REALIZABLE TRIPLES FOR STRATIFIED DOMINATION IN GRAPHS

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Abstract. A graph is 2-stratified if its vertex set is partitioned into two classes, where the vertices in one class are colored red and those in the other class are colored blue. Let F be a 2-stratified graph rooted at some blue vertex v. An F-coloring of a graph G is a red-blue coloring of the vertices of G in which every blue vertex v belongs to a copy of F rooted at v. The F-domination number $\gamma_F(G)$ is the minimum number of red vertices in an F-coloring of G. In this paper, we study F-domination where F is a red-blue-blue path of order 3 rooted at a blue end-vertex. It is shown that a triple $(\mathcal{A},\mathcal{B},\mathcal{C})$ of positive integers with $\mathcal{A} \leqslant \mathcal{B} \leqslant 2\mathcal{A}$ and $\mathcal{B} \geqslant 2$ is realizable as the domination number, open domination number, and F-domination number, respectively, for some connected graph if and only if $(\mathcal{A},\mathcal{B},\mathcal{C}) \neq (k,k,\mathcal{C})$ for any integers k and $\mathcal{C}$ with $\mathcal{C} > k \geqslant 2$.

Keywords: stratified graph, F-domination, domination, open domination

MSC 2000: 05C15, 05C69

1. Introduction

An area of graph theory that has received considerable attention in recent decades is domination. Although initiated by Berge [1] and Ore [9] in 1958 and 1962, respectively, it was a paper by Cockayne and Hedetniemi [5] in 1977 that began the popularity of the subject and has led to a theory. This subject is based on a very simple definition: A vertex v *dominates* a vertex u in a graph G if either $u = v$ or u is adjacent to v. Over the years a large number of variations of domination have surfaced. Each type of domination is based on a condition under which a vertex v dominates a vertex u in a graph G. As with standard domination, many definitions of domination state that a vertex v dominates a vertex u in a graph G if either $u = v$ or u satisfies some condition involving v. Then there are those definitions of domination that state a vertex v dominates a vertex u not if $u = v$ but if u satisfies some condition involving v. The simplest example of this is *total* or *open domination* where v dominates u if u is adjacent to v. An advantage of the former type of domination is that every graph G contains a set of vertices (called a *dominating set*) such that every vertex of G is dominated by some vertex of S; while this is not necessarily the case for the latter type of domination. For example, graphs with isolated vertices contain no *open dominating sets*.

In 1999 a new way of looking at domination was introduced in [3] that encompassed several of the best known domination parameters defined earlier (including standard domination and open domination). This gave rise to an infinite class of domination parameters, each of which is defined for every graph. This new view of domination was based on a simple but fundamental idea introduced by Rashidi [10] in 1994. A graph whose vertex set $V(G)$ is partitioned is a *stratified graph*. If $V(G)$ is partitioned into k subsets, then G is *k-stratified*. In particular, the vertex set of a 2-stratified graph is partitioned into two subsets. Typically, the vertices of one subset in a 2-stratified graph are considered to be colored red and those in the other subset are colored blue. A *red-blue coloring* of a graph G is an assignment of colors to the vertices of G, where each vertex is colored either red or blue. In a red-blue coloring, however, all vertices of G may be colored the same. A red-blue coloring in which at least one vertex is colored red and at least one vertex is colored blue and thereby produces a 2-stratification of G.

We now describe how domination was defined in [3] with the aid of stratification. Let F be a 2-stratified graph in which some blue vertex r is designated as the "root" of F. Thus F is said to *be rooted at r*. Since F is 2-stratified, necessarily F contains at least two vertices, at least one of which is colored red and at least one of which is colored blue. Of course, the root r is blue but there may be other blue vertices in F. Now let G be a graph. By an *F-coloring* of a graph G, we mean a red-blue coloring of G such that for every blue vertex u of G, there is a copy of F in G with r at u. Therefore, every blue vertex u of G belongs to a copy F' of F rooted at u. A red vertex v in G is said to F-dominate a vertex u if $u = v$ or there exists a copy F' of F rooted at u and containing the red vertex v. The set S of red vertices in a red-blue coloring of G is an *F-dominating set* of G if every vertex of G is F-dominated by some vertex of S, that is, this red-blue coloring of G is an F-coloring. The minimum number of red vertices in an F-dominating set is called the *F-domination number* $\gamma_F(G)$ of G. An F-dominating set with $\gamma_F(G)$ vertices is a *minimum F-dominating set*. The *F-domination number* of every graph G is defined since $V(G)$ is an F-dominating set.

To illustrate these concepts, consider the three 2-stratified graphs H_1, H_2, and H_3 and the graph G of Figure 1, where solid vertices denote red vertices and open vertices denote blue vertices. Each of the 2-stratified graphs H_1, H_2, and H_3 has the same 2-stratification of the path P_4 of order 4 but is rooted at a different blue vertex. A minimum H_i-dominating set of G with exactly i red vertices is also shown in that figure for $i = 1, 2, 3$. Therefore, $\gamma_{H_i}(G) = i$ for $i = 1, 2, 3$. We refer to the books [4], [7] for graph theory notation and terminology not described in this paper.

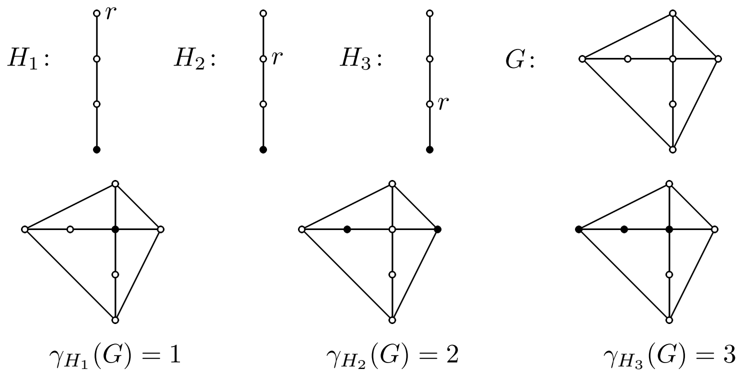

Figure 1. A minimum H_i-dominating set ($i = 1, 2, 3$) for a graph G.

2. F_3-Domination in graphs

For a graph G, the *domination number* $\gamma(G)$ of G is the minimum number of vertices in a dominating set for G. A dominating set of cardinality $\gamma(G)$ is called a *minimum dominating set*. The minimum cardinality of an open dominating set is the *open domination number* $\gamma_o(G)$ of G. An open dominating set of cardinality $\gamma_o(G)$ is a *minimum open dominating set* for G. There are five possible choices for the 2-stratified P_3 rooted at a blue vertex v shown in Figure 2. It was shown in [3] that if G is a connected graph of order at least 3, then $\gamma_{F_1}(G) = \gamma(G)$, $\gamma_{F_2}(G) = \gamma_o(G)$, $\gamma_{F_4}(G) = \gamma_r(G)$, and $\gamma_{F_5}(G) = \gamma_2(G)$, where $\gamma(G)$ is the domination number, $\gamma_o(G)$ is the open domination number, $\gamma_r(G)$ is the restrained domination number and $\gamma_2(G)$ is the 2-domination number (see [7, 8]). The parameter γ_{F_3} is new and has been studied in [6]. In this work, we continue the study of F_3-domination.

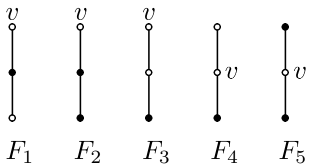

Figure 2. The five 2-stratified graphs P_3.

For simplification, we write $F = F_3$ unless otherwise stated. Since the 2-stratified graph F contains exactly one red vertex, $1 \leqslant \gamma_F(G) \leqslant n$ for every connected graph G of order n. The following result was presented in [6].

Theorem 2.1. *Let G be a connected graph of order $n \geqslant 3$. Then $\gamma_F(G) = n$ if and only if $G = K_{1,n-1}$, and $\gamma_F(G) = 1$ if and only if G contains a vertex whose neighborhood is an open dominating set of G. If G is a bipartite graph, then $\gamma_F(G) \geqslant 2$. In particular, if T is a tree, then $\gamma_F(T) = 2$ if and only if T is a double star.*

For every nontrivial connected graph G, $\gamma(G) \leqslant \gamma_o(G)$. Other than this requirement, there is no other restriction on the relative values of $\gamma(G)$, $\gamma_o(G)$, and $\gamma_F(G)$. That is, it is possible that (i) $\gamma_F(G) \leqslant \gamma(G) \leqslant \gamma_o(G)$, (ii) $\gamma(G) \leqslant \gamma_o(G) \leqslant \gamma_F(G)$, and (iii) $\gamma(G) \leqslant \gamma_F(G) \leqslant \gamma_o(G)$. This gives rise to the following natural question.

Problem 2.2. *For which triples $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of positive integers, does there exist a connected graph G such that $\gamma(G) = \mathcal{A}$, $\gamma_o(G) = \mathcal{B}$, and $\gamma_F(G) = \mathcal{C}$?*

Since $\gamma(G) \leqslant \gamma_o(G) \leqslant 2\gamma(G)$ and $\gamma_o(G) \geqslant 2$ for every nontrivial connected graph G, no triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of positive integers with $\mathcal{A} > \mathcal{B}$, $\mathcal{B} > 2\mathcal{A}$, or $\mathcal{B} = 1$ can be realized, respectively, as the domination number, the open domination number, and the F-domination number of any connected graph. For this reason, by a *triple*, we mean an ordered triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of positive integers with $\mathcal{A} \leqslant \mathcal{B} \leqslant 2\mathcal{A}$ and $\mathcal{B} \geqslant 2$. We define a triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ to be *realizable* if there exists a connected graph G such that $\gamma(G) = \mathcal{A}$, $\gamma_o(G) = \mathcal{B}$, and $\gamma_F(G) = \mathcal{C}$. Observe that $\gamma(K_3) = 1$, $\gamma_o(K_3) = 2$, and $\gamma_F(K_3) = 1$. For $\mathcal{C} \geqslant 2$, $\gamma(K_{1,\mathcal{C}-1}) = 1$, $\gamma_o(K_{1,\mathcal{C}-1}) = 2$, and $\gamma_F(K_{1,\mathcal{C}-1}) = \mathcal{C}$. Therefore, we have the following.

Observation 2.3. *Every triple $(1, 2, \mathcal{C})$ is realizable.*

In [6] the existence of graphs G was investigated for which $\gamma_F(G) = 1$ and $\gamma(G)$ and $\gamma_o(G)$ could have a wide variety of values. Also, the existence of graphs G was studied for which $\gamma(G) = \gamma_F(G) = \gamma_o(G) = k$ for various values of k. In particular, the following two results were obtained.

Theorem 2.4. *For each pair $\mathcal{A}, \mathcal{B}$ of integers with $1 \leqslant \mathcal{A} \leqslant \mathcal{B} \leqslant 2\mathcal{A}$ and $\mathcal{B} \geqslant 2$, there exists a connected graph G with $\gamma_F(G) = 1$ such that $\gamma(G) = \mathcal{A}$ and $\gamma_o(G) = \mathcal{B}$.*

Theorem 2.5. *For each integer $k \geqslant 2$, there exists a connected graph G such that $\gamma(G) = \gamma_F(G) = \gamma_o(G) = k$.*

Theorems 2.4 and 2.5 now have two immediate corollaries.

Corollary 2.6. *Every triple* $(\mathcal{A}, \mathcal{B}, 1)$ *is realizable.*

Corollary 2.7. *Every triple* (k, k, k) *is realizable for each integer* $k \geqslant 2$.

Not every triple is realizable, however. In order to show this, the following lemma from [6] is useful.

Lemma 2.8. *Let* G *be a connected graph of order at least* 3. *If* H *is a connected subgraph of* G, *then*

$$\gamma_F(G) + |V(H)| \leqslant |V(G)| + \gamma_F(H).$$

In particular, if H *is a spanning subgraph of* G, *then* $\gamma_F(G) \leqslant \gamma_F(H)$.

Proposition 2.9. *Let* $k \geqslant 2$ *be an integer. If* G *is a connected graph with* $\gamma(G) = \gamma_o(G) = k$, *then* $\gamma_F(G) \leqslant k$ *and so no triple* $(k, k, \mathcal{C})$ *is realizable for* $\mathcal{C} > k$.

Proof. Let G be a connected graph with $\gamma(G) = \gamma_o(G) = k$ and let S be a minimum open dominating set of G. Necessarily S is also a minimum dominating set. Let $v_1 \in S$. Since S is a minimum dominating set and G is connected, there exists $u_1 \notin S$ such that u_1 is dominated by v_1. Since S is a minimum dominating set, there is $u_2 \notin S$ that is not dominated by v_1. Let $v_2 \in S$ such that v_2 dominates u_2. If $k \geqslant 3$, then there is $u_3 \notin S$ that is not dominated by any vertex in $\{v_1, v_2\}$. Let $v_3 \in S$ such that v_3 dominates u_3. Continuing in this manner, we arrive at the set $U = \{u_1, u_2, \ldots, u_k\}$. We claim that U is an F-dominating set of G. Let $x \in V(G)$. If $x = u_i$ for $1 \leqslant i \leqslant k$, then x is F-dominated by itself. If $x = v_i$ for some i $(1 \leqslant i \leqslant k)$, then since S is a minimum open dominating set of G, there is a $v_j \in S$ that is adjacent to v_i. Then v_i is F-dominated by u_j. Otherwise, $x \notin U \cup S$. Since S is a dominating set, x is adjacent to some vertex v_i $(1 \leqslant i \leqslant k)$. Then x is F-dominated by u_i. Thus, $\gamma_F(G) \leqslant |U| = k$. Therefore, $(k, k, \mathcal{C})$ is nonrealizable for any $\mathcal{C} > k$. □

3. Which triples are realizable?

As we have seen, there are infinitely many realizable triples and infinitely many nonrealizable triples. We now investigate the problem of determining which triples are realizable. To simplify the notation, we classify triples into the following three categories:

A triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is of *type I* if $\mathcal{C} \leqslant \mathcal{A} \leqslant \mathcal{B}$;

A triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is of *type II* if $\mathcal{A} \leqslant \mathcal{B} \leqslant \mathcal{C}$;

A triple $(\mathcal{A},\mathcal{B},\mathcal{C})$ is of *type III* if $\mathcal{A} \leqslant \mathcal{C} \leqslant \mathcal{B}$.
Some additional information and notation from [6] will be useful to us.

Lemma 3.1. *Let v be an end-vertex of a connected graph G that is adjacent to the vertex u. Furthermore, let c be an F-coloring of G. Then v is colored red by c if either of the following two conditions are satisfied:* (1) $\deg u = 2$, (2) *u is colored red by c.*

For positive integers i, j, and t, define the graph J_i to be a copy of H_1 in Figure 3, where $V(J_i) = \{u_{i,0}, u_{i,1}, u_{i,2}, \ldots, u_{i,6}\}$ such that $u_{i,p}$ corresponds to u_p in H_1 for $0 \leqslant p \leqslant 6$; define the graph G_j to be a copy of H_2 in Figure 3 where $V(G_j) = \{v_{j,0}, v_{j,1}, v_{j,2}, v_{j,3}\}$ such that $v_{j,q}$ corresponds to v_q in H_2 for $0 \leqslant q \leqslant 3$; and define the graph I_t to be a copy of H_2 in Figure 3, where $V(I_t) = \{w_{t,0}, w_{t,1}, w_{t,2}, w_{t,3}\}$ such that $w_{t,q}$ corresponds to v_q in H_2 for $0 \leqslant q \leqslant 3$.

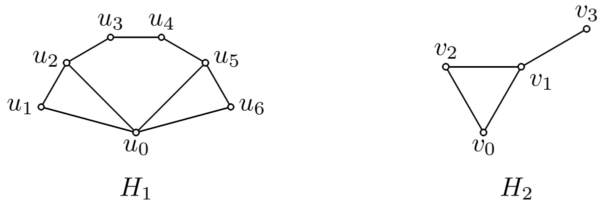

Figure 3. The graphs H_1 and H_2.

3.1 Realizable triples of type I. In this section, we show that every triple of type I is realizable.

Theorem 3.2. *Every triple $(\mathcal{A},\mathcal{B},\mathcal{C})$ of type* I *is realizable.*

P r o o f. By Corollaries 2.7 and 2.6, the result holds for $\mathcal{C} = \mathcal{A} = \mathcal{B}$ or $\mathcal{C} = 1$. Thus it suffices to consider three cases, according to whether $2 \leqslant \mathcal{C} < \mathcal{A} < \mathcal{B} \leqslant 2\mathcal{A}$, $2 \leqslant \mathcal{C} < \mathcal{A} = \mathcal{B}$, or $2 \leqslant \mathcal{C} = \mathcal{A} < \mathcal{B} \leqslant 2\mathcal{A}$. We will only prove the first case in detail.

C a s e I. $2 \leqslant \mathcal{C} < \mathcal{A} < \mathcal{B} \leqslant 2\mathcal{A}$. Let $\mathcal{A} = \mathcal{C} + k$ and $\mathcal{B} = \mathcal{C} + l$. Since $\mathcal{C} < \mathcal{A} < \mathcal{B} \leqslant 2\mathcal{A}$, it follows that $1 \leqslant k < l \leqslant \mathcal{C} + 2k$. We consider three cases, according to whether $k < l < 2k$, $2k \leqslant l < \mathcal{C} + 2k$, or $l = \mathcal{C} + 2k$.

C a s e 1. $k < l < 2k$. Let G be the graph obtained from the graphs J_i, G_j and I_t ($1 \leqslant i \leqslant l-k$, $1 \leqslant j \leqslant 2k-l$, and $1 \leqslant t \leqslant \mathcal{C}-1$) by identifying all vertices $u_{i,0}$, $v_{j,0}$ and $w_{t,3}$ and labeling the identified vertex v.

We first show that $\gamma_F(G) = \mathcal{C}$. Since $\{v\} \cup \{w_{t,2}\colon 1 \leqslant t \leqslant \mathcal{C}-1\}$ is an F-dominating set, $\gamma_F(G) \leqslant \mathcal{C}$. On the other hand, let c be a minimum F-coloring of

G. If $v \in R_c$, then $w_{t,1}$ can be F-dominated only by some vertex in $V(I_t) - \{v\}$ for $1 \leqslant t \leqslant \mathcal{C} - 1$. This implies that R_c contains at least one vertex from each set $V(I_t) - \{v\}$ for $1 \leqslant t \leqslant \mathcal{C} - 1$. Hence $\gamma_F(G) = |R_c| \geqslant 1 + (\mathcal{C} - 1) = \mathcal{C}$. Thus, we may assume $v \notin R_c$. Since $w_{t,2}$ must be F-dominated by a vertex in $V(I_t) - \{v\}$ for $1 \leqslant t \leqslant \mathcal{C} - 1$ and $u_{i,3}$ is only F-dominated by a vertex in $V(J_i) - \{v\}$ for $1 \leqslant i \leqslant l - k$, it follows that

$$\gamma_F(G) = |R_c| \geqslant (\mathcal{C} - 1) + (l - k) \geqslant \mathcal{C},$$

and so $\gamma_F(G) = \mathcal{C}$. Furthermore, observe that

$$S = \{v\} \cup \{u_{i,3}\colon 1 \leqslant i \leqslant l - k\} \cup \{v_{j,1}\colon 1 \leqslant j \leqslant 2k - l\} \cup \{w_{t,1}\colon 1 \leqslant t \leqslant \mathcal{C} - 1\}$$

is a minimum dominating set of G and $S \cup \{u_{i,2}\colon 1 \leqslant i \leqslant l - k\}$ is a minimum open dominating set of G. Therefore, $\gamma(G) = \mathcal{A}$ and $\gamma_o(G) = \mathcal{B}$.

Case 2. $2k \leqslant l < \mathcal{C} + 2k$. Let G be the graph obtained from the graphs J_i and G_j for $1 \leqslant i \leqslant l - k$ and $1 \leqslant j \leqslant \mathcal{C} + 2k - l - 1$ by (1) identifying all vertices $u_{i,0}$ and $v_{j,0}$ and labeling the identified vertex v and (2) adding $\mathcal{C} - 1$ new vertices w_t $(1 \leqslant t \leqslant \mathcal{C} - 1)$ and joining each w_t to v.

We first show that $\gamma_F(G) = \mathcal{C}$. Since $\{v\} \cup \{w_t\colon 1 \leqslant t \leqslant \mathcal{C} - 1\}$ is an F-dominating set, $\gamma_F(G) \leqslant \mathcal{C}$. On the other hand, let c be a minimum F-coloring of G. If $v \in R_c$, then $w_t \in R_c$ for $1 \leqslant t \leqslant \mathcal{C} - 1$ and so $\gamma_F(G) = |R_c| \geqslant \mathcal{C}$. Thus, we may assume that $v \notin R_c$. Since $u_{i,3}$ is only F-dominated by a vertex in $V(J_i) - \{v\}$ for $1 \leqslant i \leqslant l - k$ and $v_{j,3}$ is only F-dominated by a vertex in $V(G_j) - \{v\}$ for $1 \leqslant j \leqslant \mathcal{C} + 2k - l - 1$, it follows that

$$\gamma_F(G) = |R_c| \geqslant (l - k) + (\mathcal{C} + 2k - l - 1) = \mathcal{C} + k - 1 \geqslant \mathcal{C}$$

and so $\gamma_F(G) = \mathcal{C}$. Furthermore, since

$$S = \{v\} \cup \{u_{i,3}\colon 1 \leqslant i \leqslant l - k\} \cup \{v_{j,1}\colon 1 \leqslant j \leqslant \mathcal{C} + 2k - l - 1\}$$

is a minimum dominating set of G and $S \cup \{u_{i,2}\colon 1 \leqslant i \leqslant l - k\}$ a minimum open dominating set of G, it follows that $\gamma(G) = \mathcal{A}$ and $\gamma_o(G) = \mathcal{B}$.

Case 3. $l = \mathcal{C} + 2k$. In this case $\mathcal{B} = 2\mathcal{A}$. Let $p \geqslant 2$ be an integer. For each integer i with $1 \leqslant i \leqslant \mathcal{A} - \mathcal{C} + 1$, let M_i be the graph obtained from the path u_i, y_i, v_i by (1) adding $2p$ new vertices $r_{i,j}$ $(1 \leqslant j \leqslant 2p)$, (2) joining each vertex $r_{i,j}$ $(1 \leqslant j \leqslant p)$ to u_i and y_i, and (3) joining each vertex $r_{i,j}$ $(p + 1 \leqslant j \leqslant 2p)$ to y_i and v_i (see Figure 4). The graph M is then obtained from the $\mathcal{A} - \mathcal{C} + 1$

copies of M_i and a new vertex x by (1) joining x to y_1 and to each vertex in the set $\{u_i, v_i\colon 1 \leqslant i \leqslant \mathcal{A} - \mathcal{C} + 1\}$ and (2) joining v_i to u_{i+1} for all i with $1 \leqslant i \leqslant \mathcal{A} - \mathcal{C}$ and $v_{\mathcal{A}-\mathcal{C}+1}$ to u_1. For each integer t with $1 \leqslant t \leqslant \mathcal{C} - 1$, let $T_t\colon w_{t,1}, w_{t,2}, w_{t,3}$ be a copy of P_3. Then the graph G is obtained from the graphs M and T_t $(1 \leqslant t \leqslant \mathcal{C} - 1)$ by joining each $w_{t,1}$ $(1 \leqslant t \leqslant \mathcal{C} - 1)$ to x.

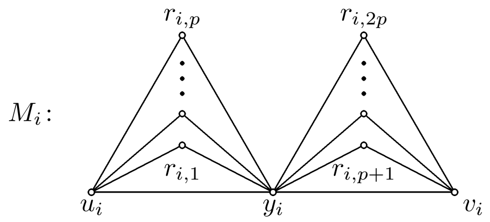

Figure 4. The graph M_i.

We first show that $\gamma_F(G) = \mathcal{C}$. Since $\{x\} \cup \{w_{t,3}\colon 1 \leqslant t \leqslant \mathcal{C} - 1\}$ is an F-dominating set, $\gamma_F(G) \leqslant \mathcal{C}$. To show that $\gamma_F(G) \geqslant \mathcal{C}$, let c be a minimum F-coloring of G. By Proposition 3.1, $w_{t,3} \in R_c$ for $1 \leqslant t \leqslant \mathcal{C} - 1$. Since x, for example, is not F-dominated by any vertex $w_{t,3}$ $(1 \leqslant t \leqslant \mathcal{C} - 1)$, it follows that $\gamma_F(G) > \mathcal{C} - 1$. Therefore, $\gamma_F(G) = \mathcal{C}$. Moreover, observe that

$$\{w_{t,2}\colon 1 \leqslant t \leqslant \mathcal{C} - 1\} \cup \{y_i\colon 1 \leqslant i \leqslant \mathcal{A} - \mathcal{C} + 1\}$$

is a minimum dominating set of G and that

$$\{w_{t,1}, w_{t,2}\colon 1 \leqslant t \leqslant \mathcal{C} - 1\} \cup \{u_i, v_i\colon 1 \leqslant i \leqslant \mathcal{A} - \mathcal{C} + 1\}$$

is a minimum open dominating set of G. Thus $\gamma(G) = \mathcal{A}$ and $\gamma_o(G) = 2\mathcal{A}$.

Case II. $2 \leqslant \mathcal{C} < \mathcal{A} = \mathcal{B}$. Let $\mathcal{A} = \mathcal{C} + k$, where $k \geqslant 1$. For $\mathcal{C} = 2$, let G be the graph obtained from the graphs G_j for $1 \leqslant j \leqslant \mathcal{A} - 1$ by identifying all vertices $v_{j,0}$ and labeling the identified vertex by v and adding one new vertex u together with the edge uv. Then $\{v, u\}$ is a minimum F-dominating set, $\gamma_F(G) = 2$. Furthermore, since $\{v\} \cup \{v_{j,1}\colon 1 \leqslant j \leqslant \mathcal{A} - 1\}$ is both a minimum dominating set and a minimum open dominating set of G, it follows that $\gamma(G) = \gamma_o(G) = \mathcal{A}$. Now assume that $\mathcal{C} \geqslant 3$. For each i with $1 \leqslant i \leqslant \mathcal{C} - 2$, let X_i be the graph obtained from the 5-cycle $x_{i,1}$, $x_{i,2}$, $x_{i,3}$, $x_{i,4}$, $x_{i,5}$, $x_{i,1}$ by adding a new vertex $x_{i,0}$ and joining $x_{i,0}$ to $x_{i,1}$, $x_{i,3}$, and $x_{i,4}$. Now, let G be the graph obtained from the graphs X_i and G_j for $1 \leqslant i \leqslant \mathcal{C} - 2$ and $1 \leqslant j \leqslant \mathcal{A} - \mathcal{C} + 1$ by (1) identifying all vertices $x_{i,0}$ and $v_{j,0}$ and labeling the identified vertex by v and (2) adding a new vertex u and the edge uv.

Since $\{v, u\} \cup \{x_{i,1}\colon\ 1 \leqslant i \leqslant \mathcal{C} - 2\}$ is a minimum F-dominating set, $\gamma_F(G) = \mathcal{C}$. Since

$$\{v\} \cup \{x_{i,1}\colon\ 1 \leqslant i \leqslant \mathcal{C} - 2\} \cup \{v_{j,1}\colon\ 1 \leqslant j \leqslant \mathcal{A} - \mathcal{C} + 1\}$$

is both a minimum dominating set and a minimum open dominating set of G, it follows that $\gamma(G) = \gamma_o(G) = \mathcal{A}$.

Case III. $2 \leqslant \mathcal{C} = \mathcal{A} < \mathcal{B} \leqslant 2\mathcal{A}$. Let $\mathcal{B} = \mathcal{A} + l$, where $1 \leqslant l \leqslant \mathcal{A}$. We consider two cases, according to whether $1 \leqslant l < \mathcal{A}$, or $l = \mathcal{A}$.

Case 1. $1 \leqslant l < \mathcal{A}$. If $\mathcal{A} = 2$ and $\mathcal{B} = 3$, then let G be the graph obtained from the graph H_1 by adding a new vertex u and the edge u_0u. Then $\{u, u_0\}$ is a minimum F-dominating set, $\{u_0, u_4\}$ is a minimum dominating set and $\{u_0, u_4, u_5\}$ is a minimum open dominating set. Therefore, $\gamma(G) = \gamma_F(G) = 2$ and $\gamma_o(G) = 3$. Thus, we can assume that $\mathcal{A} \geqslant 3$. Let G be the graph obtained from the graphs J_i and G_j for $1 \leqslant i \leqslant l$ and $1 \leqslant j \leqslant \mathcal{A} - l - 1$ by (1) identifying all vertices $u_{i,0}$ and $v_{j,0}$ and labeling the identified vertex v and (2) adding $\mathcal{A} - 1$ new vertices w_t $(1 \leqslant t \leqslant \mathcal{A} - 1)$ and joining each w_t to v. (Note that if $l = A - 1$, then there is no graph G_j in the construction of G.) Since $\{v\} \cup \{w_t\colon\ 1 \leqslant t \leqslant \mathcal{A} - 1\}$ is a minimum F-dominating set, $\gamma_F(G) = \mathcal{A}$. Furthermore, since

$$S = \{v\} \cup \{u_{i,3}\colon\ 1 \leqslant i \leqslant l\} \cup \{v_{j,1}\colon\ 1 \leqslant j \leqslant \mathcal{A} - l - 1\}$$

is a minimum dominating set of G and $S \cup \{u_{i,2}\colon\ 1 \leqslant i \leqslant l\}$ is a minimum open dominating set of G, it follows that $\gamma(G) = \mathcal{A}$ and $\gamma_o(G) = \mathcal{B}$.

Case 2. $l = \mathcal{A}$. In this case, $\mathcal{B} = 2\mathcal{A}$. Let $p \geqslant 2$ be an integer. Let M be the graph obtained from the graph M_1 in Figure 4 by adding a new vertex x and joining x to each vertex in $\{u_1, v_1, y_1\}$. For each integer j with $1 \leqslant j \leqslant \mathcal{A} - 1$, let $T_j\colon\ w_{j,1}, w_{j,2}, w_{j,3}$ be a copy of P_3. Then the graph G is obtained from the graphs M and T_j $(1 \leqslant j \leqslant \mathcal{A} - 1)$ by joining each $w_{j,1}$ $(1 \leqslant j \leqslant \mathcal{A} - 1)$ to x. Since $\{x\} \cup \{w_{j,3}\colon\ 1 \leqslant j \leqslant \mathcal{A} - 1\}$ is a minimum F-dominating set, $\gamma_F(G) = \mathcal{A}$. Since $\{y_1\} \cup \{w_{j,2}\colon\ 1 \leqslant j \leqslant \mathcal{A} - 1\}$ is a minimum dominating set of G and $\{x, y_1\} \cup \{w_{j,1}, w_{j,2}\colon\ 1 \leqslant j \leqslant \mathcal{A} - 1\}$ is a minimum open dominating set of G, it follows that $\gamma(G) = \mathcal{A}$ and $\gamma_o(G) = 2\mathcal{A}$. $\square$

3.2 Realizable triples of type II. Recall that a triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is of type II if $\mathcal{A} \leqslant \mathcal{B} \leqslant \mathcal{C}$. By Proposition 2.9, each triple $(k, k, \mathcal{C})$ of type II is nonrealizable for $\mathcal{C} > k \geqslant 2$. In this section we show that all other triples of type II are realizable, beginning with those triples for which $\mathcal{B} = 2\mathcal{A}$.

Theorem 3.3. *Every triple* $(\mathcal{A},\mathcal{B},\mathcal{C})$ *of type* II *with* $\mathcal{B}=2\mathcal{A}$ *is realizable.*

Proof. By Observation 2.3, every triple $(1,2,\mathcal{C})$ is realizable for each positive integer $\mathcal{C}$. Thus we may assume that $\mathcal{A}\geqslant 2$. Let $P\colon v_1,v_2,\ldots,v_{3\mathcal{A}-2}$ be a path of order $3\mathcal{A}-2$ and let G be the caterpillar obtained from P by adding $\mathcal{C}-\mathcal{A}-1\geqslant 1$ pendant edges at each vertex v_{3i+1} for $0\leqslant i\leqslant \mathcal{A}-1$. For $\mathcal{A}=2,3,4$, the graph G is drawn in Figure 5.

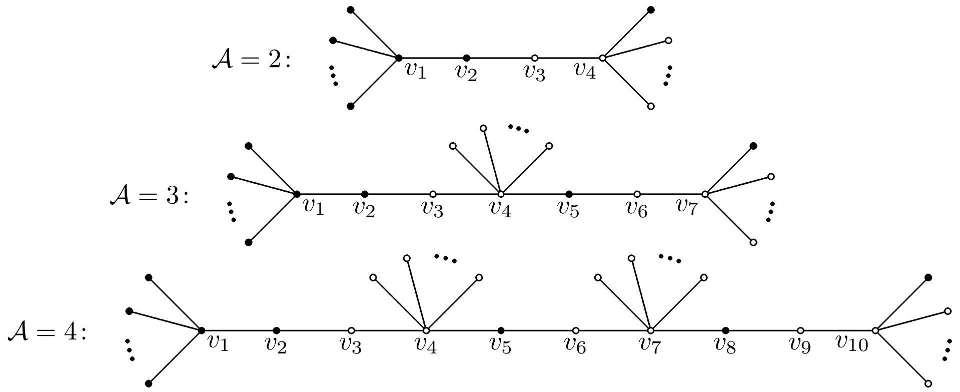

Figure 5. The graph G for $\mathcal{A}=2,3,4$.

For each vertex v_{3i+1} $(0\leqslant i\leqslant \mathcal{A}-1)$, let $W_i=N(v_{3i+1})-V(P)$. We show that $\gamma_F(G)=\mathcal{C}$. Since

$$S=W_0\cup\{v_1\}\cup\{v_{3i+2}\colon\ 0\leqslant i\leqslant \mathcal{A}-2\}\cup\{w_{\mathcal{A}-1}\},$$

where $w_{\mathcal{A}-1}\in W_{\mathcal{A}-1}$, is an F-dominating set, $\gamma_F(G)\leqslant |S|=\mathcal{C}$. To show that $\gamma_F(G)\geqslant \mathcal{C}$, let c be a minimum F-coloring of G.

First, we show that if $v_1\in R_c$, then $|R_c|\geqslant \mathcal{C}$. Suppose that $v_1\in R_c$. Then necessarily, $W_0\subseteq R_c$. We verify the following two claims.

Claim 1. At least one vertex in $\{v_{3i+2},v_{3i+3},v_{3i+4}\}\cup W_{i+1}$ must be red for each i with $0\leqslant i\leqslant \mathcal{A}-3$. Assume, to the contrary, that each vertex in $\{v_{3j+2},v_{3j+3},v_{3j+4}\}\cup W_{j+1}$ is blue for some j with $0\leqslant j\leqslant \mathcal{A}-3$. Then a vertex in W_{j+1} can only be F-dominated by v_{3j+5} and so $v_{3j+5}\in R_c$. However then, v_{3j+4} is not F-dominated by any vertex in R_c, a contradiction. Therefore, at least one vertex in $\{v_{3i+2},v_{3i+3},v_{3i+4}\}\cup W_{i+1}$ is red for $0\leqslant j\leqslant \mathcal{A}-3$.

Claim 2. At least two vertices in $\{v_{3A-4},v_{3A-3},v_{3A-2}\}\cup W_{\mathcal{A}-1}$ must be red. Since $w_{\mathcal{A}-1}\in W_{\mathcal{A}-1}$ is only F-dominated by v_{3A-3} or by a vertex in $W_{\mathcal{A}-1}$, either v_{3A-3} is red or some vertex in $W_{\mathcal{A}-1}$ is red. Furthermore, since v_{3A-2} is only

F-dominated by v_{3A-2} or by v_{3A-4}, it follows that $v_{3A-2} \in R_c$ or $v_{3A-4} \in R_c$. Therefore, at least two vertices in $\{v_{3A-4}, v_{3A-3}, v_{3A-2}\} \cup W_{\mathcal{A}-1}$ are red.

Since $\{v_1\} \cup W_0 \subseteq R_c$, it then follows by Claims 1 and 2 that

$$\gamma_F(G) = |R_c| \geqslant 1 + |W_0| + (\mathcal{A} - 2) + 2 = 1 + (\mathcal{C} - \mathcal{A} - 1) + (\mathcal{A} - 2) + 2 = \mathcal{C}.$$

Therefore, if $v_1 \in R_c$, then $|R_c| \geqslant \mathcal{C}$. We now consider two cases.

Case 1. *Suppose that* $v_{3i+1} \in R_c$ *for some* i $(0 \leqslant i \leqslant \mathcal{A} - 1)$. Let j be the smallest integer i such that $v_{3i+1} \in R_c$. If $j = 0$, then $v_1 \in R_c$ and we have seen that $|R_c| \geqslant \mathcal{C}$. Hence, we may assume that $1 \leqslant j \leqslant \mathcal{A} - 1$. Thus $v_{3j+1} \in R_c$ and $W_j \subseteq R_c$. If $j < \mathcal{A} - 1$ and $j \leqslant i \leqslant \mathcal{A} - 2$, then an argument similar to the situation where $v_1 \in R_c$ shows that at least one vertex in $\{v_{3i+2}, v_{3i+3}, v_{3i+4}\} \cup W_{i+1}$ must be red. We now show that if $0 \leqslant i \leqslant j-1$, then some vertex in $\{v_{3i+1}, v_{3i+2}, v_{3i+3}\} \cup W_i$ is red. If $j \geqslant 2$, then we first consider v_{3i+1}, where $1 \leqslant i < j$. Thus v_{3i+1} is blue and is either F-dominated by v_{3i+3} or by v_{3i-1}. If v_{3i+1} is F-dominated by v_{3i+3}, then $v_{3i+3} \in R_c$. If v_{3i+1} is F-dominated by v_{3i-1}, then $v_{3i-1} \in R_c$ and v_{3i} is blue. Hence either $v_{3i+2} \in R_c$ or $w_i \in R_c$ for some $w_i \in W_i$. For $i = 0$, the blue vertex v_1 can only be F-dominated by v_3 and the blue vertex v_2 can only be F-dominated by a vertex in W_0. Thus at least two vertices in $\{v_1, v_2, v_3\} \cup W_0$ must be red, which implies that

$$\gamma_F(G) = |R_c| \geqslant 2 + (j - 1) + 1 + (\mathcal{C} - \mathcal{A} - 1) + (\mathcal{A} - 2 - j + 1) = \mathcal{C}.$$

Case 2. *Suppose that* v_{3i+1} *is blue for every integer* i $(0 \leqslant i \leqslant \mathcal{A} - 1)$. We claim that v_{3i+1} is blue *and* v_{3i+3} is red for every integer i $(0 \leqslant i \leqslant \mathcal{A} - 2)$. We verify this by induction. First, because v_1 is blue, v_1 can only be F-dominated by v_3 and so $v_3 \in R_c$. In addition, this says that v_2 is blue and so some vertex in W_0 is red. Assume that v_{3k+1} is blue and v_{3k+3} is red, where $0 \leqslant k < \mathcal{A} - 2$. By the assumption in Case 2, v_{3k+4} is blue. Since v_{3k+4} is blue and v_{3k+3} is red, v_{3k+4} can only be F-dominated by v_{3k+6} and so v_{3k+6} is red. This verifies the claim. Thus $v_{3(\mathcal{A}-2)+3} = v_{3\mathcal{A}-3}$ is red. If $v_{3\mathcal{A}-2}$ is blue, then $v_{3\mathcal{A}-2}$ is not F-dominated by any vertex. Hence $v_{3\mathcal{A}-2} \in R_c$ and so $W_{\mathcal{A}-1} \subseteq R_c$ as well. Therefore,

$$\gamma_F(G) = |R_c| \geqslant (\mathcal{A} - 1) + 1 + 1 + (\mathcal{C} - \mathcal{A} - 1) = \mathcal{C},$$

as desired. Furthermore, since $\{v_{3i+1} \colon 0 \leqslant i \leqslant \mathcal{A} - 1\}$ is a minimum domination set and for $w_i \in W_i$ with $0 \leqslant i \leqslant \mathcal{A} - 1$, $\{v_{3i+1}, w_i \colon 0 \leqslant i \leqslant \mathcal{A} - 1\}$ is a minimum open domination set, $\gamma(G) = \mathcal{A}$ and $\gamma_o(G) = 2\mathcal{A} = \mathcal{B}$. $\square$

It remains to consider those triples $(\mathcal{A},\mathcal{B},\mathcal{C})$ of type II with $\mathcal{B} \neq 2\mathcal{A}$. For a positive integer α, let L_α be the graph shown in Figure 6. Since $\{w,y\}$ is a minimum dominating set, $\{w,x,y\}$ is a minimum open dominating set, and $\{w,x\} \cup \{w_i \colon 1 \leqslant i \leqslant \alpha\}$ is a minimum F-dominating set, $\gamma(L_\alpha) = 2$, $\gamma_o(L_\alpha) = 3$, and $\gamma_F(L_\alpha) = \alpha + 2$ for every integer $\alpha \geqslant 1$.

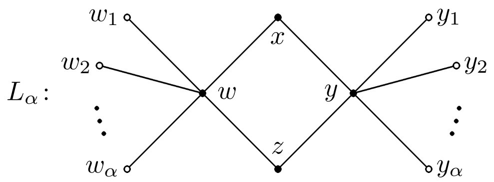

Figure 6. The graph L_α.

Theorem 3.4. *Let $(\mathcal{A},\mathcal{B},\mathcal{C})$ be a triple of type* II *such that $\mathcal{B} \neq 2\mathcal{A}$. If $(\mathcal{A},\mathcal{B},\mathcal{C}) \neq (k,k,\mathcal{C})$ where $\mathcal{C} > k \geqslant 2$, then $(\mathcal{A},\mathcal{B},\mathcal{C})$ is realizable.*

P r o o f. Observe that $\mathcal{A} \geqslant 2$. By Corollary 2.7 and Proposition 2.9, it suffices to consider two cases, according to whether $2 \leqslant \mathcal{A} < \mathcal{B} < \mathcal{C}$ or $2 \leqslant \mathcal{A} < \mathcal{B} = \mathcal{C}$.

C a s e I. $2 \leqslant \mathcal{A} < \mathcal{B} < \mathcal{C}$. If $\mathcal{A} = 2$, then $\mathcal{B} = 3$, and $\mathcal{C} \geqslant 4$. Note that the graph $L_{\mathcal{C}-2}$ has the desired properties. Thus, we may assume that $\mathcal{A} \geqslant 3$. Let $\mathcal{B} = \mathcal{A} + k$ and $\mathcal{C} = \mathcal{A} + l$. Since $\mathcal{A} < \mathcal{B} < 2\mathcal{A}$ and $\mathcal{B} < \mathcal{C}$, it follows that $1 \leqslant k \leqslant \mathcal{A} - 1$ and $k < l$. We consider three cases, according to whether $k = 1$, $2 \leqslant k \leqslant \mathcal{A} - 2$, or $k = \mathcal{A} - 1$.

C a s e 1. $k = 1$. Let G be the graph obtained from the graph $L_{\mathcal{C}-2}$ and the $\mathcal{A} - 2$ graphs G_i ($1 \leqslant i \leqslant \mathcal{A} - 2$) by identifying all the vertices $v_{i,0}$ and w and calling the new vertex v. Since $S = \{v,x\} \cup \{w_j \colon 1 \leqslant j \leqslant \mathcal{C} - 2\}$ is a minimum F-dominating set, $\gamma_F(G) = \mathcal{C}$. Observe that $S = \{v,y\} \cup \{v_{i,1} \colon 1 \leqslant i \leqslant \mathcal{A} - 2\}$ is a minimum dominating set of G and $S \cup \{x\}$ is a minimum open dominating set of G; so $\gamma(G) = \mathcal{A}$ and $\gamma_o(G) = \mathcal{B}$.

C a s e 2. $2 \leqslant k \leqslant \mathcal{A} - 2$. Let G be the graph obtained from the graph $L_{\mathcal{C}-2}$ and the graphs J_i, G_j for $1 \leqslant i \leqslant k - 1$ and $1 \leqslant j \leqslant \mathcal{A} - k - 1$ by identifying all vertices $u_{i,0}, v_{j,0}$ and w and labeling the identified vertex v. Since $\{v,x\} \cup \{w_t \colon 1 \leqslant t \leqslant \mathcal{C} - 2\}$ is a minimum F-dominating set, $\gamma_F(G) = \mathcal{C}$. Furthermore, since

$$S = \{v,y\} \cup \{u_{i,3} \colon 1 \leqslant i \leqslant k-1\} \cup \{v_{j,1} \colon 1 \leqslant j \leqslant \mathcal{A} - k - 1\}$$

is a minimum dominating set of G and $S \cup \{x\} \cup \{u_{i,2} \colon 1 \leqslant i \leqslant k - 1\}$ is a minimum open dominating set of G, it follows that $\gamma(G) = \mathcal{A}$ and $\gamma_o(G) = \mathcal{A} + k = \mathcal{B}$.

C a s e 3. $k = \mathcal{A} - 1$. Then $\mathcal{B} = \mathcal{A} + k = 2k + 1$ and $\mathcal{C} = \mathcal{A} + l = k + l + 1$. Let G be the graph obtained from the graph $L_{\mathcal{C}-2}$ and the graphs J_i for $1 \leqslant i \leqslant k - 1$ by identifying all vertices $u_{i,0}$ and w and labeling the identified vertex v. Since $\{v, x\} \cup \{w_j \colon 1 \leqslant j \leqslant \mathcal{C} - 2\}$ is a minimum F-dominating set, $\gamma_F(G) = \mathcal{C}$. Observe that

$$S = \{v, y\} \cup \{u_{i,3} \colon 1 \leqslant i \leqslant k - 1\}$$

is a minimum dominating set of G and

$$S \cup \{x\} \cup \{u_{i,2} \colon 1 \leqslant i \leqslant k - 1\}$$

is a minimum open dominating set of G; so $\gamma(G) = \mathcal{A}$ and $\gamma_o(G) = \mathcal{B}$.

C a s e II. $2 \leqslant \mathcal{A} < \mathcal{B} = \mathcal{C}$. If $\mathcal{A} = 2$, then $\mathcal{B} = 3$ and the graph L_1 has the desired properties. Thus, we may assume that $\mathcal{A} \geqslant 3$. Let $\mathcal{B} = \mathcal{A} + k$. Since $\mathcal{A} < \mathcal{B} < 2\mathcal{A}$, it follows that $1 \leqslant k < \mathcal{A}$. We consider three cases, according to whether $k = 1$, $2 \leqslant k \leqslant \mathcal{A} - 2$, or $k = \mathcal{A} - 1$.

C a s e 1. $k = 1$. Let G be the graph obtained from a copy of the graph $L_{\mathcal{B}-2}$ and the $\mathcal{A} - 2$ graphs G_i $(1 \leqslant i \leqslant \mathcal{A} - 2)$ by identifying all the vertices $v_{i,0}$ and w and calling the new vertex v. Since $\{v, x\} \cup \{w_j \colon 1 \leqslant j \leqslant \mathcal{B} - 2\}$ is a minimum F-dominating set, $\gamma_F(G) = \mathcal{B}$. Observe that

$$S = \{v, y\} \cup \{v_{i,1} \colon 1 \leqslant i \leqslant \mathcal{A} - 2\}$$

is a minimum dominating set of G and $S \cup \{x\}$ is a minimum open dominating set of G; so $\gamma(G) = \mathcal{A}$ and $\gamma_o(G) = \mathcal{B}$.

C a s e 2. $2 \leqslant k \leqslant \mathcal{A} - 2$. Let G be the graph obtained from the graph $L_{\mathcal{B}-2}$ and the graphs J_i, G_j for $1 \leqslant i \leqslant k - 1$ and $1 \leqslant j \leqslant \mathcal{A} - k - 1$ by identifying all vertices $u_{i,0}, v_{j,0}$ and w and labeling the identified vertex v. Since $\{v, x\} \cup \{w_t \colon 1 \leqslant t \leqslant \mathcal{B} - 2\}$ is a minimum F-dominating set, $\gamma_F(G) = \mathcal{B}$. Since

$$S = \{v, y\} \cup \{u_{i,3} \colon 1 \leqslant i \leqslant k - 1\} \cup \{v_{j,1} \colon 1 \leqslant j \leqslant \mathcal{A} - k - 1\}$$

is a minimum dominating set of G and $S \cup \{x\} \cup \{u_{i,2} \colon 1 \leqslant i \leqslant k - 1\}$ it follows that $\gamma(G) = \mathcal{A}$ and $\gamma_o(G) = \mathcal{A} + k = \mathcal{B}$.

C a s e 3. $k = \mathcal{A} - 1$. Then $\mathcal{B} = \mathcal{A} + k = 2k + 1$. Let G be the graph obtained from the graph $L_{\mathcal{B}-2}$ and the graphs J_i for $1 \leqslant i \leqslant k - 1$ by identifying all vertices $u_{i,0}$ and w and labeling the identified vertex v. Since $\{v, x\} \cup \{w_j \colon 1 \leqslant j \leqslant \mathcal{B} - 2\}$ is a minimum F-dominating set, $\gamma_F(G) = \mathcal{B}$. Since

$$S = \{v, y\} \cup \{u_{i,3} \colon 1 \leqslant i \leqslant k - 1\}$$

is a minimum dominating set of G and

$$S \cup \{x\} \cup \{u_{i,2}\colon\ 1 \leqslant i \leqslant k-1\}$$

is a minimum open dominating set of G, it follows that $\gamma(G) = \mathcal{A}$. and $\gamma_o(G) = \mathcal{B}$. $\square$

Combining Proposition 2.9 and Theorems 3.3 and 3.4, we have the following.

Corollary 3.5. *A triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of type* II *is realizable if and only if $(\mathcal{A}, \mathcal{B}, \mathcal{C}) \neq (k, k, \mathcal{C})$ for any integers k and $\mathcal{C}$ with $\mathcal{C} > k \geqslant 2$.*

3.3 Realizable triples of type III. Recall that a triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is of type III if $\mathcal{A} \leqslant \mathcal{C} \leqslant \mathcal{B}$. In this section we show that every triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of type III is realizable, beginning with those triples for which $\mathcal{B} = 2\mathcal{A}$.

Theorem 3.6. *Every triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of type* III *with $\mathcal{B} = 2\mathcal{A}$ is realizable.*

Proof. By Proposition 2.3, $(1, 2, 1)$ and $(1, 2, 2)$ are realizable. Thus, we may assume that $\mathcal{A} \geqslant 2$. First, suppose that $\mathcal{A} = \mathcal{C}$. Let G be the graph obtained from the cycle $C_{3\mathcal{A}}\colon\ v_1, v_2, \ldots, v_{3\mathcal{A}}, v_1$ by adding the pendant edge $u_i v_{3i+1}$ for $0 \leqslant i \leqslant \mathcal{A}-1$. Since $\{v_{3i+2}\colon\ 0 \leqslant i \leqslant \mathcal{A}-1\}$ is a minimum F-dominating set, $\gamma_F(G) = \mathcal{A}$. Observe that $S = \{v_{3i+1} : 0 \leqslant i \leqslant \mathcal{A}-1\}$ is a minimum dominating set and $S \cup \{u_i\colon\ 0 \leqslant i \leqslant \mathcal{A}-1\}$ is a minimum open dominating set; so $\gamma(G) = \mathcal{A}$ and $\gamma_o(G) = 2\mathcal{A}$.

Next, suppose that $\mathcal{A} < \mathcal{C}$. If $\mathcal{C} \geqslant \mathcal{A}+2$, then $\mathcal{C} - \mathcal{A} - 1 \geqslant 1$. Let G be the graph constructed in Theorem 3.3, that is, let G be the caterpillar obtained from the path $P\colon\ v_1, v_2, \ldots, v_{3\mathcal{A}-2}$ of order $3\mathcal{A}-2$ by adding $\mathcal{C} - \mathcal{A} - 1 \geqslant 1$ pendant edges at each vertex v_{3i+1} for $0 \leqslant i \leqslant \mathcal{A}-1$. For each vertex v_{3i+1} $(0 \leqslant i \leqslant \mathcal{A}-1)$, let $W_i = N(v_{3i+1}) - V(P)$. For $w_{\mathcal{A}-1} \in W_{\mathcal{A}-1}$,

$$S = W_0 \cup \{v_1\} \cup \{v_{3i+2}\colon\ 0 \leqslant i \leqslant \mathcal{A}-2\} \cup \{w_{\mathcal{A}-1}\},$$

is a minimum F-dominating set by the proof of Theorem 3.3. Thus $\gamma_F(G) = |S| = \mathcal{C}$. Furthermore, since $\{v_{3i+1}\colon\ 0 \leqslant i \leqslant \mathcal{A}-1\}$ is a minimum domination set and for $w_i \in W_i$ with $0 \leqslant i \leqslant \mathcal{A}-1$, the set $\{v_{3i+1}, w_i\colon\ 0 \leqslant i \leqslant \mathcal{A}-1\}$ is a minimum open domination set. Therefore, $\gamma(G) = \mathcal{A}$ and $\gamma_o(G) = 2\mathcal{A} = \mathcal{B}$.

Thus, we may assume that $\mathcal{C} = \mathcal{A}+1$. Let $P\colon\ v_1, v_2, \ldots, v_{3\mathcal{A}-2}$ be a path of order $3\mathcal{A}-2$ and let H be the caterpillar obtained from P by adding three pendant edges at each vertex v_{3i+1} for $0 \leqslant i \leqslant \mathcal{A}-1$. For each vertex v_{3i+1} $(0 \leqslant i \leqslant \mathcal{A}-1)$, let $W_i = N(v_{3i+1}) - V(P)$. The graph G is then obtained from H by joining two vertices in $W_{\mathcal{A}-1}$. For $\mathcal{A} = 2, 3, 4$, the graph G is drawn in Figure 7.

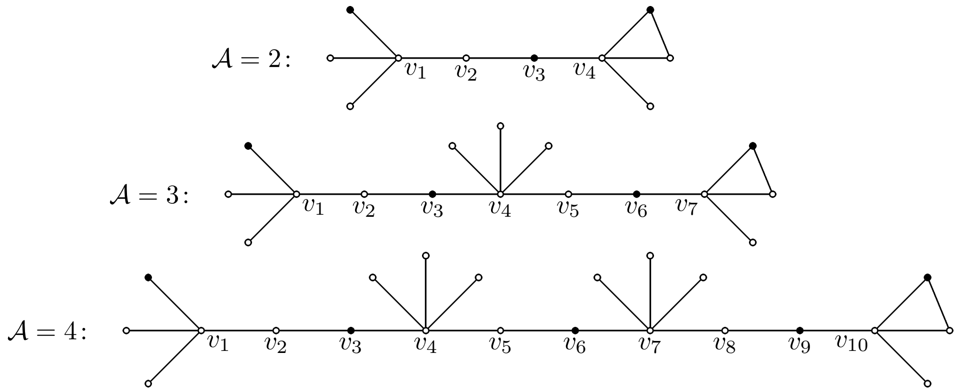

Figure 7. The graph G for $\mathcal{A}=2,3,4$.

For $w_i \in W_i$ for $i=0,\mathcal{A}-1$, where $\deg w_{\mathcal{A}-1}=2$,

$$S=\{w_0,w_{\mathcal{A}-1}\}\cup\{v_{3i}\colon\ 1\leqslant i\leqslant \mathcal{A}-1\}$$

is an F-dominating set and so $\gamma_F(G)=|S|=\mathcal{A}+1$. To show that $\gamma_F(G)\geqslant \mathcal{A}+1$, let c be a minimum F-coloring of G.

First, we show that if $v_1\in R_c$, then $|R_c|>\mathcal{A}+1=\mathcal{C}$. Suppose that $v_1\in R_c$. Then necessarily, $W_0\subseteq R_c$. We verify the following claim.

Claim. At least one vertex in $\{v_{3i+2},v_{3i+3},v_{3i+4}\}\cup W_{i+1}$ must be red for each i with $0\leqslant i\leqslant \mathcal{A}-2$. Assume, to the contrary, that each vertex in $\{v_{3j+2},v_{3j+3},v_{3j+4}\}\cup W_{j+1}$ is blue for some j with $0\leqslant j\leqslant \mathcal{A}-2$. First suppose that $0\leqslant j\leqslant \mathcal{A}-3$. Then a vertex in W_{j+1} can only be F-dominated by v_{3j+5} and so $v_{3j+5}\in R_c$. However then, v_{3j+4} is not F-dominated by any vertex in R_c, a contradiction. Next suppose that $j=\mathcal{A}-2$. Then $w_{\mathcal{A}-1}\in W_{\mathcal{A}-1}$ can only be F-dominated by v_{3A-3} or by a vertex in $W_{\mathcal{A}-1}$ and so either v_{3A-3} is red or some vertex in $W_{\mathcal{A}-1}$ is red.

Since $\{v_1\}\cup W_0\subseteq R_c$, it then follows by the claim

$$\gamma_F(G)=|R_c|\geqslant 1+|W_0|+(\mathcal{A}-1)=1+3+(\mathcal{A}-1)=\mathcal{A}+3>\mathcal{A}+1=\mathcal{C}.$$

Therefore, if $v_1\in R_c$, then $|R_c|>\mathcal{C}$. Since this is impossible, it follows that v_1 is blue. We now consider two cases.

Case 1. $v_{3i+1}\in R_c$ *for some* i $(1\leqslant i\leqslant \mathcal{A}-1)$. Let j be the smallest integer i such that $v_{3i+1}\in R_c$. Thus $v_{3j+1}\in R_c$ and $W_j\subseteq R_c$. If $j<\mathcal{A}-1$ and $j\leqslant i\leqslant \mathcal{A}-2$, then an argument similar to the situation where $v_1\in R_c$ shows that at least one vertex in $\{v_{3i+2}, v_{3i+3}, v_{3i+4}\} \cup W_{i+1}$ must be red. We now show that if $0 \leqslant i \leqslant j-1$, then some vertex in $\{v_{3i+1}, v_{3i+2}, v_{3i+3}\} \cup W_i$ is red. If $j \geqslant 2$, then we first consider v_{3i+1}, where $1 \leqslant i < j$. Thus v_{3i+1} is blue and is either F-dominated by v_{3i+3} or by v_{3i-1}. If v_{3i+1} is F-dominated by v_{3i+3}, then $v_{3i+3} \in R_c$. If v_{3i+1} is F-dominated by v_{3i-1}, then $v_{3i-1} \in R_c$ and v_{3i} is blue. Hence either $v_{3i+2} \in R_c$ or $w_i \in R_c$ for some $w_i \in W_i$. For $i = 0$, the blue vertex v_1 can only be F-dominated by v_3 and the blue vertex v_2 can only be F-dominated by a vertex in W_0. Thus at least two vertices in $\{v_1, v_2, v_3\} \cup W_0$ must be red, which implies that

$$\gamma_F(G) = |R_c| \geqslant 2 + (j-1) + 1 + 3 + (\mathcal{A} - 2 - j + 1) = \mathcal{A} + 4 > \mathcal{A} + 1 = \mathcal{C}.$$

Thus Case 1 cannot occur.

Case 2. *v_{3i+1} is blue for every integer i ($0 \leqslant i \leqslant \mathcal{A}-1$).* We claim that v_{3i+1} is blue *and* v_{3i+3} is red for every integer i ($0 \leqslant i \leqslant \mathcal{A}-2$). We verify this by induction. First, because v_1 is blue, v_1 can only be F-dominated by v_3 and so $v_3 \in R_c$. In addition, this says that v_2 is blue and so some vertex in W_0 is red. Assume that v_{3k+1} is blue and v_{3k+3} is red, where $0 \leqslant k < \mathcal{A} - 2$. By the assumption in Case 2, v_{3k+4} is blue. Since v_{3k+4} is blue and v_{3k+3} is red, v_{3k+4} can only be F-dominated by v_{3k+6} and so v_{3k+6} is red. This verifies the claim. Thus $v_{3(\mathcal{A}-2)+3} = v_{3\mathcal{A}-3}$ is red. Since $v_{3\mathcal{A}-2}$ can only be F-dominated by $v_{3\mathcal{A}-2}$ or by a vertex of degree 2 in $W_{\mathcal{A}-1}$, it follows that either $v_{3\mathcal{A}-2}$ is red or a vertex of degree 2 in $W_{\mathcal{A}-1}$ is red. Therefore,

$$\gamma_F(G) = |R_c| \geqslant 2 + (\mathcal{A} - 2) + 1 = \mathcal{A} + 1 = \mathcal{C},$$

as desired. Furthermore, since $\{v_{3i+1} \colon 0 \leqslant i \leqslant \mathcal{A}-1\}$ is a minimum domination set and $\{v_{3i+1}, w_i \colon 0 \leqslant i \leqslant \mathcal{A}-1\}$, where $w_i \in W_i$, is a minimum open domination set, it follows that $\gamma(G) = \mathcal{A}$ and $\gamma_o(G) = 2\mathcal{A} = \mathcal{B}$. $\square$

Theorem 3.7. *Every triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of type* III *with $\mathcal{B} < 2\mathcal{A}$ is realizable.*

Proof. By Corollary 2.7 and Case II ($2 \leqslant \mathcal{A} < \mathcal{B} = \mathcal{C} < 2\mathcal{A}$) in Theorem 3.4, we need only consider the two cases $2 \leqslant \mathcal{A} < \mathcal{C} < \mathcal{B} < 2\mathcal{A}$ and $2 \leqslant \mathcal{A} = \mathcal{C} < \mathcal{B} < 2\mathcal{A}$.

Case I. $2 \leqslant \mathcal{A} < \mathcal{C} < \mathcal{B} < 2\mathcal{A}$. Necessarily, $\mathcal{A} \geqslant 3$ in this case. Let $\mathcal{C} = \mathcal{A} + k$ and $\mathcal{B} = \mathcal{A} + l$. Since $\mathcal{A} < \mathcal{C} < \mathcal{B} < 2\mathcal{A}$, it follows that $1 \leqslant k < l < \mathcal{A}$. Thus, we consider two cases, according to whether $2 \leqslant l \leqslant \mathcal{A} - 2$, or $l = \mathcal{A} - 1$.

Case 1. $2 \leqslant l \leqslant \mathcal{A} - 2$. Let G be the graph obtained from the graph $L_{\mathcal{C}-2}$ and the graphs J_i, G_j for $1 \leqslant i \leqslant l-1$ and $1 \leqslant j \leqslant \mathcal{A} - l - 1$ by identifying all vertices $u_{i,0}, v_{j,0}$ and w and labeling the identified vertex v.

Since $\{v,x\}\cup\{w_t\colon 1\leqslant t\leqslant \mathcal{C}-2\}$ is an F-dominating set, $\gamma_F(G)\leqslant \mathcal{C}$. On the other hand, let c be an F-coloring of G. Since y is only F-dominated by v or y, it follows that either $v\in R_c$ or $y\in R_c$. Thus either $W_1=\{v\}\cup\{w_t\colon 1\leqslant t\leqslant \mathcal{C}-2\}\subseteq R_c$ or $W_2=\{y\}\cup\{y_t\colon 1\leqslant t\leqslant \mathcal{C}-2\}\subseteq R_c$. In either case, $|R_c|\geqslant \mathcal{C}-1$. If $\gamma_F(G)=\mathcal{C}-1$, then either $R_c=W_1$ or $R_c=W_2$. However then, the blue vertex x is not F-dominated by any vertex in R_c, which is a contradiction. Therefore $\gamma_F(G)=\mathcal{C}$. Furthermore, since

$$S=\{v,y\}\cup\{u_{i,3}\colon 1\leqslant i\leqslant l-1\}\cup\{v_{j,1}\colon 1\leqslant j\leqslant \mathcal{A}-l-1\}$$

is a minimum dominating set of G and

$$S\cup\{x\}\cup\{u_{i,2}\colon 1\leqslant i\leqslant l-1\}$$

is a minimum open dominating set of G, it follows that $\gamma(G)=\mathcal{A}$ and $\gamma_o(G)=\mathcal{A}+l=\mathcal{B}$.

Case 2. $l=\mathcal{A}-1$. Then $\mathcal{B}=\mathcal{A}+l=2l+1$ and $\mathcal{C}=\mathcal{A}+k=k+l+1$. Let G be the graph obtained from the graph $L_{\mathcal{C}-2}$ and the graphs J_i for $1\leqslant i\leqslant l-1$ by identifying all vertices $u_{i,0}$ and w and labeling the identified vertex v.

Since $\{v,x\}\cup\{w_j\colon 1\leqslant j\leqslant \mathcal{C}-2\}$ is a minimum F-dominating set, $\gamma_F(G)=\mathcal{C}$. Since

$$S=\{v,y\}\cup\{u_{i,3}\colon 1\leqslant i\leqslant l-1\}$$

is a minimum dominating set of G and

$$S\cup\{x\}\cup\{u_{i,2}\colon 1\leqslant i\leqslant l-1\}$$

is a minimum open dominating set of G, it follows that $\gamma(G)=\mathcal{A}$ and $\gamma_o(G)=\mathcal{B}$.

Case II. $2\leqslant \mathcal{A}=\mathcal{C}<\mathcal{B}$. If $\mathcal{A}=2$, then since $\mathcal{A}<\mathcal{B}<2\mathcal{A}$, it follows that $\mathcal{B}=3$. Let G be obtained from the graph K_4-e and the path $P_2\colon x,y$ by joining x to a vertex of degree 2 in K_4-e. Then $\gamma(G)=\gamma_F(G)=2$ and $\gamma_o(G)=3$. Thus we may assume that $\mathcal{A}\geqslant 3$. Let $\mathcal{B}=\mathcal{A}+k$. In the remaining proof , we consider three cases, according to whether $k=1$, $2\leqslant k\leqslant \mathcal{A}-2$, or $k=\mathcal{A}-1$. The proof is similar to that in Case I and is therefore omitted. □

Combining Theorems 3.6 and 3.7, we have the following.

Corollary 3.8. *Every triple of type* III *is realizable.*

By Theorem 3.2 and Corollaries 3.5 and 3.8, we have the main result of this paper.

Theorem 3.9. *A triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is realizable if and only if $(\mathcal{A}, \mathcal{B}, \mathcal{C}) \neq (k, k, \mathcal{C})$ for any integers k and $\mathcal{C}$ with $\mathcal{C} > k \geqslant 2$.*

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