ON TWO TRANSFORMATIONS OF GRAPHS

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Abstract. The paper describes the properties of two transformations of graphs. One of them was introduced by F. Gliviak for the sake of study of metric properties of graphs, the other is related to it.

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In [1] F. Gliviak introduced a certain graph transformation. He used it only in particular cases as a tool for proving certain theorems concerning metric properties of graphs. However, this transformation seems to be interesting and therefore we will study it and define another transformation related to it.

All graphs considered are finite undirected graphs without loops and multiple edges. We use the standard terminology as e.g. in [2]. The symbol K_n denotes a complete graph with n vertices, $K_{m,n}$ denotes a complete bipartite graph, C_n is a circuit of length n, P_n is a path of length n (having n edges). By G we denote the complement of a graph G. The symbol $G_1 \cup G_2$ will denote the union of graphs; if it is used, then always vertex-disjoint graphs G_1, G_2 are meant. The vertex set of a graph G is denoted by $V(G)$, its edge set by $E(G)$. The symbol $d_G(x,y)$ denotes the distance between vertices x, y in a graph G, the symbol $N_G[x]$ denotes the closed neighbourhood of a vertex x in G, i.e. the set consisting of x and of all vertices which are adjacent to x in G. The subgraph induced by a set A is denoted by $\langle A\rangle$.

Definition 1. Let G be a graph. Take a graph G' isomorphic to G and vertex-disjoint with G. Choose an isomorphic mapping π of G onto G' and extend it to a mapping of $V(G) \cup V(G')$ onto itself by putting $\pi\bigl(\pi(x)\bigr) = x$ for each $x \in V(G)$. Then we define a graph $S(G)$ as follows. The vertex set of $S(G)$ is $V\bigl(S(G)\bigr) =$

$V(G) \cup V(G')$. The graphs G and G' are induced subgraphs of $S(G)$. A vertex x of G is adjacent to a vertex y of G' if and only if the vertices x, $\pi(y)$ are not adjacent in G.

This is the transformation introduced in [1]. The graph $S(G)$ is regular of degree $n-1$ and has $2n$ vertices, where n is the number of vertices of the graph G. To simplify the notation, similarly as in [1], we will sometimes write x' instead of $\pi(x)$ for $x \in V(G)$ and y instead of $\pi(y')$ for $y' \in V(G')$. We denote $V = V(G)$, $V' = V(G')$.

Now we define a transformation $R(G, U)$.

Definition 2. Let G be a finite undirected graph and $U \subset V(G)$. The vertex set of the graph $R(G, U)$ is $V\bigl(R(G, U)\bigr) = V(G)$. If two vertices are both in U or both in $V(G) - U$, then they are adjacent in $R(G, U)$ if and only if they are adjacent in G. If one vertex is in U and the other in $V(G) - U$, then they are adjacent in $R(G, U)$ if and only if they are not adjacent in G.

Some properties of $R(G, U)$ are clear from the definition.

(P1) $R(G, \emptyset) = R\bigl(G, V(G)\bigr) = G$

(P2) $R\bigl(G, V(G) - U\bigr) = R(G, U)$ for any $U \subset V(G)$.

(P3) $R\bigl(R(G, U_1), U_2\bigr) = R(G, U_1 \triangle U_2)$ for any $U_1 \subset V(G)$ and $U_2 \subset V(G)$. Here $\triangle$ denotes the symmetric difference. In particular, $R\bigl(R(G, U), U\bigr) = G$.

(P4) $R(\overline{G}, U) = \overline{R}(G, U)$ for any $U \subset V(G)$.

(P5) Let $U = \{u_1, \ldots, u_k\}$. Then there exists a finite sequence $H_0, H_1, \ldots, H_k$ of graphs such that $H_0 = G$, $H_{i+1} = R\bigl(H_i, \{u_{i+1}\}\bigr)$ for $i = 0, 1, \ldots, k$ and $H_k = R(G, U)$.

The following theorem is an extension of a theorem in [1]. By diam G we denote the diameter of G.

Theorem 1. *Let G be a graph with n vertices. If $G \cong K_n$ or $G \cong K_p \cup K_{n-p}$ for some p, $1 \leqslant p \leqslant n-1$, then $S(G) \cong K_n \cup K_n$ and hence $S(G)$ is disconnected. Otherwise diam $G = 3$. The equalities $d(u, v) = d(u', v') = 3$ occur for vertices u, v of G such that $N_G[v] = V - N_G[u]$. The equalities $d(u, v') = d(u', v) = 3$ occur for vertices u, v of G such that $N_G[u] = N_G[v]$. Any other pair of vertices of $S(G)$ has a distance at most 2.*

P r o o f. The assertion for $G \cong K_n$ or $G \cong K_p \cup K_{n-p}$ is clear from the definition of $S(G)$. Suppose the opposite case. Let u, v be two vertices of G. The equalities $d_{S(G)}(u, v) = d_{S(G)}(u', v')$ and $d_{S(G)}(u, v') = d_{S(G)}(u', v)$ also follow immediately from the definition of $S(G)$. Suppose that $N_G[v] \neq V - N_G[u]$. Then either $N_G[u] \cap N_G[v] \neq \emptyset$, or $V - \bigl(N_G[u] \cup N_G[v]\bigr) \neq \emptyset$. In the former case choose $w \in N_G[u] \cap N_G[v]$; we have a path of length 2 with the vertices u, w, v. In the latter case choose $w \in V - \bigl(N_G[u] \cup N_G[v]\bigr)$; then we have a path of length 2 with the vertices u, w', v.

In both cases $d_{S(G)}(u,v) \leqslant 2$. Now let $N_G[v] = V - N_G[u]$. If $\langle N_G[u]\rangle$ is not a clique in G, then there exist non-adjacent vertices x, y in $N_G[u]$; we have a path with the vertices u, x, y', v in $S(G)$. Similarly if $\langle N_G[v]\rangle$ is not a clique in G. If both $\langle N_G[u]\rangle$ and $\langle N_G[v]\rangle$ are cliques in G, then there exists $x \in N_G[u]$ and $y \in N_G[v]$ such that x is adjacent to y in G; otherwise G would be isomorphic to $K_p \cup K_{n-p}$ for some p. We have a path with the vertices u, x, y, v. We have $d_{S(G)}(u,v) \leqslant 3$ and, as $N_G[u] \cap N_G[v] \neq \emptyset$, it cannot be less; therefore $d_{S(G)}(u,v) = 3$. Now consider the vertices $u \in V$, $v' \in V'$ and first suppose $N_G[u] \neq N_G[v]$, i.e. either $N_G[u] - N_G[v] \neq \emptyset$, or $N_G[v] - N_G[u] \neq \emptyset$. In the former case choose $w \in N_G[u] - N_G[v]$; we have a path with the vertices u, w', v' and thus $d_{S(G)}(u,v') \leqslant 2$. In the latter case choose $w \in N_G[v] - N_G[u]$; we have a path with the vertices u, w', v' and again $d_{S(G)}(u,v') \leqslant 2$. Now let $N_G[u] = N_G[v]$; this case contains the particular case $u = v$. If $\langle N_G[u]\rangle$ is not a clique in G, then there exist non-adjacent vertices x, y in it and we have a path with the vertices u, x, y', v'. If $\langle N_G[u]\rangle$ is a clique in G, then $V - N_G[u] \neq \emptyset$; otherwise $G \cong K_n$. If $\langle V - N_G[u]\rangle$ is not a clique in G, we may choose non-adjacent vertices x, y in $V - N_G[u]$ and we have a path with the vertices u, x', y, v'. If both $\langle N_G[u]\rangle$ and $\langle V - N_G[u]\rangle$ are cliques in G, then there exist adjacent vertices $x \in N_G[u]$, $y \in V - N_G[u]$; otherwise G would be isomorphic to $K_p \cup K_{n-p}$ for some p. We have a path with the vertices u, x, y, v'. In all the above described cases $d_{S(G)}(u,v') = 3$. □

Now we prove some lemmas.

Lemma 1. *Let G be such a graph that the graph $S(G)$ is connected. Let x, y be two vertices of $S(G)$. Then the following two assertions are equivalent:*

(i) *There exists a vertex z of $S(G)$ such that $d_{S(G)}(x,z) = d_{S(G)}(y,z) = 3$.*

(ii) $N_{S(G)}[x] = N_{S(G)}[y]$.

Proof. (i) $\Longrightarrow$ (ii). Note that $N_{S(G)}[x] = N_G[x] \cup \big(V' - N_{G'}[\pi(x)]\big)$ for $x \in V$ and $N_{S(G)}[x] = \big(V - N_G[\pi(x)]\big) \cup N_{G'}[x]$ for $x \in V'$. Consider the case $x \in V$, $y \in V$. If $z \in V$, then by Theorem 1 we have $N_G[x] = V - N_G[z] = N_G[y]$. If $z \in V'$, then $N_G[x] = N_G[\pi(z)] = N_G[y]$. In both cases $N_G[x] = N_G[y]$ and this implies $N_{S(G)}[x] = N_{S(G)}[y]$. If $x \in V'$, $y \in V'$, we obtain analogously $N_G[x] = N_G[y]$, which again implies $N_{S(G)}[x] = N_{S(G)}[y]$. Now consider the case $x \in V$, $y \in V'$. If $z \in V$, then $N_G[x] = V - N_G[z] = V - N_G[\pi(y)]$; thus also $N_G[y] = V' - N_G[\pi(x)]$, which again implies the assertion. If $z \in V'$, the proof is analogous.

(ii) $\Longrightarrow$ (i). Let $N_{S(G)}[x] = N_{S(G)}[(y)]$. By Theorem 1 we have $d_{S(G)}\big(x,\pi(x)\big) = 3$. As y has its closed neighborhood equal to that of x, we have $d_{S(G)}\big(y,\pi(x)\big) = d_{S(G)}\big(x,\pi(x)\big) = 3$. □

Now we have a partition N of $V \cup V'$ such that two vertices x, y belong to the same class of N if and only if $N_{S(G)}[x] = N_{S(G)}[y]$.

Lemma 2. *Let G be such a graph that $S(G)$ is connected. For each vertex x of $S(G)$ the set of all vertices of $S(G)$ having the distance 3 from x is the class of N containing $\pi(x)$.*

Proof. By Theorem 1 we have $d_{S(G)}\bigl(x,\pi(x)\bigr)=3$. If y is in the class of N containing $\pi(x)$, then $N_{S(G)}[y]=N_{S(G)}[\pi(x)]$ and also $d_{S(G)}(x,y)=3$. On the other hand, if $d_{S(G)}(x,y)=3$, then $N_{S(G)}[y]=N_{S(G)}[\pi(x)]$ by Lemma 1 and y is in the same class of N as $\pi(x)$. □

Theorem 2. *Let G be such a graph that $S(G)$ is connected. Let φ be an automorphism of $S(G)$, let x be a vertex of $S(G)$. Then $N_{S(G)}\bigl[\varphi(\pi(x)\bigr]=N_{S(G)}\bigl[\pi(\varphi(x))\bigr]$.*

This follows immediately from Lemmas 1 and 2. □

Theorem 3. *Let G, H be two graphs such that $S(G)\cong S(R)$. Then there exists a subset $U\subset V(G)$ such that $H\cong R(G,U)$.*

Proof. If $S(G)$ is disconnected, then G is either a complete graph K_n, or the disjoint sum of the form K_p+K_{n-p}; the same holds for H. It is easy to see that in this case the assertion holds. Thus let $S(G)$ be connected. For the sake of simplicity we may suppose that G and H have a common vertex set V and when constructing $S(G)$ from G and $S(H)$ from H we also use the same set V'. Let φ be an isomorphism of $S(G)$ onto $S(H)$. Put $U=\bigl\{x\in V|\varphi(x)\in V\bigr\}$. If x, y are two vertices of U, then x, y are adjacent in G if and only if $\varphi(x)$, $\varphi(y)$ are adjacent in H. If x, y are two vertices of $V-U$, then x, y are adjacent in G if and only if $\varphi(x)$, $\varphi(y)$ are adjacent in H', i.e. $\pi\bigl(\varphi(x)\bigr)$, $\pi\bigl(\varphi(y)\bigr)$ are adjacent in H. If $x\in U$, $y\in V-U$, then x, y are adjacent in G if and only if $\varphi(x)$, $\varphi(y)$ are adjacent in $S(R)$, i.e. $\varphi(x)$, $\pi\bigl(\varphi(y)\bigr)$ are not adjacent in H. This implies the assertion. □

If two graphs G, H on the same vertex set V have the property that $H=R(G,U)$ for some $U\subset V$, we write $(G,H)\in\varrho$ and say that G, H are ϱ-equivalent. For two abstract graphs (i.e. isomorphism classes of graphs) we write $(G,H)\in\varrho^*$ and say that G, H are ϱ^*-equivalent, if $H\cong R(G,U)$. All graphs which are ϱ^*-equivalent to a given graph form a ϱ^*-class.

We will also generalize the concept of degree of a vertex. For each subset U of $V(G)$ the degree of U (denoted by $\deg_G U$) is the number of edges of G which have exactly one end vertex in U.

Lemma 3. *If $G\cong R(K_n,U)$ for some U, then $G\cong K_n$ or $G\cong K_p+K_{n-p}$ for some p, $1\leqslant p\leqslant n-1$.*

Lemma 4. *If $G\cong R(\overline{K}_n,U)$ for some U, then $G\cong\overline{K}_n$ or $G\cong K_{p,n-p}$ for some p, $1\leqslant p\leqslant n-1$.*

These assertions are easy to prove.

Theorem 4. *Let C be a ϱ-class of a graph G with n vertices. The class C contains at least one graph H in which $\deg_H U \geqslant \frac{1}{2}|U|\bigl(n-|U|\bigr)$ for each $U \subset V(H)$ and at least one graph H' in which $\deg_H U \leqslant \frac{1}{2}|U|\bigl(n-|U|\bigr)$ for each $U \subset V(H')$.*

Proof. When we transform a graph G into a graph $R(G,U)$, then we delete $\deg_G U$ edges and add $|U|\bigl(n-|U|\bigr)$ edges. The difference between the numbers of edges of $R(G,U)$ and G is $|U|\bigl(n-|U|\bigr)-2\deg_G U$. If $\deg_G U \leqslant \frac{1}{2}|U|\bigl(n-|U|\bigr)$ for some subset $U \subset V(G)$, then the graph $R(G,U)$ has more edges than G. If H is a graph with the maximum number of edges among the graphs from C, then necessarily $\deg_H U \geqslant \frac{1}{2}|U|\bigl(n-|U|\bigr)$ for all subsets U of $V(H)$. The proof of the existence of H' is analogous. $\square$

Corollary. *In each ϱ^*-class of a graph G with n vertices there is a graph with at least $\frac{1}{2}n(n-1)$ vertices and a graph with at most $\frac{1}{2}n(n-1)$ vertices.*

Proof. Let H be the graph from Theorem 4. In particular, the inequality must hold for all one-element subsets $U=\{u\}$ of $V(H)$, namely $\deg_H u \geqslant \frac{1}{2}n(n-1)$. This implies the assertion. Similarly for H'. $\square$

Lemma 5. *Let C be a ϱ^*-class of a graph G with n vertices, n odd. Then the numbers of edges of all graphs from C have the same parity.*

Proof. As was said in the proof of Theorem 4, the difference between the numbers of edges of $R(G,U)$ and G is equal to $|U|\bigl(n-|U|\bigr)-2\deg_G U$. If n is odd, the numbers $|U|$ and $n-|U|$ have different parities and their product is even; therefore also $|U|\bigl(n-|U|\bigr)-2\deg_G U$ is even. As G and U were chosen arbitrarily, the assertion holds. $\square$

It follows from (P4) that for each ϱ^*-class C of a graph G with n vertices there exists exactly one such ϱ^*-class $\overline{C}$ which consists of graphs isomorphic to the complements of graphs from C. We will call $\overline{C}$ a class complementary to C. If $C=\overline{C}$, we say that C is self-complementary.

Theorem 5. *Let n be a positive integer. There exists a self-complementary ϱ^*-class of graphs with n vertices if and only if $n \not\equiv 3 \pmod 4$.*

Proof. If $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$, then, by the results of G. Ringel [3] and H. Sachs [4] there exists a self-complementary graph (i.e. isomorphic to its complement) having n vertices. The ϱ^*-class of such a graph is evidently a self-complementary class. Now let $n \equiv 2 \pmod 4$. Take a vertex set V with n vertices and choose a vertex $u \in V$. Construct a self-complementary graph G_0 with the vertex set $V-\{u\}$. Let G be the disjoint union of G_0 and the graph having the unique vertex u. The graph $R(G,\{u\})$ is then isomorphic to the complement $\overline{G}$ of G and this implies the assertion. Finally, consider $n \equiv 3 \pmod 4$. Suppose that there exists a graph G with n vertices such that $\overline{G} \cong R(G,U)$ for some $U \subset V(G)$. As n is odd, by Proposition 3 the numbers of edges of G and G' have the same parity. The sum of these numbers is the number $\frac{1}{2}n(n-1)$ of edges of K_n. But for $n \equiv 3 \pmod 4$ the number $\frac{1}{2}n(n-1)$ is odd, while a sum of two numbers of the same parity is even. This is a contradiction. $\square$

Now we turn to the independence number. The independence number $\alpha(G)$ is the maximum number of vertices of an independent set in the graph G.

Theorem 6. *Let G be a graph. Then $\alpha\big(S(G)\big) \leqslant 2\alpha(G)$ and there exists a graph H such that $(G,H) \in \varrho^*$ and $\alpha(H) = \alpha\big(S(G)\big)$.*

Proof. The case $\alpha\big(S(G)\big) = 1$ is impossible, because $S(G)$ is never a complete graph. If $\alpha\big(S(G)\big) = 2$, then $S(G) = K_n + K_n$ and $G \cong K_n$ or $G \cong K_{n-p} + K_p$ for some p, $1 \leqslant p \leqslant n-1$; therefore $\alpha(G) = 1$ or $\alpha(G) = 2$. If $\alpha\big(S(G)\big) \geqslant 3$, let M be an independent set in $S(G)$ with $\alpha\big(S(G)\big)$ elements. Evidently it cannot contain any pair $\big\{x, \pi(x)\big\}$ as a subset, because any other vertex is adjacent to x or to $\pi(x)$. Let $M_1 = M \cap V$, $M_2 = M \cap V'$. Then we have $|M_1| + |M_2| = |M|$ and thus either $2|M_1| \geqslant |M|$, or $2|M_2| \geqslant |M|$. In the former case we have $\alpha\big(S(G)\big) \leqslant 2|M_1| \leqslant 2\alpha(G)$, because M_1 is an independent set in G. In the latter case $\alpha\big(S(G)\big) \leqslant 2|M_2| \leqslant 2\alpha(G') = 2\alpha(G)$, because M_2 is an independent set in G' and $G' \cong G$. The set M is independent also in the graph $R(G, M_1)$ and, as $S(G)$ contains a subgraph isomorphic to $R(G, M_1)$, we have $\alpha\big(R(G, M_1)\big) = \alpha\big(S(G)\big)$. $\square$

An analogous theorem holds for the clique number $c(G)$, i.e. the maximum number of vertices of a clique in G.

Theorem 7. *Let G be a graph. Then $c\big(S(G)\big) \leqslant 2c(G)$ and there exists a graph H such that $(G,H) \in \varrho^*$ and $c(R) = c\big(S(G)\big)$.*

Proof is analogous to the proof of Theorem 6. $\square$

A simple assertion holds for the domination number (the minimum number of vertices of a dominating set) and for the domatic number (the maximum number of dominating sets into which the vertex set can be partitioned).

Lemma 6. *Let G be a graph with n vertices. For the graph $S(G)$ the domination number $\gamma\big(S(G)\big) = 2$, the domatic number is $d\big(S(G)\big) = n$.*

Proof. Evidently no vertex of $S(G)$ is adjacent to all others, therefore $\gamma\big(S(G)\big) \geqslant 2$. Each pair $\{x, \pi(x)\}$ is a dominating set in $S(G)$ and thus $\gamma\big(S(G)\big) = 2$. This implies also $d\big(S(G)\big) = n$. $\square$

At the end we will present a characterization of graphs $S(G)$.

Theorem 8. *Let H be a graph, let N be the partition of $V(H)$ such that two vertices x, y are in the same class of N if and only if $N_H[x] = N_H[y]$. The graph H is isomorphic to the graph $S(G)$ for some connected graph G if and only if the following two conditions are fulfilled:*

(C1) *For each class $C \in N$ there exists exactly one class $\pi(C) \in N$ such that $\pi(C)$ is the set of all vertices of H whose distance from each vertex of C is 3 and $\big|\pi(C)\big| = |C|$.*

(C2) *If $C \in N$, $C' \in N$, $C' \neq \pi(C)$, then either each vertex of C is adjacent to each vertex of C' and each vertex of $\pi(C)$ is adjacent to each vertex of $\pi(C')$, or each vertex of C is adjacent to each vertex of $\pi(C')$ and each vertex of $\pi(C)$ is adjacent to each vertex of C'.*

Proof. If $H \cong S(G)$ for a connected graph G, then (C1) follows from Theorem 1 and (C2) follows directly from the definition of $S(G)$. Now suppose that (C1) and (C2) hold. We shall reconstruct the graph G. We find all classes of N and form all pairs $\big\{C, \pi(C)\big\}$ of them. Choose a partition $\{V, V'\}$ of $V(R)$ such that for each $C \in N$ either $C \subset V$, $\pi(C) \subset V'$, or $C \subset V'$, $\pi(C) \subset V$. Then the subgraph of H induced by the set V is G and $M \cong S(G)$. □

We add one assertion whose proof follows immediately from the preceding results. This assertion shows what makes the graphs $S(G)$ interesting.

Lemma 7. *Let G be a graph with the property that $N_G[x] = N_G[y]$ implies $x = y$ and $N_G[x] \neq V(G) - N_G[y]$ for any two vertices x, y. Then $S(G)$ has the diameter 3 and for each vertex x there exists exactly one vertex $\pi(x)$ such that $d_{S(G)}\big(x, \pi(x)\big) = 3$.*

The graphs with the property that for each vertex x there exists exactly one vertex whose distance from x is equal to the diameter form an interesting class of graphs. Sometimes they are called centrally symmetric graphs.

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