EXTENDED WEYL TYPE THEOREMS

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Abstract. An operator T acting on a Banach space X possesses property (gw) if $\sigma_a(T)\setminus\sigma_{\mathrm{SBF}_+^-}(T)=E(T)$, where $\sigma_a(T)$ is the approximate point spectrum of T, $\sigma_{\mathrm{SBF}_+^-}(T)$ is the essential semi-B-Fredholm spectrum of T and $E(T)$ is the set of all isolated eigenvalues of T. In this paper we introduce and study two new properties (b) and (gb) in connection with Weyl type theorems, which are analogous respectively to Browder's theorem and generalized Browder's theorem.

Among other, we prove that if T is a bounded linear operator acting on a Banach space X, then property (gw) holds for T if and only if property (gb) holds for T and $E(T)=\Pi(T)$, where $\Pi(T)$ is the set of all poles of the resolvent of T.

Keywords: B-Fredholm operator, Browder's theorem, generalized Browder's theorem, property (b), property (gb)

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1. Introduction

Throughout this paper, X will denote an infinite-dimensional complex Banach space, $L(X)$ the Banach algebra of all bounded linear operators acting on X. For $T\in L(X)$, let T^*, $N(T)$, $R(T)$, $\sigma(T)$ and $\sigma_a(T)$ denote respectively the adjoint, the null space, the range, the spectrum and the approximate point spectrum of T. Let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of T defined by $\alpha(T)=\dim N(T)$ and $\beta(T)=\operatorname{codim} R(T)$. Recall that an operator $T\in L(X)$ is called upper semi-Fredholm if $\alpha(T)<\infty$ and $R(T)$ is closed, while $T\in L(X)$ is called lower semi-Fredholm if $\beta(T)<\infty$. Let $\mathrm{SF}_+(X)$ and $\mathrm{SF}_-(X)$ denote the class of all upper semi-Fredholm operators and the class of all lower semi-Fredholm operators, respectively. If $T\in L(X)$ is either an upper or a lower semi-Fredholm operator, then T is called a

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semi-Fredholm operator, and the index of T is defined by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called a Fredholm operator. Let $F(X)$ denote the class of all Fredholm operators. Define $\mathrm{SF}_+^-(X) = \{T \in \mathrm{SF}_+(X)\colon \operatorname{ind}(T) \leqslant 0\}$. The class of Weyl operators is defined by $W(X) = \{T \in F(X)\colon \operatorname{ind}(T) = 0\}$. The classes of operators defined above generate the following spectra: The Weyl spectrum is defined by $\sigma_{\mathrm{W}}(T) = \{\lambda \in \mathbb{C}\colon T - \lambda I \notin W(X)\}$, while the Weyl essential approximate spectrum is defined by $\sigma_{\mathrm{SF}_+^-}(T) = \{\lambda \in \mathbb{C}\colon T - \lambda I \notin \mathrm{SF}_+^-(X)\}$.

Following Coburn [10], we say that Weyl's theorem holds for $T \in L(X)$ if $\sigma(T) \setminus \sigma_{\mathrm{W}}(T) = E^0(T)$, where $E^0(T) = \{\lambda \in \operatorname{iso}\sigma(T)\colon 0 < \alpha(T - \lambda I) < \infty\}$. Here and elsewhere in this paper, for $A \subset \mathbb{C}$, isoA is the set of isolated points of A. According to Rakocevic [14], an operator $T \in L(X)$ is said to satisfy a-Weyl's theorem if $\sigma_a(T) \setminus \sigma_{\mathrm{SF}_+^-}(T) = E_a^0(T)$, where $E_a^0(T) = \{\lambda \in \operatorname{iso}\sigma_a(T)\colon 0 < \alpha(T - \lambda I) < \infty\}$.

For $T \in L(X)$ and a nonnegative integer n define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular, $T_0 = T$). If for some integer n the range space $R(T^n)$ is closed and T_n is an upper (a lower) semi-Fredholm operator, then T is called an upper (a lower) semi-B-Fredholm operator. In this case the index of T is defined as the index of the semi-B-Fredholm operator T_n, see [9]. Moreover, if T_n is a Fredholm operator, then T is called a B-Fredholm operator. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. An operator T is said to be a B-Weyl operator [5, Definition 1.1] if it is a B-Fredholm operator of index zero. The B-Weyl spectrum $\sigma_{\mathrm{BW}}(T)$ of T is defined by $\sigma_{\mathrm{BW}}(T) = \{\lambda \in \mathbb{C}\colon T - \lambda I \text{ is not a B-Weyl operator}\}$.

Recall that the ascent $a(T)$ of an operator T is defined by $a(T) = \inf\{n \in \mathbb{N}\colon N(T^n) = N(T^{n+1})\}$, and the descent $\delta(T)$ of T is defined by $\delta(T) = \inf\{n \in \mathbb{N}\colon R(T^n) = R(T^{n+1})\}$ with $\inf \emptyset = \infty$. An operator $T \in L(X)$ is called Drazin invertible if it has a finite ascent and descent. The Drazin spectrum $\sigma_{\mathrm{D}}(T)$ of an operator T is defined by $\sigma_{\mathrm{D}}(T) = \{\lambda \in \mathbb{C}\colon T - \lambda I \text{ is not Drazin invertible}\}$.

Define also the set $\mathrm{LD}(X)$ by $\mathrm{LD}(X) = \{T \in L(X)\colon a(T) < \infty \text{ and } R(T^{a(T)+1}) \text{ is closed}\}$ and $\sigma_{\mathrm{LD}}(T) = \{\lambda \in \mathbb{C}\colon T - \lambda I \notin \mathrm{LD}(X)\}$. Following [8], an operator $T \in L(X)$ is said to be left Drazin invertible if $T \in \mathrm{LD}(X)$. We say that $\lambda \in \sigma_a(T)$ is a left pole of T if $T - \lambda I \in \mathrm{LD}(X)$, and that $\lambda \in \sigma_a(T)$ is a left pole of T of finite rank if λ is a left pole of T and $\alpha(T - \lambda I) < \infty$. Let $\Pi_a(T)$ denotes the set of all left poles of T and let $\Pi_a^0(T)$ denotes the set of all left poles of T of finite rank. From [8, Theorem 2.8] it follows that if $T \in L(X)$ is left Drazin invertible, then T is an upper semi-B-Fredholm operator of index less than or equal to 0.

Let $\Pi(T)$ be the set of all poles of the resolvent of T and let $\Pi^0(T)$ be the set of all poles of the resolvent of T of finite rank, that is $\Pi^0(T) = \{\lambda \in \Pi(T)\}\colon \alpha(T - \lambda I) < \infty\}$. According to [12], a complex number λ is a pole of the resolvent of T if and only if $0 < \max\,(a(T - \lambda I), \delta(T - \lambda I)) < \infty$. Moreover, if this is true then

$a(T-\lambda I)=\delta(T-\lambda I)$. According also to [12], the space $R((T-\lambda I)^{a(T-\lambda I)+1})$ is closed for each $\lambda \in \Pi(T)$. Hence we have always $\Pi(T) \subset \Pi_a(T)$ and $\Pi^0(T) \subset \Pi_a^0(T)$.

We say that Browder's theorem holds for $T \in L(X)$ if $\sigma(T) \setminus \sigma_{\mathrm{W}}(T) = \Pi^0(T)$, and that a-Browder's theorem holds for $T \in L(X)$ if $\sigma_a(T) \setminus \sigma_{\mathrm{SF}_+^-}(T) = \Pi_a^0(T)$. Following [5], we say that generalized Weyl's theorem holds for $T \in L(X)$ if $\sigma(T) \setminus \sigma_{\mathrm{BW}}(T) = E(T)$, where $E(T) = \{\lambda \in \operatorname{iso}\sigma(T)\colon 0 < \alpha(T-\lambda I)\}$ is the set of all isolated eigenvalues of T, and that generalized Browder's theorem holds for $T \in L(X)$ if $\sigma(T) \setminus \sigma_{\mathrm{BW}}(T) = \Pi(T)$. It is proved in [2, Theorem 2.1] that generalized Browder's theorem is equivalent to Browder's theorem. In [8, Theorem 3.9], it is shown that an operator satisfying generalized Weyl's theorem satisfies also Weyl's theorem, but the converse does not hold in general. Nonetheless and under the assumption $E(T) = \Pi(T)$, it is proved in [6, Theorem 2.9] that generalized Weyl's theorem is equivalent to Weyl's theorem.

Let $\mathrm{SBF}_+(X)$ be the class of all upper semi-B-Fredholm operators, $\mathrm{SBF}_+^-(X) = \{T \in \mathrm{SBF}_+(X)\colon \operatorname{ind}(T) \leqslant 0\}$. The upper B-Weyl spectrum of T is defined by $\sigma_{\mathrm{SBF}_+^-}(T) = \{\lambda \in \mathbb{C}\colon T-\lambda I \notin \mathrm{SBF}_+^-(X)\}$. We say that generalized a-Weyl's theorem holds for $T \in L(X)$ if $\sigma_a(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T) = E_a(T)$, where $E_a(T) = \{\lambda \in \operatorname{iso}\sigma_a(T)\colon 0 < \alpha(T-\lambda I)\}$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$ and that $T \in L(X)$ obeys generalized a-Browder's theorem if $\sigma_a(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T) = \Pi_a(T)$. It is proved in [2, Theorem 2.2] that generalized a-Browder's theorem is equivalent to a-Browder's theorem, and it is known from [8, Theorem 3.11] that an operator satisfying generalized a-Weyl's theorem satisfies a-Weyl's theorem, but the converse does not hold in general and under the assumption $E_a(T) = \Pi_a(T)$ it is proved in [6, Theorem 2.10] that generalized a-Weyl's theorem is equivalent to a-Weyl's theorem.

Following [15], we say that $T \in L(X)$ possesses property (w) if $\sigma_a(T) \setminus \sigma_{\mathrm{SF}_+^-}(T) = E^0(T)$. The property (w) has been studied in [3], [14]. In [3, Theorem 2.8], it is shown that property (w) implies Weyl's theorem, but the converse is not true in general.

We say that $T \in L(X)$ possesses property (gw) if $\sigma_a(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T) = E(T)$. Property (gw) has been introduced and studied in [1]. Property (gw) extends property (w) to the context of B-Fredholm theory, and it is proved in [1] that an operator possessing property (gw) possesses property (w) but the converse is not true in general.

In this paper we define and study two new properties (b) and (gb) (see Definition 2.1) in connection with Weyl type theorems [8], which play roles analogous to Browder's theorem and generalized Browder's theorem, respectively. We prove in Theorem 2.3 that an operator possessing property (gb) possesses property (b) but the converse is not true in general as shown by Example 2.4, however, under the assumption that $\Pi_a(T) = \Pi(T)$ we prove in Theorem 2.10 that the two properties are equivalent.

We show also in Theorem 2.15 that an operator possessing property (gw) possesses property (gb) and in Theorem 2.13 we show that an operator possessing property (w) possesses property (b), but the converses of those theorems are not true in general. Conditions for the equivalence of properties (gw) and (gb), and properties (w) and (b), are given in Theorem 2.15 and Theorem 2.13, respectively. Precisely we prove that property (gw) holds for $T \in L(X)$ if and only if property (gb) holds for T and $E(T) = \Pi(T)$, and that property (w) holds for $T \in L(X)$ if and only if property (b) holds for T and $E_0(T) = \Pi_0(T)$.

We prove also that an operator $T \in L(X)$ possessing property (gb) satisfies generalized Browder's theorem, and an operator $T \in L(X)$ possessing property (b) satisfies Browder's theorem but the converses do not hold in general.

In the last part, as a conclusion, we give a diagram summarizing the different relations between Weyl type theorems, extending a similar diagram given in [8].

2. Properties (b) and (gb)

For $T \in L(X)$, let $\Delta^g(T) = \sigma(T) \setminus \sigma_{\mathrm{BW}}(T)$, $\Delta_a(T) = \sigma_a(T) \setminus \sigma_{\mathrm{SF}_+^-}(T)$ and $\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T)$.

Definition 2.1. A bounded linear operator $T \in L(X)$ is said to possess property (b) if $\Delta_a(T) = \Pi^0(T)$, and is said to possess property (gb) if $\Delta_a^g(T) = \Pi(T)$.

Lemma 2.2. *Let $T \in L(X)$ be an upper semi-B-Fredholm operator. If $\alpha(T) < \infty$, then T is an upper semi-Fredholm operator.*

Proof. Since $T \in \mathrm{SBF}_+(X)$, there exists an integer n such that $R(T^n)$ is closed and $T_n\colon R(T^n) \to R(T^n)$ is an upper semi-Fredholm operator. Since $\alpha(T) < \infty$, it follows from [16, Lemma 3.3] that $\alpha(T^n) < \infty$. As we know that $R(T^n)$ is closed, hence T^n is an upper semi-Fredholm operator. Thus T is also an upper semi-Fredholm operator. □

Theorem 2.3. *Let $T \in L(X)$. If T possesses property* (gb), *then T possesses property* (b).

Proof. Suppose that T possesses property (gb), then $\Delta_a^g(T) = \Pi(T)$. If $\lambda \in \Delta_a(T)$, then $\lambda \in \Delta_a^g(T) = \Pi(T)$. Hence λ is a pole of the resolvent of T. Since $T - \lambda I \in \mathrm{SF}_+(X)$, hence $\alpha(T - \lambda I)$ is finite, so $\lambda \in \Pi^0(T)$.

Conversely, if $\lambda \in \Pi^0(T)$ then λ is a pole of the resolvent of T and $\alpha(T - \lambda I) < \infty$.

Since T possesses property (gb), we have $\lambda \in \Delta_a^g(T)$, hence $T - \lambda I \in \mathrm{SBF}_+(X)$ and $\operatorname{ind}(T - \lambda I) \leqslant 0$. Since $\alpha(T - \lambda I) < \infty$ we conclude from Lemma 2.2 that $T - \lambda I \in \mathrm{SF}_+(X)$. Thus $\lambda \in \sigma_a(T) \setminus \sigma_{\mathrm{SF}_+^-}(T)$. Finally, we have $\Delta_a(T) = \Pi^0(T)$, and T possesses property (b). $\square$

The converse of Theorem 2.3 does not hold in general as shown by the following example:

Example 2.4. Let $T \in L(\ell^2(\mathbb{N}))$ be the unilateral right shift. It is known from [13, Theorem 3.1] that $\sigma(T) = D$ is the closed unit disc in $\mathbb{C}$, $\sigma_a(T) = C(0,1)$ is the unit circle of $\mathbb{C}$ and T has empty eigenvalues set. Moreover, $\sigma_{\mathrm{SF}_+^-}(T) = C(0,1)$ and $\Pi(T) = \emptyset$. Define S on the Banach space $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $S = 0 \oplus T$. Then $N(S) = \ell^2(\mathbb{N}) \oplus \{0\}$, $\sigma_{\mathrm{SF}_+^-}(S) = \sigma_a(S) = \{0\} \cup C(0,1)$, $\sigma_{\mathrm{SBF}_+^-}(S) = C(0,1)$, $\Pi_a(S) = \{0\}$ and $\Pi(S) = \Pi^0(S) = \emptyset$. Hence $\sigma_a(S) \setminus \sigma_{\mathrm{SF}_+^-}(S) = \Pi^0(S)$ and $\sigma_a(S) \setminus \sigma_{\mathrm{SBF}_+^-}(S) = \{0\} \neq \Pi(S)$. Consequently, S possesses property (b) but does not possess property (gb).

Theorem 2.5. *Let $T \in L(X)$. If T possesses property* (b)*, then T satisfies a-Browder's theorem. In particular, T satisfies Browder's theorem.*

Proof. Suppose that property (b) holds for T. Since $\Pi^0(T) \subset \Pi_a^0(T)$, then $\Delta_a(T) \subset \Pi_a^0(T)$. Conversely if $\lambda \in \Pi_a^0(T)$, then λ is an isolated point of $\sigma_a(T)$, see [8, Remark 2.7]. By [8, Theorem 2.8], we have $\lambda \notin \sigma_{\mathrm{SF}_+^-}(T)$. Consequently $\lambda \in \Delta_a(T)$. Hence $\Delta_a(T) = \Pi_a^0(T)$ and T satisfies a-Browder's theorem. From [11, Theorem 3.9], it follows that T satisfies Browder's theorem. $\square$

The converse of Theorem 2.5 is not true as shown by the following example [3, example 2.14]:

Example 2.6. Let $T \in L(\ell^2(\mathbb{N}))$ be the unilateral right shift and $S \in L(\ell^2(\mathbb{N}))$ the operator defined by $S(\xi_1, \xi_2, \xi_3, \ldots) = (0, \xi_2, \xi_3, \xi_4, \ldots)$.

Consider the operator $R = T \oplus S$, then $\sigma(R) = D(0,1)$ is the closed unit disc in $\mathbb{C}$, $\operatorname{iso}\sigma(R) = \emptyset$ and $\sigma_a(R) = C(0,1) \cup \{0\}$, where $C(0,1)$ is the unit circle of $\mathbb{C}$, and $\sigma_{\mathrm{SF}_+^-}(R) = C(0,1)$. This implies that $\sigma_a(R) \setminus \sigma_{\mathrm{SF}_+^-}(R) = \{0\}$, $\Pi^0(R) = \emptyset$ and $E_a^0(R) = \{0\}$. Hence R satisfies a-Weyl's theorem. It follows from [8, Corollary 3.5] that R satisfies a-Browder's theorem and so it satisfies also Browder's theorem, but R does not possess property (b) because $\sigma_a(R) \setminus \sigma_{\mathrm{SF}_+^-}(R) \neq \Pi^0(R)$.

However, from Theorem 2.5 we have immediately the following result:

Corollary 2.7. *Let $T \in L(X)$. Then T possesses property* (b) *if and only if T satisfies a-Browder's theorem and $\Pi^0(T) = \Pi_a^0(T)$.*

Corollary 2.8. *Let $T \in L(X)$. If T possesses property* (gb), *then T satisfies generalized a-Browder's theorem. In particular, T satisfies generalized Browder's theorem.*

Proof. Assume that T possesses property (gb). From Theorem 2.3, T possesses property (b), and by Theorem 2.5, T satisfies a-Browder's theorem. Since a-Browder's theorem is equivalent to generalized a-Browder's theorem, see [2, Theorem 2.2], T satisfies generalized a-Browder's theorem, too. Hence, T satisfies generalized Browder's theorem by [8, Theorem 3.8]. $\square$

The converse of the preceding corollary does not hold in general. Indeed, if we consider the operator R defined in Example 2.6, then $\sigma_{\mathrm{SBF}_+^-}(R) = C(0,1)$, $E_a(R) = \{0\}$ and $\Pi(R) = \emptyset$. This implies that $\sigma_a(R) \setminus \sigma_{\mathrm{SBF}_+^-}(R) = E_a(R)$ and R satisfies generalized a-Weyl's theorem. By [8, Corollary 3.3], R satisfies generalized a-Browder's theorem and so R satisfies generalized Browder's theorem. But R does not possess property (gb) because $\sigma_a(R) \setminus \sigma_{\mathrm{SBF}_+^-}(R) \neq \Pi(R)$.

However, we have the following result:

Corollary 2.9. *Let $T \in L(X)$. Then T possesses property* (gb) *if and only if T satisfies generalized a-Browder's theorem and $\Pi(T) = \Pi_a(T)$.*

Proof. Assume that property (gb) holds for T, i.e. $\Delta_a^g(T) = \Pi(T)$. Then by Corollary 2.8, T satisfies generalized a-Browder's theorem, i.e. $\Delta_a^g(T) = \Pi_a(T)$. Hence $\Pi(T) = \Pi_a(T)$.

Conversely, assume that T satisfies generalized a-Browder's theorem and $\Pi(T) = \Pi_a(T)$, then $\Delta_a^g(T) = \Pi_a(T)$ and $\Pi(T) = \Pi_a(T)$, which implies that $\Delta_a^g(T) = \Pi(T)$ and T possesses property (gb). $\square$

Theorem 2.10. *Let $T \in L(X)$. The following statements are equivalent:*

(i) *T possesses property* (gb);

(ii) *T possesses property* (b) *and $\Pi(T) = \Pi_a(T)$.*

Proof. Assume that property (gb) holds for T, then property (b) holds for T and $\Pi(T) = \Pi_a(T)$.

Conversely, assume that property (b) holds for T and $\Pi(T) = \Pi_a(T)$. From Corollary 2.7, T satisfies a-Browder's theorem. As we know from [2, Theorem 2.2] that a-Browder's theorem is equivalent to generalized a-Browder's theorem, it follows that T satisfies generalized a-Browder's theorem. Hence we have $\Delta_a^g(T) = \Pi_a(T)$. As by assumption $\Pi_a(T) = \Pi(T)$, we have $\Delta_a^g(T) = \Pi(T)$. Therefore T possesses property (gb). $\square$

In the next theorem we give a characterization of operators possssing property (b).

Theorem 2.11. *Let $T \in L(X)$. Then T possesses property* (b) *if and only if*

(i) *T satisfies Browder's theorem;*

(ii) $\operatorname{ind}(T - \lambda I) = 0$ *for all* $\lambda \in \Delta_a(T)$.

P r o o f. Suppose that T possesses property (b). Then by Theorem 2.5, T satisfies Browder's theorem, that is $\sigma(T) \setminus \sigma_{\mathrm{W}}(T) = \Pi^0(T)$. Let $\lambda \in \Delta_a(T)$, then $\lambda \in \Pi^0(T)$, so $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{W}}(T)$. Therefore $T - \lambda I$ is a Weyl operator.

Conversely, assume that T satisfies Browder's theorem and $\operatorname{ind}(T - \lambda I) = 0$ for all $\lambda \in \Delta_a(T)$. If $\lambda \in \Delta_a(T)$, then $T - \lambda I$ is upper semi-Fredholm such that $\operatorname{ind}(T - \lambda I) = 0$. Hence $T - \lambda I$ is a Weyl operator. Since T satisfies Browder's theorem, we have $\lambda \in \Pi^0(T)$. On the other hand, if $\lambda \in \Pi^0(T)$, then $T - \lambda I$ is a Weyl operator and $\lambda \in \sigma_a(T)$, so $\lambda \in \Delta_a(T)$. Consequently, $\Delta_a(T) = \Pi^0(T)$ and T possesses property (b). $\square$

In the next theorem we prove a similar characterization for the property (gb).

Theorem 2.12. *Let $T \in L(X)$. Then T possesses property* (gb) *if and only if*

(i) *T satisfies generalized Browder's theorem;*

(ii) $\operatorname{ind}(T - \lambda I) = 0$ *for all* $\lambda \in \Delta_a^g(T)$.

P r o o f. Suppose that T possesses property (gb), then by Corollary 2.8, T satisfies generalized Browder's theorem.

If $\lambda \in \Delta_a^g(T)$, as T possesses property (gb), then $\lambda \in \Pi(T)$. Thus λ is isolated in $\sigma(T)$. From [7, Theorem 4.2] it follows that $T - \lambda I$ is a B-Fredholm operator and $\operatorname{ind}(T - \lambda I) = 0$. Conversely, assume that generalized Browder's theorem holds for T and $\operatorname{ind}(T - \lambda I) = 0$ for all $\lambda \in \Delta_a^g(T)$. If $\lambda \in \Delta_a^g(T)$, then $T - \lambda I$ is an upper semi-B-Fredholm operator such that $\operatorname{ind}(T - \lambda I) = 0$. Hence $\lambda \in \Delta^g(T)$. Since T satisfies generalized Browder's theorem, we have $\lambda \in \Pi(T)$. On the other hand, if $\lambda \in \Pi(T)$, then $T - \lambda I$ is a B-Weyl operator and so $\lambda \in \Delta_a^g(T)$. Finally, $\Delta_a^g(T) = \Pi(T)$ and T possesses property (gb). $\square$

Now we give conditions for the equivalence of property (b) and property (w).

Theorem 2.13. *Let $T \in L(X)$. Then the following statements are equivalent:*

(i) *T possesses property* (w);

(ii) *T possesses property* (b) *and* $E^0(T) = \Pi^0(T)$;

(iii) *T possesses property* (b) *and* $E^0(T) = \Pi_a^0(T)$.

P r o o f. (i) $\Rightarrow$ (ii) Suppose that T possesses property (w). Then by [3, Theorem 2.9], T satisfies Weyl's theorem and $\operatorname{ind}(T - \lambda I) = 0$ for all $\lambda \in \Delta_a(T)$. Consequently, by virtue of [4, Corollary 5] T satisfies Browder's theorem and $\operatorname{ind}(T - \lambda I) =$

0 for all $\lambda \in \Delta_a(T)$. From Theorem 2.11 it follows that T possesses property (b). As T satisfies Weyl's theorem, we conclude that $E^0(T) = \Pi^0(T)$.

(ii) $\Rightarrow$ (iii) Follows directly from Corollary 2.7.

(iii) $\Rightarrow$ (i) Assume that T possesses property (b) and $E^0(T) = \Pi^0_a(T)$. Then from Corollary 2.7 we have $\Pi^0(T) = \Pi^0_a(T)$. Hence $\Delta_a(T) = E^0(T)$, and so T possesses property (w). $\square$

From Theorem 2.13, if $T \in L(X)$ possesses property (w), then T possesses property (b). But the converse is not true in general as shown by the following example:

Example 2.14. Let $T \in L(\ell^2(\mathbb{N}))$ be defined by $T(x_1, x_2, x_3, \ldots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \ldots)$.

Then property (b) holds for T because $\sigma_a(T) = \sigma_{\mathrm{SF}_+^-}(T) = \{0\}$ and $\Pi^0(T) = \emptyset$, while property (w) does not hold for T because $E^0(T) = \{0\}$.

Similarly to Theorem 2.13, we give conditions for the equivalence of property (gb) and property (gw).

Theorem 2.15. *Let $T \in L(X)$. Then the following assertions are equivalent:*

(i) *T possesses property* (gw)*;*

(ii) *T possesses property* (gb) *and $E(T) = \Pi(T)$;*

(iii) *T possesses property* (gb) *and $E(T) = \Pi_a(T)$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that T possesses property (gw). Then by [1, Theorem 2.4], T satisfies generalized Weyl's theorem and $\operatorname{ind}(T - \lambda I) = 0$ for all $\lambda \in \Delta^g_a(T)$. Consequently, by [5, Corollary 2.6] T satisfies generalized Browder's theorem and $\operatorname{ind}(T - \lambda I) = 0$ for all $\lambda \in \Delta^g_a(T)$. Theorem 2.12, implies that T possesses property (gb). As T satisfies generalized Weyl's theorem, $E(T) = \Pi(T)$.

(ii) $\Rightarrow$ (iii) Follows directly from Corollary 2.9.

(iii) $\Rightarrow$ (i) Assume that T possesses property (gb) and $E(T) = \Pi_a(T)$. Then from Corollary 2.9 we have $\Pi(T) = \Pi_a(T)$. Hence $\Delta^g_a(T) = E(T)$, and so T possesses property (gw). $\square$

From Theorem 2.15, if $T \in L(X)$ possesses property (gw), then T possesses property (gb). But the converse does not hold in general as shown by the following example [8, Example 3.12]:

Example 2.16. Let Q be defined for each $x = (\xi_i) \in \ell^1$ by

$$Q(\xi_1, \xi_2, \xi_3, \ldots, \xi_k, \ldots) = (0, \alpha_1\xi_1, \alpha_2\xi_2, \ldots, \alpha_{k-1}\xi_{k-1}, \ldots),$$

where (α_i) is a sequence of complex numbers such that $0 < |\alpha_i| \leqslant 1$ and $\sum\limits_{i=1}^{\infty} |\alpha_i| < \infty$.

Define T on $X = \ell^1 \oplus \ell^1$ by $T = Q \oplus 0$. Then $\sigma(T) = \sigma_a(T) = \{0\}$, $E(T) = \{0\}$. It follows from [8, Example 3.12] that $R(T^n)$ is not closed for any $n \in \mathbb{N}$. This implies that $\sigma_{\mathrm{SBF}_+^-}(T) = \{0\}$ and $\Pi(T) = \emptyset$. We then have $\sigma_a(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T) \neq E(T)$, $\sigma_a(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T) = \Pi(T)$. Hence T possesses property (gb), but T does not possess property (gw).

3. Conclusion

In this last part, we give a summary of the known Weyl type theorems as in [8], including the properties introduced in [15], [1] and in this paper. We use the abbreviations gaW, aW, gW, W, (gw) and (w) to signify that an operator $T \in L(X)$ obeys generalized a-Weyl's theorem, a-Weyl's theorem, generalized Weyl's theorem, Weyl's theorem, property (gw) and property (w). Similarly, the abbreviations gaB, aB, gB, B, (gb) and (b) have analogous meaning with respect to Browder's theorem or the new properties introduced in this paper.

The following table summarizes the meaning of various theorems and properties.

gaW	$\sigma_a(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T) = E_a(T)$	gaB	$\sigma_a(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T) = \Pi_a(T)$
aW	$\sigma_a(T) \setminus \sigma_{\mathrm{SF}_+^-}(T) = E_a^0(T)$	aB	$\sigma_a(T) \setminus \sigma_{\mathrm{SF}_+^-}(T) = \Pi_a^0(T)$
gW	$\sigma(T) \setminus \sigma_{\mathrm{BW}}(T) = E(T)$	gB	$\sigma(T) \setminus \sigma_{\mathrm{BW}}(T) = \Pi(T)$
W	$\sigma(T) \setminus \sigma_{\mathrm{W}}(T) = E^0(T)$	B	$\sigma(T) \setminus \sigma_{\mathrm{W}}(T) = \Pi^0(T)$
(gw)	$\sigma_a(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T) = E(T)$	(gb)	$\sigma_a(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T) = \Pi(T)$
(w)	$\sigma_a(T) \setminus \sigma_{\mathrm{SF}_+^-}(T) = E^0(T)$	(b)	$\sigma_a(T) \setminus \sigma_{\mathrm{SF}_+^-}(T) = \Pi^0(T)$

Table

In the following diagram, which extends the similar diagram presented in [8], arrows signify implications between various Weyl type theorems, Browder type theorems, property (gw) and property (gb). The numbers near the arrows are references to the results in the present paper (numbers without brackets) or to the bibliography therein (the numbers in square brackets).

$$
\begin{tikzcd}
\mathrm{gW} & \mathrm{gaB} & & & & & \\
\mathrm{gw} \arrow[u, "{[1]}"'] \arrow[r, "2.15"] \arrow[d, "{[1]}"'] & \mathrm{gb} \arrow[u, "2.8"'] \arrow[r, "2.8"] \arrow[d, "2.3"'] & \mathrm{gB} \arrow[d, "{[8]}"'] & \mathrm{gW} \arrow[l, "{[8]}"'] \arrow[d, "{[8]}"'] & \mathrm{gaW} \arrow[l, "{[8]}"'] \arrow[r, "{[8]}"] \arrow[d, "{[8]}"'] & \mathrm{gaB} \arrow[r, "{[8]}"] \arrow[d, Leftrightarrow, "{[2]}"'] & \mathrm{gB} \arrow[d, Leftrightarrow, "{[2]}"'] \\
\mathrm{w} \arrow[r, "2.13"'] \arrow[d, "{[3]}"'] & \mathrm{b} \arrow[r, "2.5"'] \arrow[d, "2.5"'] & \mathrm{B} & \mathrm{W} \arrow[l, "{[4]}"] & \mathrm{aW} \arrow[l, "{[14]}"] \arrow[r, "{[8]}"'] & \mathrm{aB} \arrow[r, "{[11]}"'] & \mathrm{B} \\
\mathrm{W} & \mathrm{aB} & & & & &
\end{tikzcd}
$$

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