ with δ and β nonnegative on J.*

If u satisfies the integral inequality

$$u(t) \le \alpha(t) + \delta(t) \int_a^t \beta(s) u(s) ds, \quad \forall t \in J,$$

then

$$u(t) \le \alpha(t) + \delta(t) \int_a^t \alpha(s)\beta(s) \exp\left(\int_s^t \delta(r)\beta(r) dr \right).$$

Proof. See a proof of this lemma in Theorem 11 of [15]. □

Observe that there are no assumptions on the signs of the functions α and u.

Theorem 3.1. *(Existence of solutions in the energy space). Assume $0 \le \max\{p, q+1\} < 2/(n-2)$ if $\alpha < 0$ and $\beta < 0$ (focusing case), otherwise $0 \le \max\{p, q+1\} < 2/n$. Then for any $(u_0, v_0) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$, there are $T_{max} > 0$ and a unique solution $(u, v) \in C([0, T_{max}); H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n))$ of (1.1) satisfying $(u(0), v(0)) = (u_0, v_0)$. Moreover, there holds the blow up alternative:*
(i) $T_{max} = \infty$

or

(ii) $T_{max} < \infty$ *and*

$$\lim_{t\to T_{max}} (\|\nabla u(t)\|_{L^2(\mathbb{R}^n)} + \|\nabla v(t)\|_{L^2(\mathbb{R}^n)}) = \infty.$$

When (i) occurs, we say that the solution is global. When (ii) occurs, we say that the solution blows up in finite time T.

The proof of this theorem is similar to that for the Schrödinger equation and it combines Strichartz estimates with the contraction mapping principle.

Lemma 3.2. *Let* u *and* v *be solutions of* (1.1)*, then*

$$\begin{aligned}\frac{\partial}{\partial t}\left\{\int Im\,(\overline{u}\,x\cdot\nabla u+\overline{v}\,x\cdot\nabla v)\,dx\right\} =&2E(0)+\frac{\alpha(2-np)}{p+1}\int\left(|u|^{2p+2}+|v|^{2p+2}\right)dx\\ &+\frac{2\beta\,(2-n(1+q))}{q+2}\int|u\,v|^{q+2}dx.\end{aligned} \tag{3.1}$$

Proof. Differentiating in the variable t and integrating by parts we obtain

$$\frac{\partial}{\partial t}\left\{\int \mathrm{Im}\,(\overline{u}\,x\cdot\nabla u)\,dx\right\} = 2\mathrm{Im}\int \overline{u}_t\,x\cdot\nabla u dx - n\int \mathrm{Im}\,(\overline{u}u_t)\,dx, \tag{3.2}$$

using the first equation in (1.1) we have

$$\int \mathrm{Im}\,(\overline{u}u_t)\,dx = -\int|\nabla u|^2dx+\alpha\int|u|^{2p+2}dx+\beta\int|u|^{2+q}\,|v|^{2+q}dx, \tag{3.3}$$

similarly

$$\begin{aligned}\mathrm{Im}\int u_t\,x\cdot\nabla\overline{u}dx =&\mathrm{Re}\int\Delta u\,x\cdot\nabla\overline{u}\,dx+\alpha\mathrm{Re}\int|u|^{2p}u\,x\cdot\nabla\overline{u}\,dx\\ &+\beta\mathrm{Re}\int|u|^q\,|v|^{2+q}u\,x\cdot\nabla\overline{u}\,dx.\end{aligned} \tag{3.4}$$

Using integration by parts twice, it is easy to see that

$$\int\Delta u\,x\cdot\nabla\overline{u}\,dx = (n-2)\int|\nabla u|^2dx - \int\Delta\overline{u}\,x\cdot\nabla u\,dx$$

and therefore

$$\mathrm{Re}\int\Delta u\,x\cdot\nabla\overline{u}\,dx = \frac{(n-2)}{2}\int|\nabla u|^2dx. \tag{3.5}$$

Integrating by parts again gives

$$\begin{aligned}2\mathrm{Re}\int|u|^{2p}u\,x\cdot\nabla\overline{u}\,dx =&-n\int|u|^{2p+2}dx-\int|u|^2\,x\cdot\nabla(|u|^{2p})dx\\ =&-n\int|u|^{2p+2}dx-\frac{2p}{2p+2}\int x\cdot\nabla(|u|^{2p+2})dx\\ =&-n\int|u|^{2p+2}dx+\frac{2pn}{2p+2}\int|u|^{2p+2}dx\\ =&\frac{-2n}{2p+2}\int|u|^{2p+2}dx.\end{aligned} \tag{3.6}$$

Similarly

$$2\mathrm{Re}\int |u|^q\,|v|^{2+q}u\,x\cdot\nabla\overline{u}\,dx = -\,n\int |uv|^{q+2}dx - \frac{q}{q+2}\int |v|^{q+2}\,x\cdot\nabla(|u|^{q+2})dx \\ -\int |u|^{q+2}\,x\cdot\nabla(|v|^{q+2})dx. \tag{3.7}$$

Combining (3.4)-(3.7) it follows that

$$\mathrm{Im}\int u_t\,x\cdot\nabla\overline{u} = \frac{(n-2)}{2}\int |\nabla u|^2\,dx - \frac{n\,\alpha}{2p+2}\int |u|^{2p+2}dx - \frac{n\beta}{2}\int |uv|^{q+2}dx \\ -\frac{q\beta}{2(q+2)}\int |v|^{q+2}\,x\cdot\nabla(|u|^{q+2})dx - \frac{\beta}{2}\int |u|^{q+2}\,x\cdot\nabla(|v|^{q+2})dx. \tag{3.8}$$

The symmetry of (1.1) in u and v and one integration by parts gives

$$\mathrm{Im}\int u_t\,x\cdot\nabla\overline{u} + v_t\,x\cdot\nabla\overline{v}\,dx = \frac{(n-2)}{2}\int \left(|\nabla u|^2 + |\nabla v|^2\right)\,dx \\ -\frac{n\,\alpha}{2p+2}\int \left(|u|^{2p+2} + |v|^{2p+2}\right)\,dx - n\beta\int |uv|^{q+2}dx \\ \frac{q\beta\,n}{2(q+2)}\int |v|^{q+2}|u|^{q+2}dx + \frac{\beta\,n}{2}\int |u|^{q+2}\,|v|^{q+2}dx. \tag{3.9}$$

Now from (3.2), (3.3) and (3.9) is not hard to see that

$$\frac{\partial}{\partial t}\left\{\int \mathrm{Im}\,(\overline{u}\,x\cdot\nabla u + \overline{v}\,x\cdot\nabla v)\,dx\right\} = 2\int \left(|\nabla u|^2 + |\nabla v|^2\right)\,dx \\ -\frac{n\alpha p}{p+1}\int \left(|u|^{2p+2} + |v|^{2p+2}\right)dx - \frac{2n\beta(q+1)}{q+2}\int |uv|^{q+2}dx. \tag{3.10}$$

We conclude the proof of Lemma by using the conservation law (1.7). □

The following Lemma is an obvious result.

Lemma 3.3. *Let u and v be solutions of the coupled system* (1.1), *we have*

$$\frac{\partial}{\partial t}|u|^2 \;=\; 2\,Im(\Delta\overline{u}\,u) \quad \textit{and} \quad \frac{\partial}{\partial t}|v|^2 \;=\; 2\,Im(\Delta\overline{v}\,v). \tag{3.11}$$

The following lemma will be useful to prove the asymptotic behaviour of solutions of (1.1)

Lemma 3.4. *Let $u_0, v_0 \in L^2(|x|^2dx)\cap H^1(\mathbb{R}^n)$ and $u(t), v(t)$ solutions of* (1.1), *then if $0\le t\le T$, we have*

$$\left(\int |x|^2|u(x,t)|^2dx\right)^{1/2} \le \left(\int |x|^2|u_0|^2dx\right)^{1/2} + 2\int_0^t \|\nabla u(t')\|_{L^2}dt', \tag{3.12}$$

and

$$\left(\int |x|^2|v(x,t)|^2dx\right)^{1/2} \le \left(\int |x|^2|v_0|^2dx\right)^{1/2} + 2\int_0^t \|\nabla v(t')\|_{L^2}dt'. \tag{3.13}$$

Proof. Using Lemma 3.3 we obtain

$$\frac{\partial}{\partial t}\int |x|^2|u(t)|^2\,dx = \int |x|^2\frac{\partial\,|u(t)|^2}{\partial t}\,dx = 2\int |x|^2\mathrm{Im}\,(u\Delta\overline{u})\;dx, \tag{3.14}$$

integrating by parts once, we have

$$\int |x|^2 u\Delta\overline{u}\,dx = -2\int ux\cdot\nabla\overline{u}\,dx - \int |x|^2|\nabla u|^2\,dx, \tag{3.15}$$

inserting (3.15) in (3.14) we arrive to

$$\begin{aligned}\frac{\partial}{\partial t}\int |x|^2|u(t)|^2\,dx &= -4\mathrm{Im}\int ux\cdot\nabla\overline{u}\,dx\\ &= 4\mathrm{Im}\int \overline{u}x\cdot\nabla u\,dx.\end{aligned} \tag{3.16}$$

Let $\Omega(t) = \|xu\|_{L^2}$, then using Cauchy-Schwartz, the inequality (3.16) implies

$$\frac{d\Omega(t)^2}{dt} = 2\Omega(t)\frac{d\Omega(t)}{dt} \le 4\Omega(t)\|\nabla u\|_{L^2}, \tag{3.17}$$

and from (3.17) integrating, we have

$$\Omega(t) \le \Omega(0) + 2\int_0^t \|\nabla u\|_{L^2}dt'.$$

Similarly we obtain the inequality (3.13). □

In this paper we will use the operators J and L defined by

$$Jw \;=\; e^{i|x|^2/4t}(2it)\nabla(e^{-i|x|^2/4t}\,w) \;=\; (x+2it\nabla)w, \qquad Lw \;=\; (i\partial_t\,+\Delta\,)w.$$

With this notation the system (1.1) is

$$\begin{cases} Lu \;=\; -F(u,v) \;=-(\alpha|u|^{2p}+\beta|u|^q|v|^{q+2})u,\\ Lv \;=\; -F(v,u). \end{cases} \tag{3.18}$$

We note that (see Remark after proof Theorem 3.2).

$$J(Lu) \;=\; L(Ju) \tag{3.19}$$

Lemma 3.5. *Let u and v be solutions of coupled system* (1.1)*, then we have*

$$Im(\int J(|u|^{2p}u)\cdot\overline{Ju}\,dx) = -\frac{2(np-2)}{(p+1)}t\int |u|^{2p+2}\,dx - \frac{2}{(p+1)}\frac{\partial}{\partial t}\left\{t^2\int |u|^{2p+2}\,dx\right\},$$

and

$$Im(\int J(|v|^{2p}v)\cdot\overline{Jv}\,dx) = -\frac{2(np-2)}{(p+1)}t\int |v|^{2p+2}\,dx - \frac{2}{(p+1)}\frac{\partial}{\partial t}\left\{t^2\int |v|^{2p+2}\,dx\right\}.$$

Proof. Using the definition of J, the scalar product of vectors and differentiating gives

$$\begin{aligned}J(|u|^{2p}u)\cdot\overline{Ju} =&|x|^2|u|^{2p+2} - 2it|u|^{2p}u\, x\cdot\nabla\overline{u} + 2it\overline{u}\nabla(|u|^{2p}u)\cdot x + 4t^2\nabla(|u|^{2p}u)\cdot\nabla\overline{u}\\ =&|x|^2|u|^{2p+2} + 2it|u|^{2p}x\cdot(\overline{u}\nabla u - u\nabla\overline{u}) + 2it|u|^2\nabla(|u|^{2p})\cdot x\\ &+ 4t^2|u|^{2p}|\nabla u|^2 + 4t^2u\nabla(|u|^{2p})\cdot\nabla\overline{u},\end{aligned}$$

taking the imaginary part we have

$$\begin{aligned}\mathrm{Im}\left(J(|u|^{2p}u)\cdot\overline{Ju}\right) =&2t|u|^2\nabla(|u|^{2p})\cdot x + 4t^2\mathrm{Im}\left(u\nabla(|u|^{2p}).\nabla\overline{u}\right)\\ =&2t\frac{p}{p+1}\nabla(|u|^{2p+2})\cdot x + 4t^2\mathrm{Im}\left(u\nabla(|u|^{2p}).\nabla\overline{u}\right),\end{aligned} \tag{3.20}$$

and after integration over $\mathbb{R}^n$, we obtain

$$\mathrm{Im}\int J(|u|^{2p}u)\overline{Ju}\,dx = \frac{2tp}{p+1}\int\nabla(|u|^{2p+2})\cdot x\,dx + 4t^2\mathrm{Im}\int u\nabla(|u|^{2p}).\nabla\overline{u})\,dx. \tag{3.21}$$

Integrating by parts, we have

$$\int\nabla(|u|^{2p+2})\cdot x\,dx = -n\int|u|^{2p+2}\,dx,$$

and

$$\int u\nabla(|u|^{2p}).\nabla\overline{u} = -\int|u|^{2p}|\nabla u|^2\,dx - \int|u|^{2p}u\Delta\overline{u}\,dx.$$

Substituting into the equation (3.21) and applying Lemma 3.3, we arrive to

$$\begin{aligned}\mathrm{Im}\int J(|u|^{2p}u)\overline{Ju}\,dx =& -\frac{2tpn}{p+1}\int|u|^{2p+2}\,dx - 4t^2\int|u|^{2p}\mathrm{Im}\left(\Delta\overline{u}\,u\right)\,dx\\ =& -\frac{2tpn}{p+1}\int|u|^{2p+2}\,dx - 2t^2\int|u|^{2p}\frac{\partial}{\partial t}|u|^2\,dx\\ =& -\frac{2tpn}{p+1}\int|u|^{2p+2}\,dx - \frac{2t^2}{p+1}\int\frac{\partial}{\partial t}|u|^{2p+2}\,dx,\end{aligned}$$

we concludes the proof by observing that

$$t^2\frac{\partial}{\partial t}\left(|u|^{2p+2}\right) = \frac{\partial}{\partial t}\left(t^2\,|u|^{2p+2}\right) - 2t\,|u|^{2p+2}.$$

□

Lemma 3.6. *Let u and v be solutions of coupled system* (1.1), *then we have*

$$\begin{aligned}&Im(\int J(|u|^q|v|^{q+2}u)\cdot\overline{Ju}\,dx) + Im(\int J(|v|^q|u|^{q+2}v)\cdot\overline{Jv}\,dx) =\\ &-\frac{4t\,(n(q+1)-2)}{q+2}\int|u\,v|^{q+2}\,dx - \frac{4}{q+2}\frac{\partial}{\partial t}\left\{t^2\int\left(|u\,v|^{q+2}\right)dx\right\}.\end{aligned} \tag{3.22}$$

Proof. From the definition of J we have

$$J(|u|^q|v|^{q+2}u) = |u|^q|v|^{q+2}u\,x + 2it\nabla(|u|^q|v|^{q+2}u), \tag{3.23}$$

making the scalar product of (3.23) with $\overline{Ju} = x\overline{u} - 2it\nabla\overline{u}$ and differentiating gives

$$\begin{aligned} J(|u|^q|v|^{q+2}u)\cdot\overline{Ju} = {} & |x|^2|u|^q|v|^{q+2}|u|^2 - 2it|u|^q|v|^{q+2}u\,x\cdot\nabla\overline{u} + 2it\overline{u}\,x\cdot\nabla(|u|^q|v|^{q+2}u) \\ & + 4t^2\nabla(|u|^q|v|^{q+2}u)\cdot\nabla\overline{u} \\ = {} & |x|^2|u|^q|v|^{q+2}|u|^2 + 2it|u|^q\,|v|^{q+2}x\cdot(\overline{u}\nabla u - u\nabla\overline{u}) + 2it|u|^2\,x\cdot\nabla(|u|^q|v|^{q+2}) \\ & + 4t^2|u|^q|v|^{q+2}|\nabla u|^2 + +4t^2 u\,\nabla(|u|^q|v|^{q+2})\cdot\nabla\overline{u}. \end{aligned} \tag{3.24}$$

Taking the imaginary part of (3.24) and differentiating again, we obtain

$$\begin{aligned} \mathrm{Im}(J(|u|^q|v|^{q+2}u)\cdot\overline{Ju}) = {} & 2t|u|^2\,x\cdot\nabla(|u|^q|v|^{q+2}) + 4t^2\mathrm{Im}\left(u\,\nabla(|u|^q|v|^{q+2})\cdot\nabla\overline{u}\right) \\ = {} & 2t|v|^{q+2}\,x\cdot|u|^2\nabla(|u|^q) + 2t|u|^{q+2}\,x\cdot\nabla(|v|^{q+2}) \\ & + 4t^2\mathrm{Im}\left(u\,\nabla(|u|^q|v|^{q+2})\cdot\nabla\overline{u}\right) \\ = {} & \frac{2tq}{2+q}|v|^{q+2}\,x\cdot\nabla(|u|^{q+2}) + 2t|u|^{q+2}\,x\cdot\nabla(|v|^{q+2}) \\ & + 4t^2\mathrm{Im}\left(u\,\nabla(|u|^q|v|^{q+2})\cdot\nabla\overline{u}\right). \end{aligned} \tag{3.25}$$

Observe that

$$\int u\,\nabla(|u|^q|v|^{q+2})\cdot\nabla\overline{u}dx = -\int |u|^q|v|^{q+2}\,|\nabla u|^2dx - \int |u|^q|v|^{q+2}\Delta\overline{u}\,u\,dx,$$

using the Lemma 3.3 it follows that

$$\begin{aligned} 4t^2\mathrm{Im}\int u\,\nabla(|u|^q|v|^{q+2})\cdot\nabla\overline{u}dx = {} & -4t^2\int |u|^q\,|v|^{q+2}\mathrm{Im}\,(\Delta\overline{v}\,v)\;dx \\ = {} & -2t^2\int |v|^{q+2}\,|u|^q\,\frac{\partial}{\partial t}|u|^2\,dx \\ = {} & -\frac{4t^2}{q+2}\int |v|^{q+2}\,\frac{\partial}{\partial t}|u|^{q+2}\,dx. \end{aligned} \tag{3.26}$$

Combining (3.25), (3.26) and integrating by parts in $\mathbb{R}^n$, it is not difficult to see that

$$\begin{aligned} \int \mathrm{Im}(J(|u|^q|v|^{q+2}u)\cdot\overline{Ju})dx + \int \mathrm{Im}(J(|v|^q|u|^{q+2})\cdot\overline{Jv})dx = {} & \frac{2tq}{q+2}\int x\cdot\nabla(|u\,v|^{q+2})dx \\ + 2t\int x\cdot\nabla(|u\,v|^{q+2})dx - \frac{4t^2}{q+2}\int\frac{\partial}{\partial t}\left(|u\,v|^{q+2}\right)dx & \\ = -\frac{4tn(q+1)}{q+2}\int |u\,v|^{q+2}\,dx - \frac{4t^2}{q+2}\int\frac{\partial}{\partial t}\left(|u\,v|^{q+2}\right)dx, & \end{aligned} \tag{3.27}$$

the proof of lemma follows using the following identity

$$t^2\frac{\partial}{\partial t}\left(|u\,v|^{q+2}\right) = \frac{\partial}{\partial t}\left(t^2\,|u\,v|^{q+2}\right) - 2t\,|u\,v|^{q+2}.$$

□

Theorem 3.2. *(Pseudo-Conformal Law) Let* u *and* v *be solutions of the coupled system* (1.1)*, then*

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\int |Ju|^2+|Jv|^2-\frac{4\alpha t^2}{(p+1)}\int\left[|u|^{2(p+1)}+|v|^{2(q+1)}\right]dx-\frac{8\beta t^2}{(q+2)}\int |u\,v|^{q+2}\,dx\right\}\\&=\frac{4\alpha(np-2)\,t}{(p+1)}\int\left[|u|^{2(p+1)}+|v|^{2(q+1)}\right]dx+\frac{8\beta t}{(q+2)}[(q+1)n-2]\int |u\,v|^{q+2}\,dx.\end{aligned}\tag{3.28}$$

Proof. From (3.18) and (3.19), we get

$$L(Ju)=\,J(Lu)\,=\,-\alpha J(|u|^{2p}u)-\beta J(|u|^q|v|^{q+2}u)\tag{3.29}$$

and by the definition of L, we have

$$i\frac{\partial}{\partial t}(Ju)+\Delta(Ju)\,=\,-\alpha J(|u|^{2p}u)-\beta J(|u|^q|v|^{q+2}u).\tag{3.30}$$

Making the scalar product of (3.30) with $\overline{Ju}$, taking two times the imaginary part, after integration in $\mathbb{R}^n$, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\int |Ju(x)|^2\,dx-2\mathrm{Im}\int|\nabla(Ju(x))|^2\,dx\,=\,&-2\alpha\mathrm{Im}\int J(|u|^{2p}u)\cdot\overline{Ju}\,dx\\&-2\beta\mathrm{Im}\int J(|u|^q|v|^{q+2}u)\cdot\overline{Ju}\,dx.\end{aligned}\tag{3.31}$$

Therefore

$$\frac{\partial}{\partial t}\int |Ju(x)|^2\,dx\,=\,-2\alpha Im(\int J(|u|^{2p}u)\cdot\overline{Ju}\,dx)-2\beta Im(\int J(|u|^q|v|^{q+2}u)\cdot\overline{Ju}\,dx).\tag{3.32}$$

Similarly

$$\frac{\partial}{\partial t}\int |Jv(x)|^2\,dx\,=\,-2\alpha Im(\int J(|v|^{2p}v)\cdot\overline{Jv}\,dx)-2\beta Im(\int J(|v|^q|u|^{q+2}v)\cdot\overline{Jv}\,dx).\tag{3.33}$$

Adding (3.32) and (3.33) and applying the lemmas 3.5 and 3.6 we concludes the proof. ☐

Remark. *Let* $u\in\mathcal{S}(\mathbb{R}^n)$*, we consider the following multiplication differential operator*

$$\widehat{Pu}(\xi)=\sum_{l=1}^n\zeta_l\xi^{\theta_l}\widehat{u}(\xi),\quad \xi\in\mathbb{R}^n,\tag{3.34}$$

where $\zeta_l\in\mathbb{R}$ *and the multi-index* $\theta_l=(\theta_l^j)_{j=1,\dots,n}\in(\mathbb{Z}^+)^n$*. In order to the differential operators*

$$L=\partial_t-iP,\quad J=x+tQ,\quad x\in\mathbb{R}^n,$$

commutes, where Q *is also a multiplication differential operator, is easy to see that we need*

$$\begin{aligned}Q(u)=&i(P(xu)-xP(u)),\quad x=(x_j)_{j=1,\dots,n}\in\mathbb{R}^n\\=&i(P(x_ju)-x_jP(u))_{j=1,\dots,n},\end{aligned}\tag{3.35}$$

and using properties of Fourier transform we have

$$\widehat{Qu}(\xi)=\left(\sum_{l=1}^n\zeta_l\theta_l^j\xi^{\theta_l-e_j}\widehat{u}(\xi)\right)_{j=1,\dots,n},\quad \xi\in\mathbb{R}^n,\tag{3.36}$$

where the canonical unit vetor $e_j = \overbrace{(0,\dots,0,1,0,\dots,0)}^{j}$*. Observe that in this case* J *also commutes with* cL *for any constant* $c \in \mathbb{C}$ *and reciprocally* L *commutes with* cJ *for any constant* $c \in \mathbb{C}$*.*

In our case, if we consider

$$Pu = \Delta u \Rightarrow \widehat{Pu}(\xi) = -\sum_{l=1}^{n} \xi^{2e_l}\widehat{u}(\xi),$$

and by definition of Q *(see* (3.36)*) we obtain*

$$\begin{aligned}\widehat{Qu}(\xi) &= -\left(2\xi^{2e_j-e_j}\widehat{u}(\xi)\right)_{j=1,\dots,n}\\ &= -2\xi\widehat{u}(\xi),\end{aligned}$$

and therefore

$$Qu = 2i\nabla u.$$

In the case $n = 1$*, considering the operator* $\partial_t + \partial_x^{2k+1}$*,* $x \in \mathbb{R}$*, then*

$$\widehat{Pu}(\xi) = (-1)^{k+1}\xi^{2k+1}\widehat{u}(\xi), \quad \xi \in \mathbb{R},$$

and $\widehat{Qu}(\xi) = (-1)^{k+1}(2k+1)\xi^{2k}\widehat{u}(\xi)$*, thus*

$$Qu = (-1)^k(2k+1)\partial_x^{2k}u,$$

in the particular case $k = 1$ *(KdV equation), we obtain*

$$J = x - 3t\partial_x^2.$$

4. A priori estimates in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$

Here we will give conditions about of the global existence. We begin with the following result well-known result

Lemma 4.1. *(The Gagliardo-Nirenberg inequality) Let* $f : \mathbb{R}^n \mapsto \mathbb{R}$*. Fix* $1 \le q, r \le \infty$ *and a natural number* m*. Suppose also that a real number* λ *and a natural number* j *are such that*

$$\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n}\right)\lambda + \frac{1-\lambda}{q}$$

and

$$\frac{j}{m} \le \lambda \le 1.$$

Then

(1) *every function* $f : \mathbb{R}^n \mapsto \mathbb{R}$ *that lies in* $L^q(\mathbb{R}^n)$ *with* m*th derivative in* $L^r(\mathbb{R}^n)$ *also has* j*th derivative in* $L^p(\mathbb{R}^n)$*;*

(2) *and, furthermore, there exists a constant* C *depending only on* m, n, j, q, r *and* λ *such that*

$$\|D^j f\|_{L^p} \le C\|D^m f\|_{L^r}^{\lambda}\|f\|_{L^q}^{1-\lambda}. \tag{4.1}$$

In the particular case $j=0$, $r=q=2$ and $m=1$, we have

$$\|f\|_{L^p} \leq C\|Df\|_{L^2}^{\lambda}\|f\|_{L^2}^{1-\lambda}, \tag{4.2}$$

where

$$0 \leq \lambda := \lambda(r) = \frac{(r-2)n}{2r} \leq 1.$$

Considering the energy equation (1.7), we can to obtain an "a priori" estimate to

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}^2 + \|\nabla v(t)\|_{L^2(\mathbb{R}^n)}^2 \tag{4.3}$$

if $(2p+2)\lambda(2p+2) \leq 2$ and $(4+2q)\lambda(4+2q) \leq 2$, i.e. if

$$0 < p \leq \frac{2}{n}, \qquad 0 < q \leq \frac{2}{n} - 1, \tag{4.4}$$

or if

$$0 < p \leq \frac{2}{n}, \qquad \text{and} \quad \beta \leq 0,$$

or if

$$0 < q \leq \frac{2}{n} - 1, \qquad \text{and} \quad \alpha \leq 0,$$

where in the equality, we obtain "a priori" estimate only to $\|u_0\|_{L^2} \leq C$ and $\|v_0\|_{L^2} \leq C$ (small data).

We observe that if $\mathcal{X} \leq 0$, then from (1.7) it follows that

$$\int \left(|\nabla u(x,t)|^2 + |\nabla v(x,t)|^2\right) dx \leq E(u_0, v_0), \qquad \forall t \geq 0. \tag{4.5}$$

In the next section we will see that in some cases when $\mathcal{X} \leq 0$, we can also get us a better asymptotic growth to (4.3).

5. Asymptotic Growth in the Energy Space

In this section we will prove the Theorem 1.2.

From Theorem 3.2 we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\left[\int \left(|J(u)|^2 + |J(v)|^2\right) dx - tf(t)\right] =& \frac{4t\alpha(np-2)}{p+1}\int |u|^{2p+2} + |v|^{2p+2}\, dx \\ &+ \frac{8t\beta[n(q+1)-2]}{q+2}\int |uv|^{q+2}\, dx, \end{aligned} \tag{5.1}$$

where the function

$$f(t) = 4t\mathcal{X}(t) = \frac{4\alpha t}{(p+1)}\int \left[|u|^{2(p+1)} + |v|^{2(p+1)}\right] dx + \frac{8\beta t}{(q+2)}\int |uv|^{q+2}\, dx. \tag{5.2}$$

We consider two cases

Case I If

$$\beta\, n(q+1) \leq \beta\, np \Longleftrightarrow \begin{cases} p \geq q+1 & \text{if} \quad \beta > 0, \\ \text{or} & \\ p \leq q+1 & \text{if} \quad \beta < 0. \end{cases}$$

In this case is

$$8t\beta[n(q+1)-2] \le 8t\beta(np-2),$$

and (5.1) implies

$$\begin{aligned} \frac{\partial}{\partial t}\left[\int \left(|J(u)|^2+|J(v)|^2\right)\,dx - tf(t)\right] \le & \frac{4t\alpha(np-2)}{p+1}\int |u|^{2p+2}+|v|^{2p+2}\,dx \\ & + \frac{8t\beta(np-2)}{q+2}\int |uv|^{q+2}\,dx \\ =& (np-2)f(t), \end{aligned} \tag{5.3}$$

integrating the inequality above

$$\begin{aligned} \int (|J(u)|^2+|J(v)|^2)\,dx - tf(t) &\le a_0 + (np-2)\int_0^t f(t')dt' \\ &\le a_0 + (np-2)\int_0^1 f(t')dt' + (np-2)\int_1^t f(t')dt', \end{aligned} \tag{5.4}$$

where

$$a_0 = \int |x|^2\left(|u_0(x)|^2+|v_0(x)|^2\right)dx, \tag{5.5}$$

which gives

$$F(t) := -tf(t) \le b_0 + \int_1^t \left(\frac{2-np}{t'}\right)F(t')dt', \tag{5.6}$$

where

$$b_0 = b_0(n,p) := a_0 + (np-2)\int_0^1 f(t')dt'. \tag{5.7}$$

The Gronwall inequality in (5.6) with $np \le 2$, implies

$$F(t) \le b_0 e^{-\int_1^t (np-2)/t'dt'} = b_0 t^{2-np}, \quad t \ge 1. \tag{5.8}$$

From conservation of energy (1.7) we can deduce

$$\int \left(|\nabla u|^2+|\nabla v|^2\right)\,dx = E(0) + \frac{f(t)}{4\,t}, \tag{5.9}$$

and from (5.8) and (5.9) it follows that

$$\int \left(|\nabla u|^2+|\nabla v|^2\right)\,dx \ge E(0) - \frac{b_0}{4\,t^{np}}, \quad t \ge 1.$$

On the other hand, if $f(t) = 4t\mathfrak{X}(t) \le 0$ (e.g. $\alpha \le 0$ and $\beta \le 0$) the above inequality and (4.5) imply (1.15). and from inequalities (5.4)-(5.8) we obtain

$$\begin{aligned} \int (|J(u)|^2+|J(v)|^2)\,dx + |tf(t)| &\le b_0 + (2-np)\int_1^t \frac{b_0 t'^{2-np}}{t'}dt' \\ &= b_0 t^{2-np} \quad \text{if} \quad np \le 2 \quad \text{and} \quad t \ge 1. \end{aligned} \tag{5.10}$$

And by definition of J it follows that

$$|J(u)|^2 = |x|^2|u|^2 + 4t^2|\nabla u|^2 - 4t\mathrm{Im}\overline{u}\,x\cdot\nabla u.$$

Hence if $np \leq 2$, using Cauchy-Schwartz we get

$$\begin{aligned}\int |x|^2 \left(|u|^2+|v|^2\right) dx + 4t^2 \int \left(|\nabla u|^2+|\nabla v|^2\right)\, dx \leq b_0 t^{2-np} + 4t \int \operatorname{Im}\overline{u}\, x\cdot \nabla u dx \\ + 4t \int \operatorname{Im}\overline{v}\, x\cdot\nabla v dx \\ \leq b_0 t^{2-np} + 4t\|x\, u\|_{L^2}\|\nabla u\|_{L^2} + 4t\|x\, v\|_{L^2}\|\nabla v\|_{L^2},\end{aligned} \tag{5.11}$$

and from (5.11) we have

$$\left(\|x\, u\|_{L^2} - 2t\|\nabla u\|_{L^2}\right)^2 + \left(\|x\, v\|_{L^2} - 2t\|\nabla v\|_{L^2}\right)^2 \leq b_0\, t^{2-np}, \tag{5.12}$$

and consequently

$$2t\left(\|\nabla u\|_{L^2} + \|\nabla v\|_{L^2}\right) \leq \|x\, u\|_{L^2} + \|x\, v\|_{L^2} + 2b_0^{1/2} t^{1-np/2}, \tag{5.13}$$

therefore using Lemma 3.4 we obtain

$$2t\left(\|\nabla u\|_{L^2} + \|\nabla v\|_{L^2}\right) \leq 2b_0^{1/2} t^{1-np/2} + a_0 + 2\int_0^t \left(\|\nabla u\|_{L^2} + \|\nabla v\|_{L^2}\right) dt'. \tag{5.14}$$

Let $\mathcal{W}(t) = \|\nabla u(t)\|_{L^2} + \|\nabla v(t)\|_{L^2}$, the above inequality gives

$$\begin{aligned} t\mathcal{W}(t) &\leq b_0^{1/2} t^{1-np/2} + \frac{a_0}{2} + \int_0^t \mathcal{W}(t')dt' \\ &= b_0^{1/2} t^{1-np/2} + \frac{a_0}{2} + \int_0^1 \mathcal{W}(t')dt' + \int_1^t \mathcal{W}(t')dt' \\ &:= b_0^{1/2} t^{1-np/2} + c_0 + \int_1^t \left(\frac{1}{t'}\right) t'\mathcal{W}(t')dt', \end{aligned} \tag{5.15}$$

where

$$c_0 = \frac{a_0}{2} + \int_0^1 \mathcal{W}(t')dt', \tag{5.16}$$

and a_0 as defined in (5.5), and by Gronwall's inequality (see Lema 3.1), we concludes that if $np \leq 2$ and $t \geq 1$, then

$$\begin{aligned} t\mathcal{W}(t) \leq & b_0^{1/2} t^{1-np/2} + c_0 + \int_1^t \left(b_0^{1/2} t'^{1-np/2} + c_0\right) \frac{1}{t'} \exp\left\{\int_{t'}^t \frac{1}{r} dr\right\} dt' \\ \leq & b_0^{1/2} t^{1-np/2} + c_0 + t\int_1^t \left(b_0^{1/2} t'^{1-np/2} + c_0\right) \frac{1}{t'^2} dt'. \end{aligned} \tag{5.17}$$

Consequently, if $np \leq 2$ and $t \geq 1$ we estimate $\mathcal{W}(t)$ by

$$\mathcal{W}(t) \leq \left(\frac{2b_0^{1/2}}{np} + c_0\right) - \frac{b_0^{1/2}(2-np)}{np} t^{-np/2}.$$

Using this inequality and (5.12) is easy to verify the estimate (1.14).

Case II If

$$\alpha\, n(q+1) \geq \alpha\, np \Longleftrightarrow \begin{cases} p \leq q+1 & \text{if} \quad \alpha > 0, \\ p \geq q+1 & \text{if} \quad \alpha < 0. \end{cases}$$

In this case is

$$4t\alpha[n(q+1)-2] \geq 4t\alpha(np-2),$$

and (5.1) implies

$$\begin{aligned}\frac{\partial}{\partial_t}\left[\int\left(|J(u)|^2+|J(v)|^2\right)\,dx-tf(t)\right] \leq&\frac{4t\alpha[n(q+1)-2]}{p+1}\int|u|^{2p+2}+|v|^{2p+2}\,dx\\ &+\frac{8t\beta[n(q+1)-2]}{q+2}\int|uv|^{q+2}\,dx\\ =&-[2-n(q+1)]f(t),\end{aligned}\tag{5.18}$$

and similarly as the above case we can show that if $n(q+1)\leq 2$, then

$$\begin{aligned}\int\left(|\nabla u|^2+|\nabla v|^2\right)\,dx=&E(0)+\frac{f(t)}{4\,t}\\ \geq&E(0)-\frac{b_1}{4t^{n(q+1)}},\quad t\geq 1,\end{aligned}\tag{5.19}$$

where

$$b_1=b_1(n,q):=a_0-[2-n(q+1)]\int_0^1 f(t')dt'.\tag{5.20}$$

Similarly as in Case I, if $f(t)=4t\mathfrak{X}(t)\leq 0$, from the inequalities above we obtain

$$\begin{aligned}\int(|J(u)|^2+|J(v)|^2)\,dx+|tf(t)|\leq&b_1+(2-n(q+1))\int_1^t\frac{b_1t'^{2-n(q+1)}}{t'}dt'\\ =&b_1t^{2-n(q+1)}\quad\text{if}\quad n(q+1)\leq 2\quad\text{and}\quad t\geq 1.\end{aligned}\tag{5.21}$$

Let $\mathcal{W}(t)=\|\nabla u(t)\|_{L^2}+\|\nabla v(t)\|_{L^2}$, as in Case I, we obtain

$$\begin{aligned}t\mathcal{W}(t)\leq&b_1^{1/2}t^{1-n(q+1)/2}+c_0+\int_1^t\left(b_1^{1/2}t'^{1-n(q+1)/2}+c_0\right)\frac{1}{t'}\exp\left\{\int_{t'}^t\frac{1}{r}dr\right\}dt'\\ \leq&b_1^{1/2}t^{1-n(q+1)/2}+c_0+t\int_1^t\left(b_1^{1/2}t'^{1-n(q+1)/2}+c_0\right)\frac{1}{t'^2}dt'.\end{aligned}\tag{5.22}$$

Consequently, if $n(q+1)\leq 2$ and $t\geq 1$ we estimate $W(t)$ by

$$\mathcal{W}(t)\leq\left(\frac{2b_1^{1/2}}{n(q+1)}+c_0\right)-\frac{b_1^{1/2}(2-n(q+1))}{n(q+1)}t^{-n(q+1)/2}.$$

Finally using this inequality and (5.12) is easy to verify the estimate (1.16).

Remark. *Let* $P(t)=\|x\,u(t)\|^2_{L^2_x}+\|x\,v(t)\|^2_{L^2_x}$ *and* $W(t)=\|\nabla u(t)\|^2_{L^2}+\|\nabla v(t)\|^2_{L^2}$, *then*

i) *If* $E(0)>1$ *and* $P(0)\ll 1$ *is very small, then it is not difficult to see that in the right side of* (1.13) *we have that*

$$\left(c_0+\frac{2b_0^{1/2}}{np}\right)-\frac{b_0^{1/2}(2-np)}{np}t^{-np/2}<E(0).$$

ii) *With the conditions of Theorem 1.2, i.e. if $np \leq 2$ and $p \geq q+1$ if $\beta > 0$ or $p \leq q+1$ if $\beta < 0$ and $\mathcal{X} \leq 0$ (see* (1.8)*, e.g., $\alpha \leq 0$ and $\beta \leq 0$) and using Lemma 3.4 and Cauchy-Schwartz inequality we have*

$$2t^2W(t) - b_0 t^{2-np} \leq P(t) \leq b_0 t^{2-np} + 2a_0 + 8\,t\int_0^t W(t')\,dt',$$

and similarly if $n(q+1) \leq 2$ and $p \geq q+1$ if $\alpha > 0$ or $p \leq q+1$ if $\alpha < 0$ and $\mathcal{X} \leq 0$, then

$$2t^2W(t) - b_0 t^{2-n(q+1)} \leq P(t) \leq b_0 t^{2-n(q+1)} + 2a_0 + 8\,t\int_0^t W(t')\,dt'.$$

iii) *Using equality* (3.16) *in the first inequality from* (5.11)*, we obtain*

$$\begin{aligned} P(t) + 4t^2W(t) \leq& b_0 t^{2-np} + 4t\int Im\overline{u}\, x\cdot\nabla u dx + 4t\int Im\overline{v}\, x\cdot\nabla v dx \\ \leq& b_0 t^{2-np} + tP'(t), \end{aligned}$$

hence

$$P(t) + 4t^2W(t) \leq b_0 t^{2-np} + tP'(t),$$

it follows that

$$4W(t) - \frac{b_0}{t^{np}} \leq \frac{d}{dt}\left(\frac{P(t)}{t}\right).$$

6. BLOW-UP IN $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$

In this section we will prove the Theorem 1.3. Using Lemma 3.2 and equality (3.16) we get

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\int |x|^2(|u(t)|^2 + |v(t)|^2)\,dx =& 4\frac{\partial}{\partial t}\left\{\mathrm{Im}\int(\overline{u}x\cdot\nabla u + \overline{v}x\cdot\nabla v)\;dx\right\} \\ =& 8E(0) + \frac{4\alpha(2-np)}{p+1}\int\left(|u|^{2p+2} + |v|^{2p+2}\right)dx \\ &+ \frac{8\beta\;(2-n(1+q))}{q+2}\int |u\,v|^{q+2}dx. \end{aligned} \tag{6.1}$$

We consider two cases

Case I If

$$\beta\, p \leq \beta\,(q+1) \Longleftrightarrow \begin{cases} p-q \leq 1 & \text{if} \quad \beta > 0, \\ p-q \geq 1 & \text{if} \quad \beta < 0. \end{cases}$$

In this case is

$$8\beta[2-n(q+1)] \leq 8\beta(2-np),$$

and (6.1) gives

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\int |x|^2(|u(t)|^2+|v(t)|^2)\,dx \leq & 8E(0)+\frac{4\alpha(2-np)}{p+1}\int \left(|u|^{2p+2}+|v|^{2p+2}\right)dx \\ & +\frac{8\beta\,(2-np)}{q+2}\int |u\,v|^{q+2}dx \\ \leq & 8E(0)-\frac{(np-2)f(t)}{t}.\end{aligned} \tag{6.2}$$

From conservation of the energy (1.7) we can deduce

$$-\frac{f(t)}{4\,t}=E(0)-\int \left(|\nabla u|^2+|\nabla v|^2\right)\,dx, \tag{6.3}$$

therefore

$$-\frac{f(t)}{t}\leq 4E(0). \tag{6.4}$$

Combining (6.2), (6.4) and that $np\geq 2$, we have

$$\frac{\partial^2}{\partial t^2}\int |x|^2(|u(t)|^2+|v(t)|^2)\,dx \leq 4npE(0). \tag{6.5}$$

Integrating and using (3.16) we can show that

$$\frac{\partial}{\partial t}\int |x|^2(|u(t)|^2+|v(t)|^2)\,dx\leq 4\mathrm{Im}\int \left(\overline{u}_0\,x\cdot\nabla u_0+\overline{v}_0\,x\cdot\nabla v_0\right)\,dx+4npE(0)t, \tag{6.6}$$

integrating again we obtain

$$\begin{aligned}\int |x|^2(|u(t)|^2+|v(t)|^2)dx\leq & \int |x|^2(|u_0|^2+|v_0|^2)dx+4t\mathrm{Im}\int \left(\overline{u}_0x\cdot\nabla u_0+\overline{v}_0x\cdot\nabla v_0\right)\,dx \\ & +2npE(0)t^2 \\ := & A_0+B_0t+C_0t^2:=P_0(t).\end{aligned} \tag{6.7}$$

It is not difficult to see that there exists a $T>0$ such that $\int |x|^2(|u(T)|^2+|v(T)|^2)dx=0$ in the following cases:

(1)

$$E(0)=0 \quad \text{and} \quad \mathrm{Im}\int \left(\overline{u}_0x\cdot\nabla u_0+\overline{v}_0x\cdot\nabla v_0\right)\,dx<0,$$

(2)

$$E(0)<0,$$

(3)

$$E(0)>0,$$

and

$$\left(\mathrm{Im}\int \left(\overline{u}_0x\cdot\nabla u_0+\overline{v}_0x\cdot\nabla v_0\right)\,dx\right)^2>\frac{npE(0)}{2}\int |x|^2(|u_0|^2+|v_0|^2)dx.$$

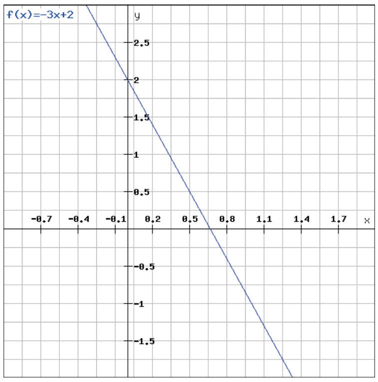

FIGURE 1. The graph of $P_0(t)$ corresponding to the case (1).

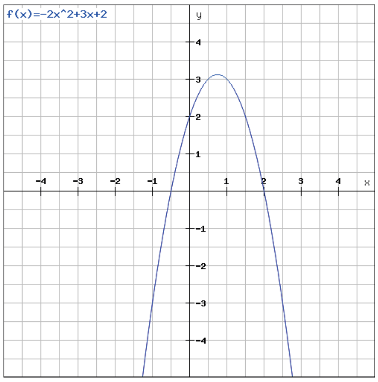

FIGURE 2. The graph of $P_0(t)$ corresponding to the case (2).

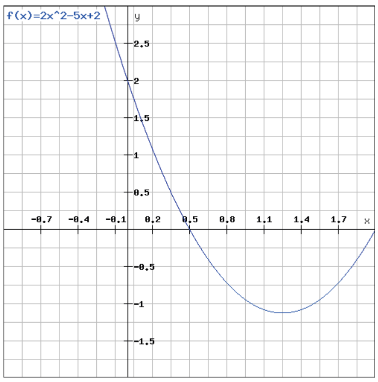

FIGURE 3. The graph of $P_0(t)$ corresponding to the case (3).

The following graphs are examples corresponding to cases (1), (2) and (3) above.

Now the Heisenberg inequality (Uncertainty inequality)

$$\|f\|_{L^2}^2 \le \frac{2}{n}\| xf\|_{L^2}\| \nabla f\|_{L^2}, \tag{6.8}$$

implies that if the initial data u_0 and v_0 satisfies (1), (2) or (3) then, there exists $0 < T^* \leq T$ such that

$$\lim_{t\to T^*} \| \nabla u(t)\|_{L^2} = \infty, \qquad \lim_{t\to T^*} \| \nabla v(t)\|_{L^2} = \infty.$$

Case II If

$$\alpha\,(q+1) < \alpha p \iff \begin{cases} p-q>1 & \text{if} \quad \alpha>0, \\ p-q<1 & \text{if} \quad \alpha<0. \end{cases}$$

In this case is

$$4\alpha(2-np) \leq 4\alpha[2-n(q+1)],$$

and (6.1) gives

$$\frac{\partial^2}{\partial t^2}\int |x|^2(|u(t)|^2+|v(t)|^2)\,dx \leq 8E(0) - \frac{(n(q+1)-2)f(t)}{t}.$$

As in Case I, using (6.4) and $n(q+1)\geq 2$, we have

$$\frac{\partial^2}{\partial t^2}\int |x|^2(|u(t)|^2+|v(t)|^2)\,dx \leq 4n(1+q)E(0).$$

Integrating two times and using (3.16) we obtain

$$\begin{aligned}\int |x|^2(|u(t)|^2+|v(t)|^2)dx \leq& \int |x|^2(|u_0|^2+|v_0|^2)dx + 4t\mathrm{Im}\int (\overline{u}_0 x\cdot\nabla u_0 + \overline{v}_0 x\cdot \nabla v_0)\,dx \\ &+ 2n(1+q)E(0)t^2 \\ :=& A_0 + B_0 t + C_1 t^2.\end{aligned}$$

It is not difficult to see that there exists a $T>0$ such that $\int |x|^2(|u(T)|^2+|v(T)|^2)dx = 0$ in the following cases:

(1)
$$E(0)=0 \quad \text{and} \quad \mathrm{Im}\int (\overline{u}_0 x\cdot\nabla u_0 + \overline{v}_0 x\cdot\nabla v_0)\,dx < 0,$$

(2)
$$E(0)<0,$$

(3)
$$E(0)>0,$$

and

$$\left(\mathrm{Im}\int (\overline{u}_0 x\cdot\nabla u_0 + \overline{v}_0 x\cdot\nabla v_0)\,dx\right)^2 > \frac{n(q+1)E(0)}{2}\int |x|^2(|u_0|^2+|v_0|^2)dx.$$

Using the Heinseberg inequality (6.8) again we concludes in this case that if the initial data u_0 and v_0 satisfies (1), (2) or (3) then, there exists $0<T^*\leq T$ such that

$$\lim_{t\to T^*} \| \nabla u(t)\|_{L^2} = \infty, \qquad \lim_{t\to T^*} \| \nabla v(t)\|_{L^2} = \infty.$$

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