ON HERMITE-HERMITE MATRIX POLYNOMIALS

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Abstract. In this paper the definition of Hermite-Hermite matrix polynomials is introduced starting from the Hermite matrix polynomials. An explicit representation, a matrix recurrence relation for the Hermite-Hermite matrix polynomials are given and differential equations satisfied by them is presented. A new expansion of the matrix exponential for a wide class of matrices in terms of Hermite-Hermite matrix polynomials is proposed.

Keywords: matrix functions, Hermite matrix polynomials, recurrence relation, Hermite matrix differential equation, Rodrigues's formula

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1. Introduction and preliminaries

Orthogonal matrix polynomials form an emergent field whose development is reaching important results from both the theoretical and practical points of view. Some recent results in this field can be found in [1], [6], [10]. Important connections between orthogonal matrix polynomials and matrix differential equations appear in [1], [2], [4], [7], [11]. Special functions, as a branch of mathematics are of utmost importance to scientists and engineers in many areas of applications [8], [12]. Theory of special functions plays an important role in the formalism of mathematical physics. Hermite and Chebyshev polynomials in [9] are among the most important special functions, with very diverse applications to physics, engineering and mathematical physics ranging from abstract number theory to problems of physics and engineering. Recently, the Hermite matrix polynomials have been introduced and studied in a number of papers [1], [5], [7], [10]. For the most part the relations with which we deal here are included because they are amusing or particularly pretty. It would be unwise, however, to pass up the subject as one of other value. The symbolic notation also suggests the study of some interesting polynomials which may not otherwise be noticed [8], [9].

This paper deals with the introduction and study of Hermite matrix polynomials taking advantage of those recently treated in [1], [2]. The organization of the paper is as follows. In Section 2 Hermite-Hermite matrix polynomials are defined, the three terms matrix recurrence relations are proved, their connections with differential equations are shown and the expansions of Hermite-Hermite matrix polynomials into series are established. Finally, we study the case relevant to mixed matrix polynomials with the type Hermite-Hermite matrix polynomials that are defined here in Section 3.

If D_0 is the complex plane cut along the negative real axis and $\log(z)$ denotes the principal logarithm of z, then $z^{\frac{1}{2}}$ represents $\exp(\frac{1}{2}\log(z))$. If A is a matrix in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$, its two-norm denoted by $\|A\|_2$ is defined by $\|A\|_2 = \|Ax\|_2/\|x\|_2$, where for a vector y in $\mathbb{C}^{\mathbb{N}}$, $\|y\|_2$ denotes the usual Euclidean norm of y, $\|y\|_2 = (y^T y)^{\frac{1}{2}}$. The set of all eigenvalues of A is denoted by $\sigma(A)$. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane, and if A is a matrix in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ such that $\sigma(A) \subset \Omega$, then the matrix functional calculus yields that

$$f(A)g(A) = g(A)f(A). \tag{1.1}$$

If A is a matrix with $\sigma(A) \subset D_0$, then $A^{\frac{1}{2}} = \sqrt{A} = \exp(\frac{1}{2}\log(A))$ denotes the image by $z^{\frac{1}{2}} = \sqrt{z} = \exp(\frac{1}{2}\log(z))$ of the matrix functional calculus acting on the matrix A. We say that A is a positive stable matrix [3], [5], [7] if

$$\operatorname{Re}(z) > 0 \quad \text{for all } z \in \sigma(A). \tag{1.2}$$

If $A(k,n)$ and $B(k,n)$ are matrices on $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ for $n \geqslant 0$, $k \geqslant 0$, it follows in an analogous way to the proof of Lemma 11 of [9] that

$$\begin{aligned} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} A(k,n-2k), \\ \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n-k). \end{aligned} \tag{1.3}$$

Similarly to (1.3), we can write

$$\begin{aligned} \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} A(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+2k), \\ \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+k). \end{aligned} \tag{1.4}$$

1.1. On Hermite matrix polynomials. We consider the Hermite matrix polynomials $H_n(x,A)$ defined by the generating function [1], [6]

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x,A)=\exp\bigl(xt\sqrt{2A}-t^2I\bigr). \tag{1.5}$$

The polynomials $H_n(x,A)$ are explicitly expressed as

$$H_n(x,A)=n!\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k\bigl(x\sqrt{2A}\bigr)^{n-2k}}{k!(n-2k)!},\quad n\geqslant 0. \tag{1.6}$$

It is clear that

$$H_{-1}(x,A)=0,\quad H_0(x,A)=I,\quad H_1(x,A)=x\sqrt{2A}$$
$$\text{and}\quad H_n(-x,A)=(-1)^nH_n(x,A).$$

Their recurrence properties can be derived either from (1.5) or from (1.6). It is easy to prove that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}H_n(x,A)&=n\sqrt{2A}H_{n-1}(x,A),\\ H_{n+1}(x,A)&=\Bigl[x\sqrt{2A}-\frac{2}{\sqrt{2A}}\frac{\mathrm{d}}{\mathrm{d}x}\Bigr]H_n(x,A).\end{aligned} \tag{1.7}$$

The differential equation satisfied by $H_n(x,A)$ can be straightforwardly deduced by introducing the shift operators

$$\begin{aligned}\widehat{P}&=\frac{1}{\sqrt{2A}}\frac{\mathrm{d}}{\mathrm{d}x},\\ \widehat{M}&=x\sqrt{2A}-\frac{2}{\sqrt{2A}}\frac{\mathrm{d}}{\mathrm{d}x}\end{aligned} \tag{1.8}$$

which act on $H_n(x,A)$ according to the rules

$$\begin{aligned}\widehat{P}H_n(x,A)&=nH_{n-1}(x,A),\\ \widehat{M}H_n(x,A)&=H_{n+1}(x,A).\end{aligned} \tag{1.9}$$

Using the identity

$$\widehat{M}\widehat{P}H_n(x,A)=nH_n(x,A) \tag{1.10}$$

from (1.10), we find that $H_n(x,A)$ satisfies the following ordinary differential equation of second order [4], [10], [11]:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{x}{2}\big(\sqrt{2A}\big)^2\frac{\mathrm{d}}{\mathrm{d}x} + \frac{n}{2}\big(\sqrt{2A}\big)^2\right]H_n(x,A) = 0. \tag{1.11}$$

We consider the operational definition of Hermite matrix polynomials [1] in the form

$$H_n(x,A) = \exp\Big(-\frac{1}{(\sqrt{2A})^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big)\big(x\sqrt{2A}\big)^n. \tag{1.12}$$

Our aim is to prove some known properties as well as new expansions formulae related to these Hermite matrix polynomials. In the following, we will apply the above results to Hermite-Hermite matrix polynomials and we will see that the results, summarized in this section, can be exploited to state quite general results.

2. On Hermite-Hermite matrix polynomials

The Hermite-Hermite matrix polynomials are defined by the series

$${}_HH_n(x,A) = n!\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k\big(\sqrt{2A}\big)^{n-2k}H_{n-2k}(x,A)}{k!(n-2k)!}. \tag{2.1}$$

It is clear that

$${}_HH_{-1}(x,A) = 0, \quad {}_HH_0(x,A) = I, \quad {}_HH_1(x,A) = 2xA.$$

Using (1.4), (1.5) and (2.1), we arrange the series in the form

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{{}_HH_n(x,A)t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k\big(\sqrt{2A}\big)^{n-2k}H_{n-2k}(x,A)}{k!(n-2k)!}t^n \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\big(\sqrt{2A}\big)^{n}H_{n}(x,A)}{k!n!}t^{n+2k} \\
&= \sum_{n=0}^{\infty}\frac{\big(\sqrt{2A}\big)^{n}H_{n}(x,A)}{n!}t^n\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}t^{2k}I \\
&= \exp\Big(xt\big(\sqrt{2A}\big)^2 - \big(t\sqrt{2A}\big)^2\Big)\exp(-t^2I) \\
&= \exp\Big(xt\big(\sqrt{2A}\big)^2 - t^2\Big(\big(\sqrt{2A}\big)^2 + I\Big)\Big).
\end{aligned}$$

We obtain an explicit representation for the Hermite-Hermite matrix polynomials by the generating function in the form

$$(2.3)\quad F(x,t,A)=\sum_{n=0}^{\infty}\frac{{}_HH_n(x,A)t^n}{n!}=\exp\bigl(xt\bigl(\sqrt{2A}\bigr)^2-t^2\bigl(\bigl(\sqrt{2A}\bigr)^2+I\bigr)\bigr);\quad |t|<\infty$$

where $F(x,t,A)$ regarded as a function of the complex variable t is an entire matrix, therefore has the Taylor series about $t=0$ and the series obtained converges for all values of x and t. Clearly, ${}_HH_n(x,A)$ is a matrix polynomial of degree n in x. Replacing x by $-x$ and t by $-t$ in (2.3), the left-hand side does not change. Therefore

$$ {}_HH_n(-x,A)=(-1)^n\,{}_HH_n(x,A).$$

2.1. Recurrence relations. Some recurrence relations will be established for the Hermite matrix polynomials. First, we obtain

Theorem 2.1. *The Hermite-Hermite matrix polynomials* ${}_HH_n(x,A)$ *satisfy the relations*

$$(2.4)\qquad \frac{\mathrm{d}^r}{\mathrm{d}x^r}\,{}_HH_n(x,A)=\frac{\bigl(\sqrt{2A}\bigr)^{2r}n!}{(n-r)!}\,{}_HH_{n-r}(x,A),\quad 0\leqslant r\leqslant n.$$

Proof. Differentiating the identity (2.2) with respect to x yields

$$(2.5)\qquad \sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{\mathrm{d}}{\mathrm{d}x}\,{}_HH_n(x,A)=t\bigl(\sqrt{2A}\bigr)^2\exp\bigl(xt\bigl(\sqrt{2A}\bigr)^2-t^2\bigl(\bigl(\sqrt{2A}\bigr)^2+I\bigr)\bigr)$$

From (2.5) and (2.2) we have

$$\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\mathrm{d}}{\mathrm{d}x}\,{}_HH_n(x,A)t^n=\bigl(\sqrt{2A}\bigr)^2\sum_{n=0}^{\infty}\frac{1}{n!}\,{}_HH_n(x,A)t^{n+1}.$$

Hence, identifying the coefficients at t^n, we obtain

$$(2.6)\qquad \frac{\mathrm{d}}{\mathrm{d}x}\,{}_HH_n(x,A)=n\bigl(\sqrt{2A}\bigr)^2\,{}_HH_{n-1}(x,A),\quad n\geqslant 1.$$

Iteration (2.6) for $0\leqslant r\leqslant n$ implies (2.4). Therefore, the expression (2.4) is established and the proof of Theorem 2.1 is completed. The above three-terms recurrence relation will be used in the following theorem.

Theorem 2.2. *Let A be a matrix in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ satisfying* (1.2). *Then we have*

$$(2.7)\qquad {}_HH_n(x,A)=x\big(\sqrt{2A}\big)^2\,{}_HH_{n-1}(x,A)-2(n-1)\,{}_HH_{n-2}(x,A),\quad n\geqslant 2.$$

P r o o f. Differentiating (2.3) with respect to x and t, we find respectively

$$\frac{\partial}{\partial x}F(x,t,A)=t\big(\sqrt{2A}\big)^2\exp\big(xt\big(\sqrt{2A}\big)^2-t^2\big(\big(\sqrt{2A}\big)^2+I\big)\big)=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\mathrm{d}}{\mathrm{d}x}\,{}_HH_n(x,A)t^n$$

and

$$\begin{aligned}\frac{\partial}{\partial t}F(x,t,A)&=\big(x\big(\sqrt{2A}\big)^2-2\big(\big(\sqrt{2A}\big)^2+I\big)t\big)\exp\big(xt\big(\sqrt{2A}\big)^2-t^2\big(\big(\sqrt{2A}\big)^2+I\big)\big)\\&=\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\,{}_HH_n(x,A)t^{n-1}.\end{aligned}$$

Therefore, $F(x,t,A)$ satisfies the partial matrix differential equation

$$\big(x\big(\sqrt{2A}\big)^2-2\big(\big(\sqrt{2A}\big)^2+I\big)t\big)\frac{\partial F}{\partial x}-t\big(\sqrt{2A}\big)^2\frac{\partial F}{\partial t}=0$$

which, by virtue of (2.3), becomes

$$\begin{aligned}\big(\sqrt{2A}\big)^2\sum_{n=0}^{\infty}\frac{n}{n!}\,{}_HH_n(x,A)t^n&=x\big(\sqrt{2A}\big)^2\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\mathrm{d}}{\mathrm{d}x}\,{}_HH_n(x,A)t^n\\&\quad-2\big(\big(\sqrt{2A}\big)^2+I\big)\sum_{n=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{n!}\,{}_HH_n(x,A)t^{n+1}.\end{aligned}$$

Since $x(\mathrm{d}/\mathrm{d}x)\,{}_HH_1(x,A)=\,{}_HH_1(x,A)$, it follows that

$$n\big(\sqrt{2A}\big)^2\,{}_HH_n(x,A)=x\big(\sqrt{2A}\big)^2\frac{\mathrm{d}}{\mathrm{d}x}\,{}_HH_n(x,A)-2n\big(\big(\sqrt{2A}\big)^2+I\big)\frac{\mathrm{d}}{\mathrm{d}x}\,{}_HH_{n-1}(x,A).$$

Using (2.6) and (2.8), we get (2.7). The proof of Theorem 2.2 is completed.

The above recurrence properties can be derived either from (2.1) or from (2.2). It is easy to prove that

$$(2.9)\qquad \begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\,{}_HH_n(x,A)&=n\big(\sqrt{2A}\big)^2\,{}_HH_{n-1}(x,A)=2nA\,{}_HH_{n-1}(x,A),\\ H_{n+1}(x,A)&=\Big[x\big(\sqrt{2A}\big)^2-\frac{2\big(\big(\sqrt{2A}\big)^2+I\big)}{\big(\sqrt{2A}\big)^2}\frac{\mathrm{d}}{\mathrm{d}x}\Big]\,{}_HH_n(x,A)\\&=\Big[2xA-\frac{2(2A+I)}{2A}\frac{\mathrm{d}}{\mathrm{d}x}\Big]\,{}_HH_n(x,A).\end{aligned}$$

The differential equation satisfied by ${}_HH_n(x,A)$ can be straightforwardly inferred by introducing the shift operators

$$\widehat{P} = \frac{1}{2A}\frac{\mathrm{d}}{\mathrm{d}x}, \qquad \widehat{M} = 2Ax - \frac{2(2A+I)}{2A}\frac{\mathrm{d}}{\mathrm{d}x} \tag{2.10}$$

which act on ${}_HH_n(x,A)$ according to the rules

$$\widehat{P}\,{}_HH_n(x,A) = n\,{}_HH_{n-1}(x,A)(x,A), \qquad \widehat{M}\,{}_HH_n(x,A) = {}_HH_{n+1}(x,A). \tag{2.11}$$

Using the identity

$$\widehat{M}\widehat{P}\,{}_HH_n(x,A) = n\,{}_HH_n(x,A) \tag{2.12}$$

from (2.12), we find that $H_n(x,A)$ satisfies the following ordinary differential equation of second order [4], [10], [11]

$$\Big[\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{x(2A)^2}{2(2A+I)}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{n(2A)^2}{2(2A+I)}\Big]\,{}_HH_n(x,A) = 0, \quad n \geqslant 0. \tag{2.13}$$

In the next result, the Hermite-Hermite matrix polynomials appear as finite series solutions of the second order matrix differential equation.

Corollary 2.1. *The Hermite-Hermite matrix polynomials are solutions of the matrix differential equation of the second order*

$$\Big[\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{x(2A)^2}{2(2A+I)}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{n(2A)^2}{2(2A+I)}\Big]\,{}_HH_n(x,A) = 0, \quad n \geqslant 0. \tag{2.14}$$

Proof. Replacing n by $n-1$ in (2.6) gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\,{}_HH_{n-1}(x,A)) = 2(n-1)A\,{}_HH_{n-2}(x,A). \tag{2.15}$$

Substituting from (2.15) into (2.4) yields

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,{}_HH_n(x,A) = 2nA\frac{\mathrm{d}}{\mathrm{d}x}\,{}_HH_{n-1}(x,A) = n(n-1)(2A)^2\,{}_HH_{n-2}(x,A). \tag{2.16}$$

From (2.7), (2.15) and (2.16) we obtain (2.14). Thus the proof of Corollary 2.1 is completed.

2.2. Expansion of Hermite-Hermite matrix polynomials. Now, we can use the expansion of Hermite-Hermite matrix polynomials together with their properties to prove the following result.

Theorem 2.3. *Let A be a positive stable matrix in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ satisfying* (1.2). *Then we have*

$$(2xA)^n = n!\sum_{k=0}^{[\frac{1}{2}n]} \frac{\bigl(2A+I\bigr)^k}{k!(n-2k)!}\, {}_HH_{n-2k}(x,A), \quad -\infty < x < \infty. \tag{2.17}$$

Proof. By (1.3) and (2.2) we can write

$$\begin{aligned}\exp\bigl(2xtA\bigr) &= \sum_{n=0}^{\infty}\frac{(2xA)^n}{n!}t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\bigl(2A+I\bigr)^k\, {}_HH_n(x,A)}{n!k!}\, t^{n+2k} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]}\frac{\bigl(2A+I\bigr)^k\, {}_HH_{n-2k}(x,A)}{k!(n-2k)!}\, t^n.\end{aligned} \tag{2.18}$$

Expanding the left-hand side of (2.18) into powers of t and identifying the coefficients of t^n on both sides gives (2.17). Therefore, the expression (2.17) is established and the proof of Theorem 2.3 is completed.

2.3. Hermite-Hermite matrix polynomials series expansions. It is well-known that the matrix exponential plays an important role in many different fields and its computation is difficult, see [2], [4], [6] for example. Using Hermite-Hermite matrix polynomial series we propose new expansions of the matrices $\exp(xB)$, $\sin(xB)$, $\cos(xB)$, $\cosh(xB)$ and $\sinh(xB)$ for matrices satisfying the spectral property

$$|\mathrm{Re}(x)| > |\mathrm{Im}(x)| \quad \text{for all } x\in\sigma(B). \tag{2.19}$$

Theorem 2.4. *Let B be a matrix in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ satisfying* (2.19). *Then*

$$\exp\bigl(xB\bigr) = \exp\bigl(B+I\bigr)\sum_{n=0}^{\infty}\frac{1}{n!}\, {}_HH_n\Bigl(x,\frac{1}{2}B\Bigr), \ -\infty < x < \infty, \tag{2.20}$$

$$\cos(xB) = \exp\bigl(-(B+I)\bigr)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\, {}_HH_{2n}\Bigl(x,\frac{1}{2}B\Bigr), \ -\infty < x < \infty, \tag{2.21}$$

$$\sin(xB) = \exp\bigl(-(B+I)\bigr)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\, {}_HH_{2n+1}\Bigl(x,\frac{1}{2}B\Bigr), \ -\infty < x < \infty, \tag{2.22}$$

$$\cosh(xB) = \exp\bigl(B+I\bigr)\sum_{n=0}^{\infty}\frac{1}{(2n)!}\, {}_HH_{2n}\Bigl(x,\frac{1}{2}B\Bigr), \ -\infty < x < \infty \tag{2.23}$$

and

$$\sinh(xB) = \exp\bigl(B+I\bigr)\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\, {}_HH_{2n+1}\Bigl(x,\frac{1}{2}B\Bigr), \ -\infty < x < \infty. \tag{2.24}$$

P r o o f. Let $A = \frac{1}{2}B$. By the spectral mapping theorem [2], [4], [6] and (2.19), it follows that

$$\sigma(A) = \Big\{\frac{b}{2}; b \in \sigma(B)\Big\}, \quad \operatorname{Re}\Big(\frac{b}{2}\Big) = \frac{1}{2}\Big\{\operatorname{Re}(b) - \operatorname{Im}(b)\Big\} > 0, \quad b \in \sigma(B). \tag{2.25}$$

Thus A is a positive stable matrix and taking $t = 1$ in (2.2), $B = 2A$ gives

$$\exp(xB - (B+I)) = \sum_{n=0}^{\infty} \frac{1}{n!}\, {}_H H_n\Big(x, \frac{1}{2}B\Big). \tag{2.26}$$

Therefore, (2.20) follows.

Considering (2.17) for the positive stable matrix $A = \frac{1}{2}B$, we obtain that

$$(xB)^{2n} = (2n)! \sum_{k=0}^{n} \frac{(B+I)^k}{k!(2n-2k)!}\, {}_H H_n\Big(x, \frac{1}{2}B\Big).$$

Taking into account the series expansion of $\cos(Bx)$ and (1.4), we can write

$$\begin{aligned}
\cos(xB) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(xB)^{2n} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^n (B+I)^k}{k!(2n-2k)!}\, {}_H H_{2n-2k}\Big(x, \frac{1}{2}B\Big) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^n (B+I)^k}{k!(2n)!}\, {}_H H_{2n}\Big(x, \frac{1}{2}B\Big) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k (B+I)^k}{k!} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, {}_H H_{2n}\Big(x, \frac{1}{2}B\Big) \\
&= \exp(-(B+I)) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, {}_H H_{2n}\Big(x, \frac{1}{2}B\Big).
\end{aligned}$$

Therefore, (2.21) follows. By similar arguments we can prove the relations (2.22), (2.23) and (2.24).

Moreover, the convergence of the matrix series appearing in (2.20)–(2.23) and (2.24) to the respective matrix function $\exp(xB)$, $\sin(xB)$, $\cos(xB)$, $\sinh(xB)$ and $\cosh(xB)$ is uniform in any bounded interval of the real axis. Therefore, the result is established.

R e m a r k 2.1. The series developments given by (2.20)–(2.24) have one important advantage as compared to the Taylor series, from the computational point of view. In fact, the advantage follows from the fact that it is not necessary to compute the powers B^n of the matrix B, as well as from the fact that using relationship (2.7), the Hermite-Hermite matrix polynomials can be computed recurrently in terms of ${}_H H_0(x, \frac{1}{2}B) = I$ and ${}_H H_1(x, \frac{1}{2}B) = xB$.

In the next theorem we obtain another representation for the Hermite-Hermite matrix polynomials.

Theorem 2.5. *Suppose that A is a matrix in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ satisfying* (1.2). *Then the Hermite-Hermite matrix polynomials have the representation*

$$ {}_{H}H_n(x,A) = \exp\Big(-\frac{1}{(\sqrt{2A})^4}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big)\big(\sqrt{2A}\big)^n H_n(x,A). \tag{2.27}$$

Proof. It is clear by (1.7) and (2.1) that

$$\begin{aligned}\exp\Big(-\frac{1}{(\sqrt{2A})^4}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big)\big(\sqrt{2A}\big)^n H_n(x,A) &= \sum_{n=0}^{\infty}\frac{(-1)^k}{k!\big(\sqrt{2A}\big)^{4k}}\frac{\mathrm{d}^{2k}}{\mathrm{d}x^{2k}}\big(\sqrt{2A}\big)^n H_n(x,A)\\ &= n!\sum_{k=0}^{\infty}\frac{(-1)^k\big(\sqrt{2A}\big)^{-2k}}{k!(n-2k)!}\big(\sqrt{2A}\big)^n H_{n-2k}(x,A)\\ &= n!\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k\big(\sqrt{2A}\big)^{n-2k}}{k!(n-2k)!}H_{n-2k}(x,A) = {}_{H}H_n(x,A).\end{aligned}$$

Therefore, the result is established. Using (2.10) and substituting for n the values $0,1,2,\ldots,n-1$, we get

$$ {}_{H}H_n(x,A) = \exp\Big(-\frac{2}{(\sqrt{2A})^4}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big)\big(x\sqrt{2A}\big)^n H_n(x,A) = \Big[2xA - \frac{2(2A+I)}{2A}\frac{\mathrm{d}}{\mathrm{d}x}\Big]^n. \tag{2.28}$$

Special case: It should be observed that in view of the explicit representation (2.1), the Hermite-Hermite matrix polynomials ${}_{H}H_n(x,A)$ reduce to the Hermite-Hermite matrix polynomials ${}_{H/\sqrt{2A}}H_n(x,A)$.

3. Another representation for the Hermite-Hermite matrix polynomials

Let us consider the matrix polynomials

$$\Psi_n(x,A) = n!\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k H_{n-2k}(x,A)}{k!(n-2k)!}. \tag{3.1}$$

It is clear that

$$\begin{gathered}\Psi_{-1}(x,A)=0,\quad \Psi_0(x,A)=I,\quad \Psi_1(x,A)=x\sqrt{2A}\\ \text{and}\quad \Psi_n(-x,A)=(-1)^n\,\Psi_n(x,A).\end{gathered}$$

Using (1.4), (1.5) and (3.1), we arrange the series

$$\begin{aligned}
(3.2)\quad & \sum_{n=0}^{\infty} \frac{\Psi_n(x,A)t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k H_{n-2k}(x,A)}{k!(n-2k)!} t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k H_n(x,A)}{k!n!} t^{n+2k} \\
&= \sum_{n=0}^{\infty} \frac{H_n(x,A)}{n!} t^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^{2k} I = \exp\big(xt\sqrt{2A} - It^2\big)\exp(-t^2 I) \\
&= \exp\big(xt\sqrt{2A} - 2t^2 I\big) = \exp\Big(\frac{x}{\sqrt{2}}\big(t\sqrt{2}\big)\sqrt{2A} - I\big(t\sqrt{2}\big)^2\Big) \\
&= \sum_{n=0}^{\infty} \frac{H_n\big(x/\sqrt{2},A\big)\big(t\sqrt{2}\big)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^{n/2} H_n\big(x/\sqrt{2},A\big)}{n!} t^n
\end{aligned}$$

from which (3.2) follows by equating the coefficients at t^n.

We obtain a new generating function which represents the Hermite-Hermite matrix polynomials $\Psi_n(x,A)$ by

$$(3.3)\qquad \sum_{n=0}^{\infty} \frac{t^n}{n!} \Psi_n(x,A) = \exp\big(xt\sqrt{2A} - 2t^2 I\big).$$

We obtain an explicit representation for $\Psi_n(x,A)$ in the form

$$(3.4)\qquad \Psi_n(x,A) = n! \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k 2^k \big(x\sqrt{2A}\big)^{n-2k}}{k!(n-2k)!}, \quad n \geqslant 0$$

and satisfy the identities

$$\begin{aligned}
(3.5)\qquad & \frac{\mathrm{d}}{\mathrm{d}x}\Psi_n(x,A) = n\sqrt{2A}\,\Psi_{n-1}(x,A), \\
& \Psi_{n+1}(x,A) = \Big[x\sqrt{2A} - \frac{4}{\sqrt{2A}}\frac{\mathrm{d}}{\mathrm{d}x}\Big]\Psi_n(x,A).
\end{aligned}$$

Furthermore, according to (3.5), the $\Psi_n(x,A)$ are said to be under the action of the shift operators

$$\begin{aligned}
(3.6)\qquad & \widehat{P} = \frac{1}{\sqrt{2A}}\frac{\mathrm{d}}{\mathrm{d}x}, \\
& \widehat{M} = x\sqrt{2A} - \frac{4}{\sqrt{2A}}\frac{\mathrm{d}}{\mathrm{d}x}
\end{aligned}$$

which act on $\Psi_n(x, A)$ according to the rules

$$\widehat{P}\Psi_n(x, A) = n\Psi_{n-1}(x, A),$$
$$\widehat{M}\Psi_n(x, A) = \Psi_{n+1}(x, A). \tag{3.7}$$

Since the identity

$$\widehat{M}\widehat{P}\Psi_n(x, A) = n\Psi_n(x, A) \tag{3.8}$$

holds we use the explicit definition of $\widehat{M}$ and $\widehat{P}$ given by (3.8) to find that $\Psi_n(x, A)$ satisfies the ordinary differential equation of the second order

$$\Big[\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{x}{4}\big(\sqrt{2A}\big)^2\frac{\mathrm{d}}{\mathrm{d}x} + \frac{n}{4}\big(\sqrt{2A}\big)^2\Big]\Psi_n(x, A) = 0. \tag{3.9}$$

We also find

$$\Psi_n(x, A) = n!\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k 2^k}{k!}\frac{1}{\big(\sqrt{2A}\big)^{2k}}\frac{\mathrm{d}^{2k}}{\mathrm{d}x^{2k}}\big(x\sqrt{2A}\big)^n, \tag{3.10}$$

which can be used as an alternative to the series (3.4) and can be viewed as an alternative to Rodrigues's formula (3.10). By (3.4) and (3.10) we can define $\Psi_n(x, A)$ through the operational rule

$$\Psi_n(x, A) = \exp\Big(-\frac{2}{(\sqrt{2A})^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big)\big(x\sqrt{2A}\big)^n. \tag{3.11}$$

Using (3.5), (3.11) and substituting for n the values $0, 1, 2, \ldots, n-1$, we get

$$\exp\Big(-\frac{2}{\big(\sqrt{2A}\big)^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big)\big(x\sqrt{2A}\big)^n = \Big[x\sqrt{2A} - \frac{4}{\sqrt{2A}}\frac{\mathrm{d}}{\mathrm{d}x}\Big]^n. \tag{3.12}$$

The use of the inverse of (3.11) allows to conclude that

$$\big(x\sqrt{2A}\big)^n = \exp\Big(\frac{2}{(\sqrt{2A})^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big)\Psi_n(x, A). \tag{3.13}$$

Using some recurrence relations for $\Psi_n(x, A)$, we easily obtain the relations

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r}\Psi_n(x, A) = \frac{\big(\sqrt{2A}\big)^r n!}{(n-r)!}\Psi_{n-r}(x, A), \quad 0 \leqslant r \leqslant n \tag{3.14}$$

and

$$\Psi_n(x,A) = x\sqrt{2A}\,\Psi_{n-1}(x,A) - 4(n-1)\Psi_{n-2}(x,A), \quad n \geqslant 2. \tag{3.15}$$

Using (1.3) and (3.3), we obtain the expansion of $\left(x\sqrt{2A}\right)$ into series

$$\left(x\sqrt{2A}\right)^n = n! \sum_{k=0}^{[\frac{1}{2}n]} \frac{2^k}{k!(n-2k)!} \Psi_{n-2k}(x,A), \; -\infty < x < \infty. \tag{3.16}$$

We propose with Hermite-Hermite matrix polynomial series a new expansion of the matrix $\exp(xB)$, $\sin(xB)$, $\cos(xB)$, $\cosh(xB)$ and $\sinh(xB)$ for matrices satisfying the spectral property

$$|\mathrm{Re}(x)| > |\mathrm{Im}(x)| \quad \text{for all } x \in \sigma(B). \tag{3.17}$$

Let $A = \frac{1}{2}B^2$. By the spectral mapping theorem and (3.17) then it follows that

$$\begin{aligned} \sigma(A) &= \left\{\frac{1}{2}b^2;\; b \in \sigma(B)\right\}, \\ \mathrm{Re}\left(\frac{1}{2}b^2\right) &= \frac{1}{2}\{(\mathrm{Re}(b))^2 - (\mathrm{Im}(b))^2\} > 0, \quad b \in \sigma(B). \end{aligned} \tag{3.18}$$

Let B be a matrix in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ satisfying (3.17). Then it is easy to prove that

$$\begin{aligned} \exp(xB) &= \exp(2) \sum_{n=0}^{\infty} \frac{1}{n!} \Psi_n\left(x, \frac{1}{2}B^2\right), \\ \cos(xB) &= \exp(-2) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \Psi_{2n}\left(x, \frac{1}{2}B^2\right), \\ \sin(xB) &= \exp(-2) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \Psi_{2n+1}\left(x, \frac{1}{2}B^2\right), \\ \cosh(xB) &= \exp(2) \sum_{n=0}^{\infty} \frac{1}{(2n)!} \Psi_{2n}\left(x, \frac{1}{2}B^2\right) \end{aligned} \tag{3.19}$$

and

$$\sinh(Bx) = \exp(2) \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \Psi_{2n+1}\left(x, \frac{1}{2}B^2\right); \; -\infty < x < \infty.$$

Moreover, the convergence of the matrix series appearing in (3.19) to the respective matrix functions $\exp(xB)$, $\cos(xB)$, $\sin(xB)$, $\cosh(xB)$ and $\sinh(xB)$ is uniform in any bounded interval of the real axis. Further examples proving the usefulness of the present methods can be easily worked out, but are not reported here for conciseness. The last identities indicate that the method described in this paper can go beyond the specific problem addressed here and can be exploited in a wider context.

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References

[1] *R. S. Batahan*: A new extension of Hermite matrix polynomials and its applications. Linear Algebra Appl. *419* (2006), 82–92. zbl
[2] *E. Defez, L. Jódar*: Some applications of Hermite matrix polynomials series expansions. J. Comput. Appl. Math. *99* (1998), 105–117. zbl
[3] *E. Defez, A. Hervás, L. Jódar, A. Law*: Bounding Hermite matrix polynomials. Math. Computer Modelling *40* (2004), 117–125. zbl
[4] *L. Jódar, R. Company*: Hermite matrix polynomials and second order matrix differential equations. J. Approx. Theory Appl. *12* (1996), 20–30. zbl
[5] *L. Jódar, E. Defez*:Proceedings International Workshop on Orthogonal Polynomials in Mathematical Physics, Leganés, 1996 (M. Alfaro, ed.). Universidad Carlos III. de Madrid, Servicio de Publicaciones, 1997, pp. 93–101. zbl
[6] *L. Jódar, E. Defez*: A connection between Laguerre's and Hermite's matrix polynomials. Appl. Math. Lett. *11* (1998), 13–17. zbl
[7] *L. Jódar, E. Defez*: On Hermite matrix polynomials and Hermite matrix functions. J. Approx. Theory Appl. *14* (1998), 36–48. zbl
[8] *N. N. Lebedev*: Special Functions and Their Applications. Dover, New York, 1972. zbl
[9] *E. D. Rainville*: Special Functions. Macmillan, New York, 1962.
[10] *K. A. M. Sayyed, M. S. Metwally, R. S. Batahan*: On generalized Hermite matrix polynomials. Electron. J. Linear Algebra *10* (2003), 272–279. zbl
[11] *K. A. M. Sayyed, M. S. Metwally, R. S. Batahan*: Gegenbauer matrix polynomials and second order matrix differential equations. Divulg. Mat. *12* (2004), 101–115. zbl
[12] *H. M. Srivastava, H. L. Manocha*: A Treatise on Generating Functions. Ellis Horwood, New York, 1984. zbl

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