ON UPPER TRACEABLE NUMBERS OF GRAPHS

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Abstract. For a connected graph G of order $n \geqslant 2$ and a linear ordering $s\colon v_1, v_2, \ldots, v_n$ of vertices of G, $d(s) = \sum\limits_{i=1}^{n-1} d(v_i, v_{i+1})$, where $d(v_i, v_{i+1})$ is the distance between v_i and v_{i+1}. The upper traceable number $t^+(G)$ of G is $t^+(G) = \max\{d(s)\}$, where the maximum is taken over all linear orderings s of vertices of G. It is known that if T is a tree of order $n \geqslant 3$, then $2n-3 \leqslant t^+(T) \leqslant \lfloor n^2/2 \rfloor - 1$ and $t^+(T) \leqslant \lfloor n^2/2 \rfloor - 3$ if $T \neq P_n$. All pairs n, k for which there exists a tree T of order n and $t^+(T) = k$ are determined and a characterization of all those trees of order $n \geqslant 4$ with upper traceable number $\lfloor n^2/2 \rfloor - 3$ is established. For a connected graph G of order $n \geqslant 3$, it is known that $n - 1 \leqslant t^+(G) \leqslant \lfloor n^2/2 \rfloor - 1$. We investigate the problem of determining possible pairs n, k of positive integers that are realizable as the order and upper traceable number of some connected graph.

Keywords: traceable graph, traceable number, upper traceable number

MSC 2000: 05C12, 05C45

1. Introduction

In 1856 William Rowan Hamilton developed the *Icosian Game*, consisted of a board with twenty holes and some lines between certain pairs of holes, where the holes are designated by the twenty consonants of the English alphabet (see Figure 1). Hamilton's Icosian game can also be interpreted as a graph called the *dodecahedron*. The problems proposed by Hamilton in his Icosian Game gave rise to major concepts in graph theory. A path in a graph G that contains every vertex of G is a *Hamiltonian path* of G and a cycle that contains every vertex of G is a *Hamiltonian cycle* of G. A graph containing a Hamiltonian cycle is called a *Hamiltonian graph*, while a graph containing a Hamiltonian path is often called *traceable*. As Hamilton himself had observed, the graph of the dodecahedron is Hamiltonian. On the other hand, Hamilton also observed that this graph had a much stronger property, that is, any path with five vertices in this graph can be extended to a Hamiltonian cycle. Hamilton proposed a number of additional problems in his Icosian Game such as showing the existence of three initial vertices that can be extended to a Hamiltonian path but which cannot in turn be extended to a Hamiltonian cycle.

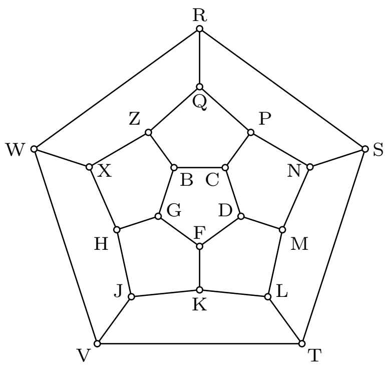

Figure 1. Hamilton's Icosian Game

The problems Hamilton proposed in his Icosian Game have inspired a number of other research topics. In the late 1960s Chartrand defined a graph G to be randomly traceable if every path in G can be extended to a Hamiltonian path in G, while G is randomly Hamiltonian if every path in G can be extended to a Hamiltonian cycle in G. For graphs of order 3 or more, these concepts are equivalent. Chartrand and Kronk [4] characterized all of these graphs in 1969. In 1973 this concept was generalized by Carsten Thomassen [11] when he studied graphs having the property that each path lies on some Hamiltonian cycle.

In 1973 Goodman and Hedetniemi [7] introduced the concept of a Hamiltonian walk in a connected graph G, defined as a closed spanning walk of minimum length. Therefore, for a connected graph G of order n, the length of a Hamiltonian walk of G is n if and only if G is Hamiltonian. During the 10-year period 1973–1983, this concept received considerable attention. For example, Hamiltonian walks were also studied by Asano, Nishizeki, and Watanabe [1], [2], Bermond [3], Nebeský [10], and Vacek [12], [13]. This concept was studied from a different point of view in 2004 by Chartrand, Saenpholphat, Thomas, and Zhang, namely in terms of sequences of vertices of a graph, as inspired by Hamilton's original work (see [5]). In this paper, we refer to the book [6] for graph theory notation and terminology not described here.

For a connected graph G of order $n \geqslant 3$ and a *cyclic ordering* $s\colon v_1, v_2, \ldots, v_n$, $v_{n+1} = v_1$ of vertices of G, the number $d(s)$ is defined in [5] as $d(s) = \sum_{i=1}^{n} d(v_i, v_{i+1})$, where $d(v_i, v_{i+1})$ is the distance between v_i and v_{i+1}. The *Hamiltonian number* $h(G)$ of G is defined in [5] by $h(G) = \min\{d(s)\}$, where the minimum is taken over all cyclic orderings s of vertices of G. Thus $h(G) \geqslant n$ for every connected graph G of order $n \geqslant 3$ and $h(G) = n$ if and only if G is Hamiltonian. The Hamiltonian number of a connected graph G is, in fact, the length of a Hamiltonian walk in G (see [5]). A related concept was introduced in [8]. For a connected graph G of order $n \geqslant 2$ and a *linear ordering* $s\colon v_1, v_2, \ldots, v_n$ of vertices of G, the number $d(s)$ is defined as

$$d(s) = \sum_{i=1}^{n-1} d(v_i, v_{i+1}). \tag{1}$$

The *traceable number* $t(G)$ of G is defined in [8] as $t(G) = \min\{d(s)\}$, where the minimum is taken over all linear orderings s of vertices of G. Thus if G is a connected graph of order $n \geqslant 2$, then $t(G) \geqslant n-1$. Furthermore, $t(G) = n-1$ if and only if G is traceable. In fact, the traceable number of a connected graph G is the minimum length of a spanning walk in G. The Hamiltonian number $h(G)$ and traceable number $t(G)$ of a graph G provide measures of traversability for G.

For a connected graph G, the *upper Hamiltonian number* $h^+(G)$ is defined in [5] as $h^+(G) = \max\{d(s)\}$, where the maximum is taken over all cyclic orderings s of vertices of G. As expected, for a connected graph G, the *upper traceable number* $t^+(G)$ is defined in [9] as

$$t^+(G) = \max\{d(s)\},$$

where $d(s)$ is described in (1) and the maximum is taken over all linear orderings s of vertices of G. For a connected graph G, let $\operatorname{diam}(G)$ denote the diameter of G (the largest distance between two vertices of G). Consequently, for every nontrivial connected graph G of order n,

$$n-1 \leqslant t(G) \leqslant t^+(G) \leqslant (n-1)\operatorname{diam}(G). \tag{2}$$

Both upper and lower bounds in (2) are sharp. Characterizations of all graphs whose upper traceable number and traceable number differ by at most 1 have been established in [9].

Theorem 1.1 [9]. *Let G be a connected graph of order $n \geqslant 3$. Then*
(a) $t^+(G) = t(G)$ *if and only if* $G = K_n$.
(b) $t^+(G) = t(G) + 1$ *if and only if* $G = K_n - e$ *or* $G = K_{1,n-1}$.

The upper traceable numbers of some well-known classes of graphs (namely, complete multipartite graphs, cycles, hypercubes) have been determined (see [9]). In particular, a formula for the upper traceable number of a tree was established. For each edge e of a tree T, the *component number* $\mathrm{cn}(e)$ of e is defined in [5] as the minimum order of a component of $T - e$. For example, the edge e_8 of the tree T of Figure 2(a) has component number 4 since the order of the smaller component of $T - e_8$ is 4. Each edge of this tree is labeled with its component number in Figure 2(b).

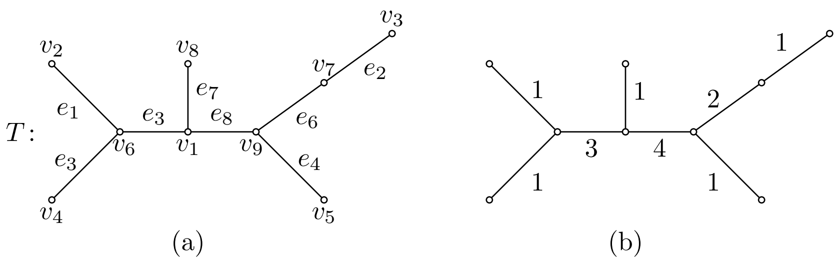

Figure 2. Component numbers of edges in a tree

The following result provides a formula for the upper traceable number of a tree in terms of the component numbers of its edges.

Theorem 1.2 [9]**.** *If T is a nontrivial tree, then*

$$t^+(T) = 2 \sum_{e \in E(T)} \mathrm{cn}(e) - 1.$$

By Theorem 1.2, the upper traceable number of a nontrivial tree is always odd. With the aid of Theorem 1.2, sharp upper and lower bounds for the upper traceable number of a tree were established in [9] in terms of its order, as we state now.

Theorem 1.3 [9]**.** *Let T be a nontrivial tree of order n. Then*

$$2n - 3 \leqslant t^+(T) \leqslant \lfloor n^2/2 \rfloor - 1.$$

Furthermore,

(a) $t^+(T) = 2n - 3$ *if and only if* $T = K_{1,n-1}$.
(b) $t^+(T) = \lfloor n^2/2 \rfloor - 1$ *if and only if* $T = P_n$.

2. Realization results for trees

By Theorem 1.3, if T is a nontrivial tree of order n with $t^+(T) = k$, then k is an odd integer and $2n-3 \leqslant k \leqslant \lfloor n^2/2 \rfloor - 1$. In this section we show that every pair n, k of integers for which $n \geqslant 2$, k is odd, and $2n-3 \leqslant k \leqslant \lfloor n^2/2 \rfloor - 1$ can be realized as the order and upper traceable number, respectively, of some tree. In order to do this, we present some additional definitions. A *double star* is a tree of diameter 3. If T is a double star with central vertices u and v such that $\deg u = a$ and $\deg v = b$, then T is denoted by $S_{a,b}$. A *caterpillar* is a tree of order 3 or more, the removal of whose end-vertices produces a path called the *spine* of the caterpillar. For a nontrivial tree T, the *component number* $\operatorname{cn}(T)$ of T is defined as

$$\operatorname{cn}(T) = \sum_{e \in E(T)} \operatorname{cn}(e).$$

Then $t^+(T) = 2\operatorname{cn}(T) - 1$.

Theorem 2.1. *For every pair n, k of integers, where $n \geqslant 2$ and k is an odd integer with*

$$2n-3 \leqslant k \leqslant \lfloor n^2/2 \rfloor - 1,$$

there exists a tree T of order n for which $t^+(T) = k$.

Proof. The result is obviously true if $2 \leqslant n \leqslant 6$. Thus we may assume that $n \geqslant 7$. Let $k = 2l - 1$, where $6 \leqslant n-1 \leqslant l \leqslant \lfloor n^2/4 \rfloor$. We first consider the case where n is odd. Assume that $n = 2a+1$, where $a \geqslant 3$. Hence $2a \leqslant l \leqslant a^2 + a$. We now construct a caterpillar T of order n for which $\operatorname{cn}(T) = l$, whose construction depends on the value of l.

Case 1. $l = 2a$. Let $T = K_{1,n-1}$ and observe that every edge of $K_{1,n-1}$ is a pendant edge. Hence $\operatorname{cn}(K_{1,n-1}) = 2a$.

Case 2. $2a+1 \leqslant l \leqslant 3a-1$. Let $l = 2a+1+i$, where $0 \leqslant i \leqslant a-2$. Let T be a double star $S_{2+i,n-2-i}$ and e the unique edge of $S_{2+i,n-2-i}$ that is not a pendant edge. Since $2+i < n-2-i$, it follows that $\operatorname{cn}(e) = 2+i$. Hence

$$\operatorname{cn}(S_{2+i,n-2-i}) = 2+i+(2a-1) = 2a+1+i$$

for $0 \leqslant i \leqslant a-2$.

Case 3. $3a \leqslant l \leqslant \frac{1}{2}a^2 + \frac{5}{2}a - 1$. First consider the function

$$f(x) = -\tfrac{1}{2}x^2 + \left(a + \tfrac{1}{2}\right)x + 2a - 1$$

defined on the set $[1, a]$ (the set of real numbers x with $1 \leqslant x \leqslant a$). Observe that f is continuous and strictly increasing on $[1, a]$. Let $b \in [1, a] \cap \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers. Note that $f(b) \in \mathbb{Z}$ and

$$f(1) + 1 = 3a \quad \text{and} \quad f(a) = \tfrac{1}{2}a^2 + \tfrac{5}{2}a - 1.$$

Let

$$P\colon\ x_b, x_{b-1}, \ldots, x_1, x_0 = y_0, y_1, y_2$$

be a path of length $b + 2$. We first construct the caterpillar T'_b of order n from P by adding $2a - b - 2$ new end-vertices, $a - b$ of which are joined to x_{b-1} and $a - 2$ of which are joined to y_1. If $b = 1$, then $T'_1 = S_{a,a+1}$ and so

$$\operatorname{cn}(T'_1) = a + (2a - 1) = f(1).$$

If $2 \leqslant b \leqslant a$, then let

$$N(x_{b-1}) = \{x_{b-2}, x_b, u_1, u_2, \ldots, u_{a-b}\},$$
$$N(y_1) = \{y_0, y_2, v_1, v_2, \ldots, v_{a-2}\}.$$

Observe that $\operatorname{cn}(x_0x_1) = \operatorname{cn}(y_0y_1) = a$ and for $1 \leqslant i \leqslant b - 2$,

$$\operatorname{cn}(x_ix_{i+1}) = a - i.$$

The remaining $2a - b$ edges are pendant edges; so

$$\operatorname{cn}(T'_b) = a + \binom{a+1}{2} - \binom{a-b+2}{2} + (2a - b) = f(b).$$

Now, since f is strictly increasing on $[1, a]$ and $f(1) + 1 \leqslant l \leqslant f(a)$, it follows that there exists a unique integer $b \in [2, a]$ such that

$$f(b-1) + 1 \leqslant l \leqslant f(b).$$

Since $f(b - 1) + 1 = f(b) - (a - b)$, it follows that $l = f(b) - j$ for some j with $0 \leqslant j \leqslant a - b$. We construct $T'_{b,j}$ of order n from T'_b by (i) first deleting the j vertices $u_1, u_2, \ldots, u_j$ and then (ii) adding j new end-vertices $w_1, w_2, \ldots, w_j$ and joining each of them to x_{b-2}. Observe that

$$\operatorname{cn}_{T'_{b,j}}(e) = \operatorname{cn}_{T'_b}(e)$$

for each edge $e \in E(T'_b) \cap E(T'_{b,j}) - \{x_{b-2}x_{b-1}\}$ and

$$\operatorname{cn}_{T'_{b,j}}(x_{b-2}x_{b-1}) = a - b + 2 - j = \operatorname{cn}_{T'_b}(x_{b-2}x_{b-1}) - j.$$

Since the j new edges are pendant edges,

$$\operatorname{cn}(T'_{b,j}) = \operatorname{cn}(T'_b) - j = f(b) - j = l.$$

Case 4. $\frac{1}{2}a^2 + \frac{5}{2}a \leqslant l \leqslant a^2 + a$. First consider the function

$$g(x) = -\tfrac{1}{2}x^2 + \left(a + \tfrac{1}{2}\right)x + \tfrac{1}{2}(a^2 + a)$$

defined on the set $[2, a]$ (of real numbers). Observe that g is continuous and strictly increasing on $[2, a]$. Let $c \in [2, a] \cap \mathbb{Z}$. Note that $g(c) \in \mathbb{Z}$ and

$$g(2) + 1 = \tfrac{1}{2}a^2 + \tfrac{5}{2}a \quad \text{and} \quad g(a) = a^2 + a.$$

Let

$$Q\colon\ x_a, x_{a-1}, \ldots, x_1, x_0 = y_0, y_1, \ldots, y_c$$

be a path of length $a + c$. We first construct the caterpillar T''_c of order n from Q by joining $a - c$ new end-vertices $u_1, u_2, \ldots, u_{a-c}$ to y_{c-1}. If $c = 2$, then $T''_2 = T'_a$ and so

$$\operatorname{cn}(T''_2) = f(a) = \tfrac{1}{2}a^2 + \tfrac{5}{2}a - 1 = g(2).$$

If $3 \leqslant c \leqslant a$, then observe that

$$\begin{aligned} \operatorname{cn}(x_i x_{i+1}) &= a - i \quad \text{for } 0 \leqslant i \leqslant a - 1, \\ \operatorname{cn}(y_i y_{i+1}) &= a - i \quad \text{for } 0 \leqslant i \leqslant c - 2, \end{aligned}$$

and the remaining $a - c + 1$ edges are pendant edges. Hence

$$\operatorname{cn}(T''_c) = 2\binom{a+1}{2} - \binom{a-c+2}{2} + (a - c + 1) = g(c).$$

Now, since g is strictly increasing on $[2, a]$ and $g(2) + 1 \leqslant l \leqslant g(a)$, it follows that there exists a unique integer $c \in [3, a]$ such that

$$g(c-1) + 1 \leqslant l \leqslant g(c).$$

Since $g(c-1) + 1 = g(c) - (a - c)$, it follows that $l = g(c) - j$ for some j with $0 \leqslant j \leqslant a - c$. We construct $T''_{c,j}$ of order n from T''_c by (i) first deleting the j

vertices $u_1, u_2, \ldots, u_j$ and then (ii) adding j new end-vertices $w_1, w_2, \ldots, w_j$ and joining each of them to y_{c-2}. Observe that

$$\mathrm{cn}_{T''_{c,j}}(e) = \mathrm{cn}_{T''_c}(e)$$

for each edge $e \in E(T''_c) \cap E(T''_{c,j}) - \{y_{c-2}y_{c-1}\}$ and

$$\mathrm{cn}_{T''_{c,j}}(y_{c-2}y_{c-1}) = a - c + 2 - j = \mathrm{cn}_{T''_c}(y_{c-2}y_{c-1}) - j.$$

Since the j new edges are pendant edges,

$$\mathrm{cn}(T''_{c,j}) = \mathrm{cn}(T''_c) - j = g(c) - j = l.$$

We now consider the case when n is even. Since the argument for this case is similar to the one employed in the case when n is odd, we only present an outline of the proof in this case. Let $n = 2a$, where $a \geqslant 4$ is an integer. Hence $2a - 1 \leqslant l \leqslant a^2$. Then we apply a similar argument to the following four cases:
(i) $l = 2a - 1$,
(ii) $2a \leqslant l \leqslant 3a - 2$,
(iii) $3a - 1 \leqslant l \leqslant \frac{1}{2}a^2 + \frac{5}{2}a - 3$, and
(iv) $\frac{1}{2}a^2 + \frac{5}{2}a - 2 \leqslant l \leqslant a^2$,
where the corresponding functions $f(x)$ and $g(x)$ are defined as

$$\begin{aligned} f(x) &= -\tfrac{1}{2}x^2 + \big(a + \tfrac{1}{2}\big)x + 2a - 3 \text{ for } x \in [1, a], \\ g(x) &= -\tfrac{1}{2}x^2 + \big(a + \tfrac{1}{2}\big)x + \tfrac{1}{2}(a^2 - a) \text{ for } x \in [3, a]. \end{aligned}$$

Hence for each odd integer $k = 2l - 1$ with $2n - 3 \leqslant k \leqslant \lfloor n^2/2 \rfloor - 1$, there exists a tree T for which $\mathrm{cn}(T) = l$ and so

$$t^+(T) = 2l - 1 = k,$$

providing the desired result. $\square$

We now illustrate the proof of Theorem 2.1 for $n = 11$ (so $a = 5$). Since $19 \leqslant k = 2l - 1 \leqslant 59$, it follows that $10 \leqslant l \leqslant 30$. In this case,

$$f(x) = -\tfrac{1}{2}x^2 + \tfrac{11}{2}x + 9 \quad \text{and} \quad g(x) = -\tfrac{1}{2}x^2 + \tfrac{11}{2}x + 15.$$

b	1	2	3	4	5
$f(b)$	14	18	21	23	24

c	2	3	4	5
$g(c)$	24	27	29	30

There are four cases.

C a s e 1. $l = 2a = 10$. Let $T = K_{1,10}$.

C a s e 2. $2a+1 \leqslant l \leqslant 3a-1$, that is, $11 \leqslant l \leqslant 14$. For $l = 11+i$, where $0 \leqslant i \leqslant 3$, let $T = S_{2+i,11-2-i} = S_{2+i,9-i}$. Thus, in this case, $T \in \{S_{2,9}, S_{3,8}, S_{4,7}, S_{5,6}\}$.

C a s e 3. $3a \leqslant l \leqslant \frac{1}{2}a^2 + \frac{5}{2}a - 1$, that is, $15 \leqslant l \leqslant 24$. In this case, $b \in \{2, 3, 4, 5\}$.

If $b = 2$, then $f(2) - 3 \leqslant l \leqslant f(2)$ and so the possible values of l are

$l = 15 = f(2) - 3$ (and so $T = T'_{2,3}$);

$l = 16 = f(2) - 2$ (and so $T = T'_{2,2}$);

$l = 17 = f(2) - 1$ (and so $T = T'_{2,1}$);

$l = 18 = f(2)$ (and so $T = T'_2 = T'_{2,0}$).

If $b = 3$, then $f(3) - 2 \leqslant l \leqslant f(3)$ and so the possible values of l are

$l = 19 = f(3) - 2$ (and so $T = T'_{3,2}$);

$l = 20 = f(3) - 1$ (and so $T = T'_{3,1}$);

$l = 21 = f(3)$ (and so $T = T'_3 = T'_{3,0}$).

If $b = 4$, then $f(4) - 1 \leqslant l \leqslant f(4)$ and so the possible values of l are

$l = 22 = f(4) - 1$ (and so $T = T'_{4,1}$);

$l = 23 = f(4)$ (and so $T = T'_4 = T'_{4,0}$).

If $b = 5$, then $l = 24 = f(5)$ and $T = T'_5 = T'_{5,0}$.

C a s e 4. $\frac{1}{2}a^2 + \frac{5}{2}a \leqslant l \leqslant a^2 + a$, that is, $25 \leqslant l \leqslant 30$. In this case, $c \in \{3, 4, 5\}$.

If $c = 3$, then $g(3) - 2 \leqslant l \leqslant g(3)$ and so the possible values of l are

$l = 25 = g(3) - 2$ (and so $T = T''_{3,2}$);

$l = 26 = g(3) - 1$ (and so $T = T''_{3,1}$);

$l = 27 = g(3)$ (and so $T = T''_3 = T''_{3,0}$).

If $c = 4$, then $g(4) - 1 \leqslant l \leqslant g(4)$ and so the possible values of l are

$l = 28 = g(4) - 1$ (and so $T = T''_{4,1}$);

$l = 29 = g(4)$ (and so $T = T''_4 = T''_{4,0}$).

If $c = 5$, then $l = 30 = g(5)$ and $T = T''_5 = T''_{5,0} = P_{11}$.

By Theorems 1.3 and 2.1, we have the following corollary.

Corollary 2.2. *A pair n, k of integers is realizable as the order and upper traceable number of a nontrivial tree if and only if $n \geqslant 2$, k is odd, and $2n - 3 \leqslant k \leqslant \lfloor n^2/2 \rfloor - 1$.*

It was shown in [9] that if T is a nontrivial tree, then

$$h^+(T) = t^+(T) + 1 = 2 \sum_{e \in E(T)} \operatorname{cn}(e). \tag{3}$$

Thus the upper Hamiltonian number of a nontrivial tree is always even. Furthermore, if T is a nontrivial tree of order n, then

$$2n - 2 \leqslant h^+(T) \leqslant \lfloor n^2/2 \rfloor . \tag{4}$$

The following corollary is a consequence of (3), (4), and Theorem 2.1.

Corollary 2.3. *A pair n, k of integers is realizable as the order and upper Hamiltonian number of a nontrivial tree if and only if $n \geqslant 3$, k is even, and $2n-2 \leqslant k \leqslant \lfloor n^2/2 \rfloor$.*

3. A characterization

By Theorem 1.3, if T is a nontrivial tree of order n, then $t^+(T) \leqslant \lfloor n^2/2 \rfloor - 1$ and $t^+(T) = \lfloor n^2/2 \rfloor - 1$ if and only if $T = P_n$. For each positive integer $n \geqslant 4$, the integer $\lfloor n^2/2 \rfloor - 2$ is even. By Theorem 1.2, there is no tree T of order $n \geqslant 4$ such that $t^+(T) = \lfloor n^2/2 \rfloor - 2$. Therefore, if T is a tree of order $n \geqslant 4$ and $T \neq P_n$, then

$$t^+(T) \leqslant \lfloor n^2/2 \rfloor - 3.$$

In this section, we characterize all trees of order $n \geqslant 4$ whose upper traceable number is $\lfloor n^2/2 \rfloor - 3$. In order to do this, we first establish a useful lemma. For a vertex v and an edge $e = uw$ in a nontrivial connected graph G, the *distance* between v and e is defined as

$$d(v, e) = \min\{d(v, u), d(v, w)\}.$$

Lemma 3.1. *Let T be a nontrivial tree of order at least 3 with* $\operatorname{diam}(T) = d$. *Then there exists an end-vertex v of T such that*

$$1 \leqslant \operatorname{cn}(T) - \operatorname{cn}(T - v) \leqslant \lceil d/2 \rceil.$$

P r o o f. Assume that T is a tree of order $n \geqslant 3$. Let $M = \max\{\operatorname{cn}(e)\colon\ e \in E(T)\}$ and choose an edge $f \in E(T)$ such that $\operatorname{cn}(f) = M$. If $M = 1$, then T and $T - v$ are stars for every end-vertex v of T. Hence $d = 2$ and $\operatorname{cn}(T) - \operatorname{cn}(T - v) = 1$, so the result holds. Thus we assume that $M \geqslant 2$. Let U be the set of end-vertices of T and

$$l = \min\{d(v, f)\colon\ v \in U\}.$$

Choose a vertex $u \in U$ such that $d(u, f) = l$. Note that $1 \leqslant l \leqslant \lfloor (d-1)/2 \rfloor$. Let $P\colon\ u = v_0, v_1, v_2, \ldots, v_l, v_{l+1}$ be the path of length $l+1$ that has the initial vertex u and terminal edge $f = v_l v_{l+1}$. Let $X = E(P) - \{uv_1\}$, $Y = E(T) - E(P)$, and $T' = T - u$. (Hence $E(T') = X \cup Y$ and $|X| = l$.)

We first show that if $e \in X$, then $0 \leqslant \mathrm{cn}_T(e) - \mathrm{cn}_{T'}(e) \leqslant 1$. Let G_1 and G_2 be the two components of $T - e$ such that u belongs to G_1. If $|V(G_1)| > |V(G_2)|$, then $\mathrm{cn}_T(e) = |V(G_2)| = \mathrm{cn}_{T'}(e)$. If $|V(G_1)| \leqslant |V(G_2)|$, then $\mathrm{cn}_T(e) = |V(G_1)|$ and

$$\mathrm{cn}_{T'}(e) = |V(G_1)| - 1 = \mathrm{cn}_T(e) - 1.$$

Hence $0 \leqslant \mathrm{cn}_T(e) - \mathrm{cn}_{T'}(e) \leqslant 1$ for each edge $e \in X$.

Next we show that if $e \in Y$, then $\mathrm{cn}_T(e) - \mathrm{cn}_{T'}(e) = 0$. Let $e \in Y$ and suppose that G_1 and G_2 are the two components of $T - e$. Necessarily, one of G_1 and G_2 contains the entire P, say G_1 does. If $|V(G_1)| > |V(G_2)|$, then $\mathrm{cn}_T(e) = |V(G_2)| = \mathrm{cn}_{T'}(e)$. Otherwise, $|V(G_1)| \leqslant |V(G_2)|$ and so $\mathrm{cn}_T(e) = |V(G_1)|$. Let H_1 and H_2 be the two components of $T - f$. Then one of H_1 and H_2 contains the entire G_2 and the edge e, say H_2 does. Then $|V(H_2)| \geqslant |V(G_2)| + 1$ and so

$$|V(H_1)| = n - |V(H_2)| \leqslant n - (|V(G_2)| + 1) = |V(G_1)| - 1.$$

This implies that

$$\mathrm{cn}_T(f) \leqslant |V(H_1)| < |V(G_1)| = \mathrm{cn}_T(e),$$

a contradiction. Hence $\mathrm{cn}_T(e) - \mathrm{cn}_{T'}(e) = 0$ for every edge $e \in Y$.

Now observe that

$$\mathrm{cn}(T) = \mathrm{cn}_T(uv_1) + \sum_{e \in X} \mathrm{cn}_T(e) + \sum_{e \in Y} \mathrm{cn}_T(e) = 1 + \sum_{e \in X} \mathrm{cn}_T(e) + \sum_{e \in Y} \mathrm{cn}_{T'}(e)$$

and

$$\mathrm{cn}(T') = \sum_{e \in X} \mathrm{cn}_{T'}(e) + \sum_{e \in Y} \mathrm{cn}_{T'}(e) \leqslant \sum_{e \in X} \mathrm{cn}_T(e) + \sum_{e \in Y} \mathrm{cn}_{T'}(e).$$

Thus $\mathrm{cn}(T) - \mathrm{cn}(T') \geqslant 1$ and

$$\mathrm{cn}(T) - \mathrm{cn}(T') = 1 + \sum_{e \in X} [\mathrm{cn}_T(e) - \mathrm{cn}_{T'}(e)] \leqslant 1 + |X| \leqslant 1 + \lfloor (d-1)/2 \rfloor = \lceil d/2 \rceil,$$

completing the proof. $\square$

For each integer $n \geqslant 4$, let T_n be the caterpillar of order n and $\mathrm{diam}(T_n) = n - 2$ such that the vertex of degree $\Delta(T_n) = 3$ is adjacent to two of the three end-vertices. The caterpillar T_n is shown in Figure 3 for $n = 7$. Next, we show that for each integer $n \geqslant 4$, the caterpillar T_n is the only tree of order n whose upper traceable number is $\lfloor n^2/2 \rfloor - 3$.

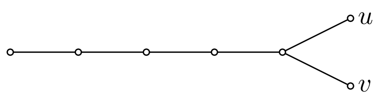

Figure 3. The caterpillar T_n for $n = 7$

Theorem 3.2. *Let T be a tree of order $n \geqslant 4$. Then*

$$t^{+}(T) = \lfloor n^2/2 \rfloor - 3 \quad \textit{if and only if} \quad T = T_n.$$

P r o o f. The result follows immediately for $4 \leqslant n \leqslant 6$, so we may assume that $n \geqslant 7$. We first show that $t^{+}(T_n) = \lfloor n^2/2 \rfloor - 3$. We consider two cases, according to the parity of n.

C a s e 1. n is odd. Then $n = 2k + 1$ for some integer $k \geqslant 3$. Observe that T_n contains three edges with component number 1 and one edge with component number 2. Moreover, for each integer i with $3 \leqslant i \leqslant k$, there are two edges with component number i. Therefore,

$$\mathrm{cn}(T_n) = 2\binom{k+1}{2} - 1 = k^2 + k - 1$$

and so

$$t^{+}(T_n) = 2\,\mathrm{cn}(T_n) - 1 = \lfloor (2k+1)^2/2 \rfloor - 3.$$

C a s e 2. n is even. Then $n = 2k$ for some integer $k \geqslant 4$. Observe that T_n contains three edges with component number 1, one edge with component number 2, and one edge with component number k. Moreover, for each integer i with $3 \leqslant i \leqslant k - 1$, there are two edges with component number i. Therefore,

$$\mathrm{cn}(T_n) = 2\binom{k}{2} - 1 + k = k^2 - 1$$

and so

$$t^{+}(T_n) = 2\,\mathrm{cn}(T_n) - 1 = \lfloor (2k)^2/2 \rfloor - 3.$$

For the converse, let T be a tree of order $n \geqslant 7$ having $t^{+}(T) = \lfloor n^2/2 \rfloor - 3$. Then $T \neq P_n$ and $\mathrm{cn}(T) = \lfloor n^2/4 \rfloor - 1$. We first show that $\mathrm{diam}(T) = n - 2$. Assume, to the contrary, that $d = \mathrm{diam}(T) \leqslant n - 3$. Then $T - v \neq P_{n-1}$ for each end-vertex v of T. Hence $t^{+}(T - v) \leqslant \lfloor (n-1)^2/2 \rfloor - 3$ and consequently $\mathrm{cn}(T - v) \leqslant \lfloor (n-1)^2/4 \rfloor - 1$. Let u be an end-vertex of T such that $1 \leqslant \mathrm{cn}(T) - \mathrm{cn}(T_n) \leqslant \lceil d/2 \rceil$. If n is odd, then $n = 2k + 1$ for some integer $k \geqslant 3$ and

$$\mathrm{cn}(T) - \mathrm{cn}(T - u) \leqslant \lceil d/2 \rceil \leqslant \lceil (n-3)/2 \rceil = k - 1.$$

However,

$$\mathrm{cn}(T) - \mathrm{cn}(T - u) \geqslant (k^2 + k - 1) - (k^2 - 1) = k,$$

a contradiction. If n is even, then $n = 2k$ for some integer $k \geqslant 4$ and

$$\operatorname{cn}(T) - \operatorname{cn}(T-u) \leqslant \lceil d/2 \rceil \leqslant \lceil (n-3)/2 \rceil = k-2.$$

On the other hand,

$$\operatorname{cn}(T) - \operatorname{cn}(T-u) \geqslant (k^2-1) - (k^2-k-1) = k,$$

a contradiction.

Hence $\operatorname{diam}(T) = n-2$ and so T is a caterpillar with three end-vertices obtained from a path $P\colon v_1, v_2, \ldots, v_{n-1}$ of order $n-1$ by joining a new vertex u to v_i for some i with $2 \leqslant i \leqslant n-2$. By symmetry, we may assume that $2 \leqslant i \leqslant \lfloor n/2 \rfloor$.

Case 1. n is odd. Then $n = 2k+1$ for some integer $k \geqslant 3$. Observe that T contains three edges with component number 1, one edge with component number i, and for each integer j with $2 \leqslant j \leqslant k$ and $j \neq i$, there are two edges with component number j. Hence

$$\operatorname{cn}(T) = 2\binom{k+1}{2} + 1 - i = k^2 + k + 1 - i$$

and so $i = 2$, that is, $T = T_n$.

Case 2. n is even. Then $n = 2k$ for some integer $k \geqslant 4$. Observe that T contains three edges with component number 1. If $i = k$, then for each j with $2 \leqslant j \leqslant k-1$, there are two edges with component number j. Hence

$$\operatorname{cn}(T) = 2\binom{k}{2} + 1 = k^2 - k + 1,$$

which is a contradiction since $k^2 - 1 = k^2 - k + 1$ only when $k = 2$. Hence $2 \leqslant i \leqslant k-1$. Then T contains one edge with component number i, one edge with component number k, and for each integer j with $2 \leqslant j \leqslant k-1$ and $j \neq i$, there are two edges with component number j. Hence

$$\operatorname{cn}(T) = 2\binom{k}{2} + 1 + k - i = k^2 + 1 - i$$

and so $i = 2$, that is, $T = T_n$. $\square$

The following is an immediate consequence of Theorems 1.3 and 3.2.

Corollary 3.3. *Let T be a tree of order $n \geqslant 5$. Then*

$$2n-3 \leqslant t^+(T) \leqslant \lfloor n^2/2 \rfloor - 5 \quad \textit{if and only if} \quad T \notin \{P_n, T_n\}.$$

4. Some results for general graphs

If G is a connected graph and H is a connected spanning subgraph of G, then $d_G(u,v) \leqslant d_H(u,v)$ for all $u,v \in V(G) = V(H)$ and so $d_G(s) \leqslant d_H(s)$ for every linear ordering s of vertices of G (or H). Therefore, we have the following observation.

Observation 4.1. *If H is a connected spanning subgraph of a nontrivial graph G, then $t^+(G) \leqslant t^+(H)$. In particular, if T is a spanning tree of G, then $t^+(G) \leqslant t^+(T)$.*

With the aid of Observation 4.1 and Theorem 1.3, we are able to establish sharp upper and lower bounds for $t^+(G)$ for a connected graph G in terms of its order. In order to do this, we first present a formula for the upper traceable number of a cycle in terms of its order (see [9]). For each integer $n \geqslant 3$,

$$t^+(C_n) = \lceil (n-1)^2/2 \rceil. \tag{5}$$

Theorem 4.2. *If G is a connected graph of order $n \geqslant 3$, then*

$$n-1 \leqslant t^+(G) \leqslant \lfloor n^2/2 \rfloor - 1. \tag{6}$$

Furthermore,

(a) $t^+(G) = n-1$ *if and only if* $G = K_n$.

(b) $t^+(G) = \lfloor n^2/2 \rfloor - 1$ *if and only if* $G = P_n$.

Proof. The inequalities in (6) and (a) follow by Theorems 1.1 and 1.3 and Observation 4.1. Thus, it remains only to verify (b). If $G = P_n$, then $t^+(G) = \lfloor n^2/2 \rfloor - 1$ by Theorem 1.3. For the converse, let G be a connected graph of order $n \geqslant 3$ such that $t^+(G) = \lfloor n^2/2 \rfloor - 1$. If G is a tree, then by Theorem 1.3, it follows that $G = P_n$. Now suppose that G is not a tree and let T be a spanning tree of G. By Observation 4.1,

$$t^+(G) \leqslant t^+(T) \leqslant \lfloor n^2/2 \rfloor - 1.$$

Thus $t^+(T) = \lfloor n^2/2 \rfloor - 1$, implying that $T = P_n$. That is, every spanning tree of G is isomorphic to P_n, implying that $G = C_n$. It follows by (5) that

$$\lfloor n^2/2 \rfloor - 1 = t^+(G) = \lceil (n-1)^2/2 \rceil.$$

However, this equality holds only for $n = 2$, a contradiction. Hence $G = P_n$ is the only connected graph of order $n \geqslant 3$ for which $t^+(G) = \lfloor n^2/2 \rfloor - 1$. □

By Theorem 4.2, if G is a connected graph of order n with $t^+(G) = k$, then $n-1 \leqslant k \leqslant \lfloor n^2/2 \rfloor - 1$. The following result shows that there are pairs n, k of integers with $n \geqslant 3$ and $n-1 \leqslant k \leqslant \lfloor n^2/2 \rfloor - 1$ that are not realizable as the order and upper traceable number of any connected graph.

Proposition 4.3. *For each integer $n \geqslant 4$, there is no connected graph of order n whose upper traceable number is $\lfloor n^2/2 \rfloor - 2$.*

Proof. Assume, to the contrary, that there exists a connected graph G of order $n \geqslant 4$ for which $t^+(G) = \lfloor n^2/2 \rfloor - 2$. Since $\lfloor n^2/2 \rfloor - 2$ is even, it follows that G is not a tree by Theorem 1.2. Let T be a spanning tree of G. By Observation 4.1,

$$t^+(G) \leqslant t^+(T) \leqslant \lfloor n^2/2 \rfloor - 1.$$

Thus $t^+(T) = \lfloor n^2/2 \rfloor - 1$ and so $T = P_n$. That is, every spanning tree of G is isomorphic to P_n. Since G is not a tree, it follows that $G = C_n$. By (5),

$$\lfloor n^2/2 \rfloor - 2 = t^+(G) = t^+(C_n) = \lceil (n-1)^2/2 \rceil .$$

However, this equality holds only for $n = 3$, which is a contradiction. □

We are prepared to show that there is no graph G of order $n \geqslant 6$ whose upper traceable number is $\lfloor n^2/2 \rfloor - 4$.

Proposition 4.4. *For each integer $n \geqslant 6$, there is no graph of order n whose upper traceable number is $\lfloor n^2/2 \rfloor - 4$.*

Proof. Assume, to the contrary, that there exists a graph G of order n for which $t^+(G) = \lfloor n^2/2 \rfloor - 4$. Then $G \neq P_n$. In fact, since $\lfloor n^2/2 \rfloor - 4$ is even, G is not a tree. Hence G contains a spanning subgraph H_n of size n. Observe that $H_n \neq C_n$ since otherwise

$$\lfloor n^2/2 \rfloor - 4 = t^+(G) \leqslant t^+(H_n) = \lceil (n-1)^2/2 \rceil ,$$

which holds only for $n \leqslant 5$. Also, since every spanning tree T of G must be isomorphic to either P_n or T_n by Corollary 3.3, it follows that H_n must be the graph obtained from a path $P\colon v_1, v_2, \ldots, v_n$ of order n by joining v_{n-2} and v_n. Moreover, $G = H_n$ since otherwise G contains a spanning tree that is neither P_n nor T_n. If $n = 2k+1$ for some integer $k \geqslant 3$, then $t^+(G) = 2k^2 + 2k - 4$. Consider the linear ordering

$$s_1\colon\ v_{k+2}, v_1, v_{2k+1}, v_2, v_{2k}, v_3, v_{2k-1}, \ldots, v_{k-1}, v_{k+3}, v_k, v_{k+1}$$

of vertices of G and observe that

$$t^+(G) \geqslant d(s_1) = 2k^2 + 2k - 3,$$

which is a contradiction. If $n = 2k$ for some integer $k \geqslant 3$, then $t^+(G) = 2k^2 - 4$. Consider the linear ordering

$$s_2\colon\ v_{k+1}, v_1, v_{2k}, v_2, v_{2k-1}, v_3, v_{2k-2}, \ldots, v_{k-1}, v_{k+2}, v_k$$

of vertices of G and observe that

$$t^+(G) \geqslant d(s_2) = 2k^2 - 3,$$

again, which is a contradiction. Hence there is no graph of order n whose upper traceable number is $\lfloor n^2/2 \rfloor - 4$. □

The proof of Proposition 4.4 actually shows that the graph H_n described in the proof is the only graph of order n that is not a tree and has upper traceable number $\lfloor n^2/2 \rfloor - 3$. Therefore, we obtain the following.

Theorem 4.5. *If G is a nontrivial connected graph of order $n \geqslant 6$, then*
(a) $t^+(G) = \lfloor n^2/2 \rfloor - 1$ *if and only if* $G = P_n$.
(b) $t^+(G) = \lfloor n^2/2 \rfloor - 3$ *if and only if* $G \in \{T_n, H_n\}$.
(c) *If* $G \notin \{P_n, T_n, H_n\}$, *then* $n - 1 \leqslant t^+(G) \leqslant \lfloor n^2/2 \rfloor - 5$.

We conclude with the following question:

Problem 4.6. *Which pairs n, k of integers with $n \geqslant 3$ and $n-1 \leqslant k \leqslant \lfloor n^2/2 \rfloor - 1$ are realizable as the order and upper traceable number, respectively, of some connected graph?*

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References

[1] *T. Asano, T. Nishizeki, T. Watanabe*: An upper bound on the length of a Hamiltonian walk of a maximal planar graph. J. Graph Theory *4* (1980), 315–336.

[2] *T. Asano, T. Nishizeki, T. Watanabe*: An approximation algorithm for the Hamiltonian walk problems on maximal planar graphs. Discrete Appl. Math. *5* (1983), 211–222.

[3] *J. C. Bermond*: On Hamiltonian walks. Congr. Numer. *15* (1976), 41–51.

[4] *G. Chartrand, H. V. Kronk*: On a special class of hamiltonian graphs. Comment. Math. Helv. *44* (1969), 84–88.

[5] *G. Chartrand, V. Saenpholphat, T. Thomas, P. Zhang*: A new look at Hamiltonian walks. Bull. Inst. Combin. Appl. *42* (2004), 37–52.

[6] *G. Chartrand, P. Zhang*: Introduction to Graph Theory. McGraw-Hill, Boston, 2005.

[7] *S. E. Goodman and S. T. Hedetniemi*: On Hamiltonian walks in graphs. SIAM J. Comput. *3* (1974), 214–221.

[8] *F. Okamoto, V. Saenpholphat, P. Zhang*: Measures of traceability in graphs. Math. Bohem. *131* (2006), 63–83.

[9] *F. Okamoto, V. Saenpholphat, P. Zhang*: The upper traceable number of a graph. To appear in Czech. Math. J.

[10] *L. Nebeský*: A generalization of Hamiltonian cycles for trees. Czech. Math. J. *26* (1976), 596–603.

[11] *C. Thomassen*: On randomly Hamiltonian graphs. Math. Ann. *200* (1973), 195–208.

[12] *P. Vacek*: On open Hamiltonian walks in graphs. Arch. Math., Brno *27A* (1991), 105–111.

[13] *P. Vacek*: Bounds of lengths of open Hamiltonian walks. Arch. Math., Brno *28* (1992), 11–16.

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