Applications of measure-valued solutions in fluid mechanics

Eduard Feireisl
based on joint work with J. Březina (Tokio), P. Gwiazda, A. Świerczewska-Gwiazda (Warsaw), M. Medviďová/Lukáčová (Mainz), E.Wiedemann (Bonn)

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

MATHFLOWS 2017, Bedlewo, January 2017

The research leading to these results has received funding from the European Research Council
under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078

What is a good weak solution of an evolutionary equation?

Desired properties

- A weak solution exists globally in time for “any” choice of the initial state
- A weak solution can be identified as a limit of suitable approximate problems, e.g. by adding artificial viscosity
- The set of weak solutions is closed; a limit of a family of weak solutions is another weak solution
- A weak solution can be identified as a limit of a numerical scheme
- A weak solution is the most general object that enjoys the weak–strong uniqueness property

Weak strong uniqueness

A weak solution coincides with a strong (classical) solution as long as the latter exists

Measure–valued solutions

Derivatives

Partial derivatives replaced by distributional derivatives

Oscillations

A parameterized measure (Young measure)

$$\nu_{t,x} \in \mathcal{P}(F),\ t - \text{time},\ x - \text{spatial variable},\ F - \text{phase space}$$

$$\mathbf{U} : Q \to F,\ f(\mathbf{U})(t,x) \text{ replaced by expectations } \langle \nu_{t,x}; f(\mathbf{U}) \rangle$$

Concentrations

Concentration measure $\mathcal{C} \in \mathcal{M}(Q)$

Example - barotropic Euler/ Navier Stokes system

Field equations

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = \left\{ \begin{array}{l} 0 \\ \\ \mathrm{div}_x \mathbb{S} \end{array} \right.$$

Periodic boundary conditions

$$x \in \mathcal{T}^N, \ N = 1, 2, 3$$

Pressure, pressure potential

$$p = p(\varrho), \ p'(\varrho) \geq 0, \ P(\varrho) = \varrho \int_1^{\varrho} \frac{p(z)}{z^2} \ \mathrm{d}z$$

Measure valued solutions

Equation of continuity

$$\int_0^T \int_{\mathcal{T}^N} \langle \nu_{t,x}; \varrho \rangle \, \partial_t \varphi + \langle \nu_{t,x}; \mathbf{m} \rangle \cdot \nabla_x \varphi \; \mathrm{d}x \mathrm{d}t = \int_0^T \int_{\mathcal{T}^N} \nabla_x \varphi \cdot \mathrm{d}\mathcal{C}_1$$

for all $\varphi \in C_c^\infty((0,T) \times \mathcal{T}^N)$

Momentum equation

$$\int_0^T \int_{\mathcal{T}^N} \langle \nu_{t,x}; \mathbf{m} \rangle \, \partial_t \varphi + \left\langle \nu_{t,x}; \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle : \nabla_x \varphi + \langle \nu_{t,x}; p(\varrho) \rangle \, \mathrm{div}_x \varphi \; \mathrm{d}x \mathrm{d}t$$

$$= \int_0^T \int_{\mathcal{T}^N} \nabla_x \varphi : \mathrm{d}\mathcal{C}_2$$

for all $\varphi \in C_c^\infty((0,T) \times \mathcal{T}^N; R^N)$

Energy dissipation

Energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{T}^N}\left(\frac{1}{2}\varrho|\mathbf{u}|^2+P(\varrho)\right)\ \mathrm{d}x\boxed{\leq}0$$

Measure-valued energy inequality

$$\int_{\mathcal{T}^N}\left\langle \nu_{\tau,x};\left(\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}+P(\varrho)\right)\right\rangle\ \mathrm{d}x+\boxed{\mathcal{D}(\tau)}$$
$$\leq\int_{\mathcal{T}^N}\left\langle \nu_0;\left(\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}+P(\varrho)\right)\right\rangle\ \mathrm{d}x$$

Dissipation defect - compatibility

$$\left|\mathcal{C}_1[0,\tau]\times\mathcal{T}^N\right|+\left|\mathcal{C}_2[0,\tau]\times\mathcal{T}^N\right|\leq\xi(\tau)\mathcal{D}(\tau),\ \xi\in L^1(0,T)$$

Convergence of a numerical scheme

EF, M. Lukáčová–Medviďová [2016]

Let $\Omega \subset R^3$ be a smooth bounded domain. Let

$$1 < \gamma < 2, \ \Delta t \approx h, \ 0 < \alpha < 2(\gamma - 1).$$

Suppose that the initial data are smooth and that the compressible **Navier-Stokes system** admits a smooth solution in $[0, T]$ in the class

$$\varrho, \ \nabla_x \varrho, \ \mathbf{u}, \nabla_x \mathbf{u} \in C([0, T] \times \overline{\Omega})$$

$$\partial_t \mathbf{u} \in L^2(0, T; C(\overline{\Omega}; R^3)), \ \varrho > 0, \ \mathbf{u}|_{\partial\Omega} = 0.$$

Then the numerical solutions resulting from Karlsen-Karper FV-FE scheme converge unconditionally,

$$\varrho_h \to \varrho \text{ (strongly) in } L^\gamma((0, T) \times K)$$

$$\mathbf{u}_h \to \mathbf{u} \text{ (strongly) in } L^2((0, T) \times K; R^3)$$

for any compact $K \subset \Omega$.

General strategy

Basic properties of numerical scheme

Show stability, consistency, discrete energy inequality

Measure valued solutions

Show convergence of the scheme to a

$$\boxed{\textbf{dissipative measure} - \textbf{valued solution}}$$

Weak-strong uniqueness

Use the weak-strong uniqueness principle in the class of measure-valued solutions. Strong and measure valued solutions emanating from the same initial data coincide as long as the latter exists

Truly measure-valued solutions

Truly measure-valued solutions for the Euler system (with E.Chiodaroli, O.Kreml, E. Wiedemann)

There is a measure-valued solution to the compressible Euler system (without viscosity) that *is not* a limit of bounded L^p weak solutions to the Euler system.

Weak (mv) - strong uniqueness

Theorem - EF, P.Gwiazda, A.Świerczewska-Gwiazda, E. Wiedemann [2015]

A measure valued and a strong solution emanating from the same initial data coincide as long as the latter exists

Relative energy (entropy)

Relative energy functional

$$\mathcal{E}\left(\varrho,\mathbf{m}\,\middle|\,r,\mathbf{U}\right)(\tau)$$

$$= \int_{\mathcal{T}^N} \left\langle \nu_{\tau,x}; \frac{1}{2}\frac{|\mathbf{m}-r\mathbf{U}|^2}{\varrho} + P(\varrho) - P'(r)(\varrho - r) - P(r) \right\rangle \,\mathrm{d}x$$

$$= \int_{\mathcal{T}^N} \left\langle \nu_{\tau,x}; \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right\rangle \,\mathrm{d}x - \int_{\Omega} \langle \nu_{\tau,x}; \mathbf{m}\rangle \cdot \mathbf{U} \,\mathrm{d}x$$

$$+ \int_{\Omega} \frac{1}{2} \langle \nu_{\tau,x}; \varrho\rangle |\mathbf{U}|^2 \,\mathrm{d}x$$

$$- \int_{\Omega} \langle \nu_{\tau,x}; \varrho\rangle P'(r) \,\mathrm{d}x + \int_{\Omega} p(r) \,\mathrm{d}x$$

Relative energy (entropy) inequality

Relative energy inequality

$$\mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, r, \mathbf{U}\right)(\tau)$$

$$\leq \int_\Omega \left\langle \nu_{0,x}; \frac{1}{2}\frac{|\mathbf{m} - r\mathbf{U}_0|^2}{\varrho} + P(\varrho) - P'(r_0)(\varrho - r_0) - P(r_0) \right\rangle \ \mathrm{d}x$$

$$+ \int_0^\tau \mathcal{R}\left(\varrho, \mathbf{m} \,\middle|\, r, \mathbf{U}\right) \ \mathrm{d}t$$

Remainder

$$
\begin{aligned}
&\mathcal{R}\left(\varrho, \mathbf{m} \,\middle|\, r, \mathbf{U}\right) \\
&= - \int_0^\tau \int_\Omega \langle \nu_{t,x}, \mathbf{m} \rangle \cdot \partial_t \mathbf{U} \ \mathrm{d}x \ \mathrm{d}t \\
&- \int_0^\tau \int_{\overline{\Omega}} \left[\left\langle \nu_{t,x}; \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle : \nabla_x \mathbf{U} + \langle \nu_{t,x}; p(\varrho) \rangle \, \mathrm{div}_x \mathbf{U} \right] \mathrm{d}x \ \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\langle \nu_{t,x}; \varrho \rangle \, \mathbf{U} \cdot \partial_t \mathbf{U} + \langle \nu_{t,x}; \mathbf{m} \rangle \cdot \mathbf{U} \cdot \nabla_x \mathbf{U} \right] \ \mathrm{d}x \ \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\left\langle \nu_{t,x}; \left(1 - \frac{\varrho}{r} \right) \right\rangle p'(r) \partial_t r - \langle \nu_{t,x}; \mathbf{m} \rangle \cdot \frac{p'(r)}{r} \nabla_x r \right] \ \mathrm{d}x \ \mathrm{d}t \\
&+ \int_0^\tau \int_{\mathcal{T}^N} \frac{1}{2} \nabla_x \left(|\mathbf{U}|^2 - P'(r) \right) \ \mathrm{d}\mathcal{C}_1 - \int_0^\tau \int_{\mathcal{T}^N} \nabla_x \mathbf{U} \ \mathrm{d}\mathcal{C}_2
\end{aligned}
$$

Complete Euler system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = 0$$

$$\partial_t \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] + \mathrm{div}_x \left(\left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] \mathbf{u} \right)$$

$$+ \mathrm{div}_x(p(\varrho, \vartheta)\mathbf{u}) = 0$$

Entropy inequality (admissibility)

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) \geq 0$$

Constitutive relations

$$p = \varrho \vartheta, \ \ e = c_v \vartheta, \ \ s = \log(\vartheta^{c_v}) - \log(\varrho)$$

A priori estimates

Energy bounds, total mass conservation

$$\int_{\mathcal{T}^N} \varrho \ \mathrm{d}x = \int_{\mathcal{T}^N} \varrho_0 \ \mathrm{d}x$$

$$\int_{\mathcal{T}^N} \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \ \mathrm{d}x = \int_{\mathcal{T}^N} \frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0, \vartheta_0) \ \mathrm{d}x$$

Entropy transport

$$s(\varrho, \vartheta)(\tau, x) \geq \inf s(\varrho_0, \vartheta_0)$$

L^1 estimates

$\|\varrho\|_{L^1}, \|\varrho\mathbf{u}\|_{L^1}, \ \|\varrho|\mathbf{u}|^2\|_{L^1}, \ \|\varrho\vartheta\|_{L^1}, \ \|\varrho s\|_{L^1}, \ \|p\|_{L^1}, \|\varrho s\mathbf{u}\|_{L^1}$ bounded

MV solutions, I

Basic state variables

$$\text{density } \varrho, \ \text{momentum } \mathbf{m}, \ \text{internal energy } E = \varrho e(\varrho, \vartheta)$$

$$\nu_{t,x} \in \mathcal{P}([0,\infty) \times R^N \times [0,\infty))$$

Equation of continuity

$$\int_0^T \int_{\mathcal{T}^N} \left[\langle \nu_{t,x}; \varrho \rangle \, \partial_t \varphi + \langle \nu_{t,x}; \mathbf{m} \rangle \cdot \nabla_x \varphi \right] \ \mathrm{d}x\mathrm{d}t = 0$$

for any $\varphi \in C_c^\infty((0,T) \times \mathcal{T}^N)$

Momentum equation

$$\int_0^T \int_{\mathcal{T}^N} \left[\langle \nu_{t,x}; \mathbf{m} \rangle \cdot \boldsymbol{\varphi} + \left\langle \nu_{t,x}; \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle : \nabla_x \boldsymbol{\varphi} \right] \ \mathrm{d}x\mathrm{d}t$$

$$+ \int_0^T \int_{\mathcal{T}^N} \langle \nu_{t,x}; p(\varrho, E) \rangle \, \mathrm{div}_x \boldsymbol{\varphi} \ \mathrm{d}x\mathrm{d}t = \int_0^T \int_{\mathcal{T}^N} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathcal{C}$$

for any $\boldsymbol{\varphi} \in C_c^\infty((0,T) \times \mathcal{T}^N; R^N)$

MV solutions, II

Entropy balance

$$\int_0^T \int_{\mathcal{T}^N} \left[\langle \nu_{t,x}; \varrho Z(s) \rangle \, \partial_t \varphi + \langle \nu_{t,x}; Z(s)\, \mathbf{m} \rangle \cdot \nabla_x \varphi \right] \mathrm{d}x\mathrm{d}t \le 0$$

for any $\varphi \in C_c^\infty((0,T) \times \mathcal{T}^N)$, $\varphi \ge 0$, and any $Z \in BC(R)$, $Z' \ge 0$

Total energy balance

$$\left[\int_\Omega \left\langle \nu_{t,x}; \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + E \right\rangle \, \mathrm{d}x \right]_{t=0}^{t=\tau} + \mathcal{D}(\tau) = 0$$

Compatibility

$$\|\mathcal{C}\|_{\mathcal{M}([0,\tau) \times \Omega; R^{3\times 3})} \le c \int_0^\tau \mathcal{D}(t) \, \mathrm{d}t$$

Relative energy

Ballistic free energy

$$H_\Theta(\varrho,\vartheta) = \varrho e(\varrho,\vartheta) - \Theta \varrho s(\varrho,\vartheta),$$

Relative energy

$$\mathcal{E}_Z\left(\varrho,\vartheta,\mathbf{u}\,\middle|\,r,\Theta,\mathbf{U}\right)$$

$$= \frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + \varrho e(\varrho,\vartheta) - \Theta\varrho Z(s(\varrho,\vartheta)) - \frac{\partial H_\Theta(r,\Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r,\Theta).$$

Weak strong uniqueness

Hypotheses

$$\vartheta Ds(\varrho,\vartheta) = De(\varrho,\vartheta) + p(\varrho,\vartheta) D\left(\frac{1}{\varrho}\right)$$

$$\frac{\partial p(\varrho,\vartheta)}{\partial \varrho} > 0,\ \frac{\partial e(\varrho,\vartheta)}{\partial \vartheta} > 0 \text{ for all } \varrho,\vartheta > 0$$

$$|p(\varrho,\vartheta)| \leq c(1 + \varrho + \varrho|s(\varrho,\vartheta)| + \varrho e(\varrho,\vartheta))$$

Conclusion [Březina, EF 2016]

Weak(MV)–strong uniqueness holds provided the initial density and temperature are strictly positive