A NOTE ON THE INDEX OF B-FREDHOLM OPERATORS

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Abstract. From Corollary 3.5 in [Berkani, M; Sarih, M.; Studia Math. 148 (2001), 251–257] we know that if S, T are commuting B-Fredholm operators acting on a Banach space X, then ST is a B-Fredholm operator. In this note we show that in general we do not have $\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$, contrarily to what has been announced in Theorem 3.2 in [Berkani, M; Proc. Amer. Math. Soc. 130 (2002), 1717–1723]. However, if there exist $U, V \in L(X)$ such that S, T, U, V are commuting and $US + VT = I$, then $\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$, where ind stands for the index of a B-Fredholm operator.

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1. Index of B-Fredholm operators

B-Fredholm operators were introduced in [1] as a natural generalization of Fredholm operators, and have been extensively studied in [1], [2], [3], [4], [5].

For a bounded linear operator T and a nonnegative integer n define $T_{[n]}$ to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular, $T_{[0]} = T$). If for an integer n the range space $R(T^n)$ is closed and $T_{[n]}$ is a Fredholm operator, then T is called a *B-Fredholm operator*. The index $\operatorname{ind}(T)$ of a B-Fredholm operator T is defined as the index of the Fredholm operator $T_{[n]}$. Thus $\operatorname{ind}(T) = \alpha(T_{[n]}) - \beta(T_{[n]})$, where $\alpha(T_{[n]})$ is the dimension of the kernel $\operatorname{Ker}(T_{[n]})$ of $T_{[n]}$, and $\beta(T_{[n]})$ is the codimension of the range $R(T_{[n]}) = R(T^{n+1})$ of $T_{[n]}$ into $R(T^n)$. By [1, Proposition 2.1] the definition of the index is independent of the integer n.

In [5] the following problem was formulated: If S, T are commuting B-Fredholm operators, then from [5, Corollary 3.5] we know that ST is a B-Fredholm operator. Is it true that $\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$? This question was answered affirmatively in [3, Theorem 3.2]. However, the proof of [5, Theorem 3.2] is incorrect, as the following example shows:

E x a m p l e 1. Let $X = l_2$, and let S, T be operators defined on X by:

$$S(x_1, x_2, \ldots, x_n, \ldots) = (x_1, 0, 0, 0, \ldots, 0, \ldots),\ \forall x = (x_i)_i \in l_2,$$
$$T(x_1, x_2, \ldots, x_n, \ldots) = (x_1, x_3, x_4, x_5, x_6, \ldots),\ \forall x = (x_i)_i \in l_2.$$

Then S is a B-Fredholm operator with index 0, T is a B-Fredholm operator with index 1, but $ST = TS = S$ is a B-Fredholm operator with index 0.

The mistake in the proof of [3, Theorem 3.2] originated in [1, Remark, i)] and was repeated in [3, Remark A, i)] where it is affirmed that if S, T are B-Fredholm operators, $ST = TS$ and $\|T - S\|$ is small, then $\operatorname{ind}(T) = \operatorname{ind}(S)$. But this is not true as shown by the following example:

E x a m p l e 2. Let $X = l_2$, $c > 0$, let S be the operator defined in Example 1 and let T be an operator defined on X by

$$T(x_1, x_2, \ldots, x_n, \ldots) = (x_1, c \cdot x_3, c \cdot x_4, c \cdot x_5, c \cdot x_6, \ldots),\ \forall x = (x_i)_i \in X.$$

Then S is a B-Fredholm operator with index 0, T is a B-Fredholm operator with index 1, $TS = ST = S$, $\|T - S\| = c$. We can choose c arbitrarily small, but the index of S is different from the index of T.

However, by [6, Theorem 4.7], if S, T are B-Fredholm operators, $ST = TS$ and $\|T - S\|$ is small and $S - T$ invertible, then $\operatorname{ind}(T) = \operatorname{ind}(S)$.

Now we give the correct version of [3, Theorem 3.2]

Theorem 1.1. *If S, T, U, V are commuting operators such that $US + VT = I$ and if S, T are B-Fredholm operators, then ST is a B-Fredholm operator and* $\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$.

P r o o f. Since S and T are commuting B-Fredholm operators, then by [5, Corollary 3.5], ST is also a B-Fredholm operator. Therefore there exists an integer n such that $R(S^n)$, $R(T^n)$ and $R((ST)^n)$ are closed and the operators $S_{[n]}$, $T_{[n]}$ and $(ST)_{[n]}$ are Fredholm operators. From [8, Lemma 2.6] we know that $R((ST)^n) = R(S^n) \cap R(T^n)$. Let $\widetilde{T}$ $(\widetilde{S})$ be the restriction of S (T, respectively) to $R((ST)^n)$. Since $(ST)_{[n]} = \widetilde{S}\widetilde{T}$ is a Fredholm operator, hence $\widetilde{S}$ and $\widetilde{T}$ are Fredholm operators and $\operatorname{ind}(ST) = \operatorname{ind}((ST)_{[n]}) = \operatorname{ind}(\widetilde{S}\widetilde{T}) = \operatorname{ind}(\widetilde{S}) + \operatorname{ind}(\widetilde{T})$, where the last equality is a consequence of the properties of Fredholm operators. Let us show that $\operatorname{ind}(S) = \operatorname{ind}(\widetilde{S})$. First we have $\operatorname{Ker}(\widetilde{S}) = \operatorname{Ker}(S) \cap R((ST)^n) = \operatorname{Ker}(S) \cap R(T^n) \cap R(S^n)$. Since $US + VT = I$, we have from [8, Lemma 2.6] that $\operatorname{Ker}(S) \subset R(T^n)$. Hence $\operatorname{Ker}(\widetilde{S}) = \operatorname{Ker}(S) \cap R(S^n)$. So $\alpha(\widetilde{S}) = \alpha(S_{[n]})$.

Similarly we have $R(\widetilde{S}) = R(S^{n+1}T^n)$. Moreover, as can be seen easily, T^n define a natural isomorphism from $R(S^n)/R(S^{n+1})$ onto $R(S^nT^n)/R(S^{n+1}T^n)$. Therefore we have $\beta(\widetilde{S}) = \beta(S_{[n]})$. Consequently, we have $\operatorname{ind}(\widetilde{S}) = \operatorname{ind}(S)$. By the same argument we have $\operatorname{ind}(\widetilde{T}) = \operatorname{ind}(T)$. Since $\operatorname{ind}(ST) = \operatorname{ind}(\widetilde{S}) + \operatorname{ind}(\widetilde{T})$, it follows that $\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$. □

Proposition 1.2. *If T is a B-Fredholm operator and if n is a strictly positive integer, then T^n is a B-Fredholm operator and* $\operatorname{ind}(T^n) = n \cdot \operatorname{ind}(T)$.

Proof. From [5, Corollary 3.5] it follows that T^n is a B-Fredholm operator. Let m be a positive integer such that $R(T^m)$ is closed and $T_{[m]}$ is a Fredholm operator. Then by [1, Proposition 2.1], $R(T^{nm})$ is closed, $T_{[nm]}$ is a Fredholm operator, and $\operatorname{ind}(T) = \operatorname{ind}(T_{[m]}) = \operatorname{ind}(T_{[nm]})$. We have $R((T^n)^m) = R(T^{nm})$ and $(T^n)_{[m]} = (T_{[nm]})^n$. As $(T^n)_{[m]}$ and $T_{[nm]}$ are Fredholm operators, it follows $\operatorname{ind}(T^n) = \operatorname{ind}((T^n)_{[m]}) = \operatorname{ind}((T_{[nm]})^n) = n \cdot \operatorname{ind}((T_{[nm]})) = n \cdot \operatorname{ind}(T)$. So $\operatorname{ind}(T^n) = n \cdot \operatorname{ind}(T)$. □

Corollary 1.3. *Let $P(X) = (X - \lambda_1 I)^{m_1} \dots (X - \lambda_n I)^{m_n}$ be a polynomial with complex coefficients. Assume that for each i, $1 \leqslant i \leqslant n$, $T - \lambda_i I$ is a B-Fredholm operator. Then $P(T) = (T - \lambda_1 I)^{m_1} \dots (T - \lambda_n I)^{m_n}$ is a B-Fredholm operator and* $\operatorname{ind}(P(T)) = \sum\limits_{i=1}^{n} m_i \cdot \operatorname{ind}(T - \lambda_i I)$.

Proof. From [5, Corollary 3.5] we know that $P(T)$ is a B-Fredholm operator. Let $P_1(X) = (X - \lambda_1 I)^{m_1}$ and $P_2(X) = (X - \lambda_2 I)^{m_2} \dots (X - \lambda_n I)^{m_n}$. It is clear that $P_1(X)$ and $P_2(X)$ are prime to each other. Therefore there exist two polynomials $U(X)$, $V(X)$ such that $U(X)P_1(X) + V(X)P_2(X) = 1$. Then we have $P(T) = P_1(T)P_2(T)$ and $U(T)P_1(T) + V(T)P_2(T) = I$. Theorem 1.1 and Proposition 1.2 show that $\operatorname{ind}(P(T)) = m_1 \cdot \operatorname{ind}(T - \lambda_1 I) + \operatorname{ind}(P_2(T))$. By induction it follows that $\operatorname{ind}(P(T)) = \sum\limits_{i=1}^{n} m_i \cdot \operatorname{ind}(T - \lambda_i I)$. □

Theorem 1.4. *Let X be a Hilbert space, T a bounded linear B-Fredholm operator on X. Then the following assertions are equivalent:*

1. *T is Fredholm.*
2. $\operatorname{ind}(TS) = \operatorname{ind}(S) + \operatorname{ind}(T)$ *for each Fredholm operator S on X.*

Proof. $1 \Rightarrow 2$ by [7, Theorem 23.1].

Now we will prove $2 \Rightarrow 1$. Suppose that T is not Fredholm. According to [1, Theorem 2.1] the space X is a direct sum of T-invariant closed subspaces Y, Z such that T/Y is Fredholm and T/Z is nilpotent. Evidently $\operatorname{ind}(T) = \operatorname{ind}(T/Y)$. (Fix a positive integer n such that $T^n = 0$ on Z. Then $T^n(X) = T^n(Y)$ is a subset of Y. Since T/Y is Fredholm, the operator T^n/Y is Fredholm too by [7, Satz 23.2]. Therefore the codimension of $T^n(X) = T^n(Y)$ in Y is finite. Since T/Y is Fredholm and the codimension of $T^n(X)$ in Y is finite, the operator $T/T^n(X)$ is Fredholm and $\operatorname{ind}(T/T^n(X)) = \operatorname{ind}(T/Y)$ by [9, Proposition 3.7.1]. Hence $\operatorname{ind}(T) = \operatorname{ind}(T/T^n(X)) = \operatorname{ind}(T/Y)$.) Since T is not Fredholm the dimension of Z must be infinite. (In the oposite case the operator T should be Fredholm, because T/Y is Fredholm and the codimension of Y is finite (see [9, Proposition 3.7.1])). Since

$T^n = 0$ on Z and the dimension of Z is infinite, the dimension of $Z \cap \operatorname{Ker} T$ is infinite, too. On the Banach space $Z \cap \operatorname{Ker} T$ there is a Fredholm operator A with index 1. (Since the dimension of $Z \cap \operatorname{Ker} T$ is infinite there is an orthonormal sequence $\{x_k\}$ in $Z \cap \operatorname{Ker} T$. Denote by C the closure of the linear span of $\{x_k\}$ and by D the orthogonal complement of C in $Z \cap \operatorname{Ker} T$. If x is an element of $Z \cap \operatorname{Ker} T$ then there is $y \in D$ and a sequence $\{c_k\} \in l_2$ such that $x = y + \sum_k c_k x_k$. Define $Ax = y + \sum_k c_{k+1} x_k$. Then A is a Fredholm operator on $Z \cap \operatorname{Ker} T$ with index 1.) Denote by W the orthogonal complement of $Z \cap \operatorname{Ker} T$ in Z. Since X is the direct sum of Y and Z, the space X is the direct sum of $Y + W$ and $Z \cap \operatorname{Ker} T$. Denote by P the projection of X to $Z \cap \operatorname{Ker} T$ along $Y + W$. Denote $Sx = APx + (I - P)x$. Then $S(Z \cap \operatorname{Ker} T) \subset A(Z \cap \operatorname{Ker} T) \subset Z \cap \operatorname{Ker} T$, $S = A$ on $Z \cap \operatorname{Ker} T$ and $S = I$ on $Y + W$. Hence S is a Fredholm operator of index 1. If $x \in Y + W$ then $TSx = TIx = Tx$. If $x \in Z \cap \operatorname{Ker} T$ then $TSx = TAx = 0 = Tx$, because $Ax \in Z \cap \operatorname{Ker} T$. We thus get $TS = T$ and $\operatorname{ind}(TS) = \operatorname{ind}(T)$ but $\operatorname{ind}(T) + \operatorname{ind}(S) = \operatorname{ind}(T) + 1$. □

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