OSCILLATION OF UNSTABLE SECOND ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH MIXED ARGUMENT

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Abstract. The aim of this paper is to present new oscillatory criteria for the second order neutral differential equation with mixed argument

$$(x(t)-px(t-\tau))''-q(t)x(\sigma(t))=0.$$

The results include also sufficient conditions for bounded and unbounded oscillation of the equations considered.

Keywords: neutral equation, mixed argument

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In this paper we study asymptotic and oscillatory properties of solutions of the second order neutral differential equation with mixed argument

$$(x(t)-px(t-\tau))''-q(t)x(\sigma(t))=0. \tag{1}$$

Throughout the paper we assume:

(H1) $\tau>0$ and $0\leqslant p<1$;

(H2) q, $\sigma\in C(\mathbb{R}_+,\mathbb{R}_+)$, $\lim\limits_{t\to\infty}\sigma(t)=\infty$;

(H3) σ is nondecreasing.

We put $z(t)=x(t)-px(t-\tau)$. By a proper solution of Eq. (1) we mean a function $x\colon [T_x,\infty)\to\mathbb{R}$ which satisfies (1) for all sufficiently large t and $\sup\{|x(t)|\colon\ t\geqslant T\}>0$ for any $T\geqslant T_x$, such that $z(t)$ is twice continuously differentiable. Such a solution is called oscillatory if it has a sequence of zeros tending to infinity; otherwise it is called nonoscillatory. Eq. (1) is said to be oscillatory if all its solutions are oscillatory.

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Recently, many papers devoted to differential equations with neutral terms have appeared. Many good results known for differential equations without neutral terms have been extended to neutral equations. The recent books by D. D. Bainov and D. P. Mishev [1], by I. Győri and G. Ladas [5], and by L. H. Erbe, Q. Kong and B. G. Zhang [4], sumarize some important work in this area and reflect the new developments in the theory of neutral equations.

Usually the authors study bounded or unbounded oscillation of Eq. (1). That is, for *the delayed equation* ($\sigma(t) \leqslant t$) they present sufficient conditions for all bounded solutions of Eq. (1) to be oscillatory. On the other hand, for *the advanced equation* ($\sigma(t) \geqslant t$), sufficient conditions for all unbounded solutions of Eq. (1) to be oscillatory are looked for.

The prototype of results we are about to establish is the following well-known oscillatory criterion for the ordinary differential equation without neutral term ($p = 0$), which is due to Čanturia & Koplatadze [2] (see also Theorem 4.3.1 in [6]):

Theorem A. *Let $p = 0$.*

(i) *If $\sigma(t) < t$ and*

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} [s - \sigma(t)] q(s) \, \mathrm{d}s > 1,$$

then all bounded solutions of (1) *are oscillatory.*

(ii) *If $\sigma(t) > t$ and*

$$\limsup_{t\to\infty} \int_{t}^{\sigma(t)} [\sigma(t) - s] q(s) \, \mathrm{d}s > 1,$$

then all unbounded solutions of (1) *are oscillatory.*

Erbe, Kong and Zhang in [4, Theorem 4.6.1] have shown that Theorem A (i) holds also for (1) with $0 < p < 1$. An attempt to improve Theorem 4.6.1 has been made in [3].

In this paper we are mainly interested in oscillation of Eq. (1), where $\sigma(t)$ is a mixed argument, that is, the function $\sigma(t)$ may oscillate around t or in other words $t - \sigma(t)$ may oscillate around zero. Nonetheless, the oscillatory criteria we are going to derive will be applicable also to delayed and advanced equations.

For simplicity and further references let us denote *the delayed part* and *the advanced part* of $\sigma(t)$ by D_σ and A_σ, respectively. So

$$D_\sigma = \{t \in (t_0, \infty);\ \sigma(t) < t\},$$
$$A_\sigma = \{t \in (t_0, \infty);\ \sigma(t) > t\}.$$

For our main result we assume that the sets D_σ and A_σ are not bounded from above.

The objective of this paper is to provide such criteria which include the coefficient p explicitly.

As is customary, all functional inequalities and equalities are assumed to hold eventually, that is they are fulfilled for all large t.

Theorem 1. *Assume that there exist two sequences $\{\alpha_k\}$ and $\{\beta_k\}$ such that*

$$\begin{aligned} &\alpha_k \in D_\sigma, \quad \alpha_k \to \infty \quad \text{as } k \to \infty,\\ &\beta_k \in A_\sigma, \quad \beta_k \to \infty \quad \text{as } k \to \infty. \end{aligned}$$

Let there exist an integer number $n \geqslant 0$ such that

$$\limsup_{k\to\infty} \int_{\sigma(\alpha_k)}^{\alpha_k} (s - \sigma(\alpha_k))q(s)\,\mathrm{d}s > \frac{1-p}{1-p^{n+1}}. \tag{2}$$

Let there exist an integer number $l \geqslant 0$ such that for all k large enough, $\sigma(\beta_k) > \beta_k + l\tau$ and

$$\limsup_{k\to\infty} \int_{\beta_k}^{\sigma(\beta_k)-l\tau} (\sigma(\beta_k) - l\tau - s)q(s)\,\mathrm{d}s > \frac{1-p}{1-p^{l+1}}. \tag{3}$$

Then Eq. (1) is oscillatory.

P r o o f. Without loss of generality we may assume that $x(t)$ is an eventually positive solution of Eq. (1) on (t_0, ∞). We set

$$z(t) = x(t) - px(t-\tau). \tag{4}$$

We have $z''(t) = q(t)x(\sigma(t)) > 0$ eventually. Therefore $z'(t)$ and $z(t)$ are of constant signs. There are two possibilities for $z(t)$:

(A) $z(t) > 0$ for $t \geqslant t_1 \geqslant t_0$,
(B) $z(t) < 0$ for $t \geqslant t_1$.

Now we shall discuss case (A). Since $z'(t)$ is of constant sign, we have two subcases

(A1) $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$,
(A2) $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$.

C a s e (A1). Eq. (1) can be rewritten in the form

$$z''(t) = q(t)x(\sigma(t)).$$

Using (4), we obtain

$$z''(t) = q(t)z(\sigma(t)) + pq(t)x(\sigma(t) - \tau),$$

provided that $p > 0$. Repeating this procedure we arrive at

$$z''(t) = q(t)\sum_{i=0}^{n} p^i z(\sigma(t) - i\tau) + p^{n+1}q(t)x(\sigma(t) - (n+1)\tau).$$

Therefore

$$z''(t) \geqslant q(t)\sum_{i=0}^{n} p^i z(\sigma(t) - i\tau).$$

For the sake of simplicity denote $\sum\limits_{i=0}^{n} p^i = c_1$. Then using monotonicity of $z(t)$ one gets

$$z''(t) \geqslant c_1 q(t)z(\sigma(t)), \quad t \geqslant t_1. \tag{5}$$

Note that (5) holds also for $p = 0$ with $c_1 = 1$.

It follows from $\sigma(\alpha_k) < \alpha_k$ and the monotonicity of σ that $\sigma(\alpha_k) \in D_\sigma$. Moreover, we claim that the interval $(\sigma(\alpha_k), \alpha_k) \subset D_\sigma$. To verify it, let us admit that there exists $u \in (\sigma(\alpha_k), \alpha_k)$ such that $\sigma(u) \geqslant u$; then monotonicity of σ yields $u \leqslant \sigma(u) \leqslant \sigma(\alpha_k)$, a contradiction.

Integration of (5) from u to α_k ($u \in [\sigma(\alpha_k), \alpha_k]$) yields

$$z'(\alpha_k) - z'(u) \geqslant \int_u^{\alpha_k} c_1 q(s)z(\sigma(s))\,\mathrm{d}s.$$

Then integrating in u from $\sigma(\alpha_k)$ to α_k and using monotonicity of $z(\sigma(t))$ we see that

$$\begin{aligned} 0 > z'(\alpha_k)(\alpha_k - \sigma(\alpha_k)) &\geqslant z(\alpha_k) - z(\sigma(\alpha_k)) + \int_{\sigma(\alpha_k)}^{\alpha_k}\int_u^{\alpha_k} c_1 q(s)z(\sigma(s))\,\mathrm{d}s\,\mathrm{d}u \\ &\geqslant -z(\sigma(\alpha_k)) + \int_{\sigma(\alpha_k)}^{\alpha_k} c_1 q(s)(s - \sigma(\alpha_k))z(\sigma(s))\,\mathrm{d}s \\ &\geqslant z(\sigma(\alpha_k))\Big(c_1\int_{\sigma(\alpha_k)}^{\alpha_k} q(s)(s - \sigma(\alpha_k))\,\mathrm{d}s - 1\Big), \end{aligned}$$

which contradicts (2). Thus the case (A1) is impossible.

Case (A2). Similarly as above it can be shown that

$$z''(t) \geqslant q(t)\sum_{i=0}^{l} p^i z(\sigma(t) - i\tau).$$

Setting $\sum\limits_{i=0}^{l} p^i = c_2$ and then using monotonicity of $z(t)$ one gets

$$z''(t) \geqslant c_2 q(t) z(\sigma(t) - l\tau), \quad t \geqslant t_1.$$

Denoting $\sigma_l(t) = \sigma(t) - l\tau$, the previous inequality can be written as

$$z''(t) \geqslant c_2 q(t) z(\sigma_l(t)). \tag{6}$$

Using monotonicity of σ and the inequality $\sigma(\beta_k) > \beta_k$, similarly as above it can be shown that $(\beta_k, \sigma(\beta_k)) \subset A_\sigma$.

Integration of (6) from β_k to u ($u \in [\beta_k, \sigma(\beta_k)]$) gives

$$z'(u) - z'(\beta_k) \geqslant \int_{\beta_k}^{u} c_2 q(s) z(\sigma_l(s))\,\mathrm{d}s.$$

Now integrating in u from β_k to $\sigma_l(\beta_k)$ and using monotonicity of $z(\sigma_l(t))$, one gets

$$\begin{aligned} 0 \geqslant -z'(\beta_k)(\sigma_l(\beta_k) - \beta_k) &\geqslant z(\beta_k) - z(\sigma_l(\beta_k)) + \int_{\beta_k}^{\sigma_l(\beta_k)} \int_{\beta_k}^{u} c_2 q(s) z(\sigma_l(s))\,\mathrm{d}s\,\mathrm{d}u \\ &\geqslant -z(\sigma_l(\beta_k)) + \int_{\beta_k}^{\sigma_l(\beta_k)} c_2 q(s)(\sigma_l(\beta_k) - s) z(\sigma_l(s))\,\mathrm{d}s \\ &\geqslant z(\sigma_l(\beta_k))\left(c_2 \int_{\beta_k}^{\sigma_l(\beta_k)} q(s)(\sigma_l(\beta_k) - s)\,\mathrm{d}s - 1\right), \end{aligned}$$

which contradicts (3). Therefore case (A2) is not possible either.

Case (B). In this case $z(t) < 0$ and $z''(t) > 0$. This implies $z'(t) < 0$. On the other hand, it follows from $z(t) < 0$ that

$$x(t) < px(t - \tau) < p^2 x(t - 2\tau) < \ldots < p^n x(t - n\tau)$$

for $t \geqslant t_1 + n\tau$ and since $0 \leqslant p < 1$, we conclude that $\lim\limits_{t\to\infty} x(t) = 0$. Therefore, $\lim\limits_{t\to\infty} z(t) = 0$, which is impossible due to monotonicity of z. The proof is complete. $\square$

In the next theorem we modify condition (2) of Theorem 1.

Theorem 2. *Assume that there exist two sequences* $\{\alpha_k\}$ *and* $\{\beta_k\}$ *such that*

$$\alpha_k \in D_\sigma, \quad \alpha_k \to \infty \quad \text{as } k \to \infty,$$
$$\beta_k \in A_\sigma, \quad \beta_k \to \infty \quad \text{as } k \to \infty.$$

Let

$$\limsup_{k\to\infty} \int_{\sigma(\alpha_k)}^{\alpha_k} (s - \sigma(\alpha_k))q(s)\,\mathrm{d}s > 1 - p. \tag{7}$$

Let there exist an integer number $l \geqslant 0$ *such that for all* k *large enough,* $\sigma(\beta_k) > \beta_k + l\tau$ *and*

$$\limsup_{k\to\infty} \int_{\beta_k}^{\sigma(\beta_k)-l\tau} (\sigma(\beta_k) - l\tau - s)q(s)\,\mathrm{d}s > \frac{1-p}{1-p^{l+1}}.$$

Then Eq. (1) *is oscillatory.*

Proof. Denote $a = \limsup\limits_{k\to\infty} \int_{\sigma(\alpha_k)}^{\alpha_k} (s-\sigma(\alpha_k))q(s)\,\mathrm{d}s$. Let an integer n be chosen such that

$$a > \frac{1-p}{1-p^{n+1}}.$$

Then the assertion of this theorem follows immediately from Theorem 1. $\square$

Remark 1. Theorems 1 and 2 guarantee oscillation of (1) provided that the amplitudes of $t-\sigma(t)$ are large enough. The best way how to choose the sequences $\{\alpha_k\}$ and $\{\beta_k\}$ is to set $\{\alpha_k\}$ and $\{\beta_k\}$ to be local minima and local maxima of the function $t-\sigma(t)$, respectively. So if $\sigma \in C^2(t_0,\infty)$ then α_k has to satisfy $\sigma'(\alpha_k) = 1$, $\sigma''(\alpha_k) > 0$; on the other hand, β_k has to obey $\sigma'(\beta_k) = 1$, $\sigma''(\beta_k) < 0$.

Remark 2. Theorems 1 and 2 extend Theorem A to neutral equations, moreover, the coefficient p is explicitly included in our criteria, so Theorems 1 and 2 also improve the corresponding criterion presented in [4].

Example 1. Consider the neutral differential equation with mixed argument

$$(x(t) - px(t-\tau))'' - ax(t - \sin t) = 0, \quad p \in (0,1), \quad a > 0, \quad \tau > 0. \tag{8}$$

Taking Remark 1 into account we put $\alpha_k = \frac{1}{2}\pi + 2k\pi$ and $\beta_k = -\frac{1}{2}\pi + 2k\pi$.

Condition (7) for Eq. (8) reduces to

$$a > 2 - 2p. \tag{9}$$

On the other hand, if $l \geqslant 0$ is such that $1 > l\tau$ then condition (3) for Eq. (8) takes the form

$$a > \frac{2-2p}{(1-l\tau)^2(1-p^{l+1})}. \tag{10}$$

Thus, applying Theorem 2, we conclude that Eq. (8) is oscillatory provided that (10) holds.

The following two results provide criteria for the asymptotic behavior of a solution of Eq. (1) even if (7) or (3) is violated.

Corollary 1. *Assume that there exists a sequence* $\{\alpha_k\}$ *such that*

$$\alpha_k \in D_\sigma, \qquad \alpha_k \to \infty \quad \text{as} \quad k \to \infty.$$

If (7) *holds then every bounded solution of Eq.* (1) *is oscillatory.*

Proof. We assume that $x(t) > 0$ is a bounded solution of Eq. (1). Then $z(t)$ given by (4) is also bounded. Taking the proofs of Theorems 1 and 2 into account we see that (7) guarantees that the case (A1) is impossible. Since $z(t)$ is bounded, the case (A2) is impossible. The case (B) can be eliminated exactly as in the proof of Theorem 1. $\square$

Example 2. Consider the partial case of (8), namely the following neutral differential equation:

$$(x(t) - 0.5x(t-0.5))'' - 1.1x(t-\sin t) = 0.$$

It is easy to see that (9) holds while (10) is violated. It follows from Corollary 1 that every bounded solution of the equation is oscillatory.

Corollary 2. *Assume that there exists a sequence* $\{\beta_k\}$ *such that*

$$\beta_k \in A_\sigma, \qquad \beta_k \to \infty \quad \text{as} \quad k \to \infty.$$

If there exists an integer number $l \geqslant 0$ *such that for all* k *large enough,* $\sigma(\beta_k) > \beta_k + l\tau$ *and* (3) *is satisfied then every unbounded solution of Eq.* (1) *is oscillatory.*

Proof. Let $x(t) > 0$ be an unbounded solution of Eq. (1). Thus, there exists a sequence $\{t_m\}$ such that $\lim\limits_{m\to\infty} t_m = \infty$ and moreover, $x(t_m) = \max\{x(s);\ t_0 \leqslant s \leqslant t_m\}$ and $\lim\limits_{m\to\infty} x(t_m) = \infty$. Then we have

$$\begin{aligned} x(t_m - \tau) &\leqslant \max\{x(s);\ t_0 \leqslant s \leqslant t_m - \tau\} \\ &\leqslant \max\{x(s);\ t_0 \leqslant s \leqslant t_m\} = x(t_m). \end{aligned}$$

Therefore for all large m, we have

$$z(t_m) = x(t_m) - px(t_m - \tau) \geqslant (1-p)x(t_m).$$

Thus $z(t_m) \to \infty$ as $m \to \infty$ and we conclude that $z(t)$ is unbounded. Therefore the case (A1) cannot occur. On the other hand, the case (A2) is excluded by (3). The proof is complete. $\square$

E x a m p l e 3. Consider the following neutral differential equation with mixed argument:

$$(x(t) - 0.4x(t-0.5))'' - x(t - 0.5 - \sin t) = 0.$$

We set $\alpha_k = \frac{1}{2}\pi + 2k\pi$ and $\beta_k = -\frac{1}{2}\pi + 2k\pi$. It is easy to see that (3) holds with $l = 0$. It follows from Corollary 2 that every unbounded solution of the equation is oscillatory. Note that (2) fails.

R e m a r k 3. We have not stipulated that $\sigma(t)$ is a mixed argument in Corollary 1 and Corollary 2. Therefore Corollary 1 and Corollary 2 can be applied also to delayed and advanced equations, respectively, and moreover we can improve some existing results.

Corollary 3. *Assume that* $\sigma(t) < t$. *If*

$$\limsup_{k\to\infty} \int_{\sigma(k)}^{k} (s - \sigma(k))q(s)\,\mathrm{d}s > 1 - p \tag{11}$$

then every bounded solution of Eq. (1) *is oscillatory.*

P r o o f. Set $\alpha_k = k$ in (7) of Corollary 1. $\square$

R e m a r k 4. Corollary 3 improves [4, Theorem 4.6.1] when the right hand side of (11) is the constant 1 instead of $1 - p$.

E x a m p l e 4. Consider the neutral delayed differential equation

$$(x(t) - px(t-\tau))'' - \frac{1}{t^2}x(\lambda t) = 0, \quad p \in (0,1), \quad \lambda \in (0,1), \quad \tau > 0. \tag{12}$$

Condition (11) for Eq. (12) takes the form

$$\limsup_{k\to\infty} \int_{\lambda k}^{k} (s - \lambda k)\frac{1}{s^2}\,\mathrm{d}s > 1 - p,$$

which reduces to

$$\ln\Big(\frac{1}{\lambda}\Big) + \lambda > 2 - p. \tag{13}$$

Therefore Corollary 3 implies that if (13) is fulfilled then all bounded solutions of Eq. (12) are oscillatory. It is easy to see that (13) holds for example for $p = 1/2$ and $\lambda = 1/4$. Note that Theorem 4.6.1 of [4] fails.

Corollary 4. *Assume that $\sigma(t) > t$. If there exists an integer number $l \geqslant 0$ such that for all k large enough, $\sigma(k) > k + l\tau$ and*

$$\limsup_{k\to\infty} \int_k^{\sigma(k)-l\tau} (\sigma(k) - l\tau - s)q(s)\,\mathrm{d}s > \frac{1-p}{1-p^{l+1}}, \tag{14}$$

then every unbounded solution of Eq. (1) is oscillatory.

P r o o f. Set $\beta_k = k$ in (3) of Corollary 2. □

E x a m p l e 5. We consider the neutral advanced differential equation

$$(x(t) - px(t-\tau))'' - \frac{1}{t^2}x(\lambda t) = 0, \quad p \in (0,1), \quad \lambda > 1, \quad \tau > 0. \tag{15}$$

It is easy to see that $\sigma(k) > k + l\tau$ for all integer l. Condition (14) for Eq. (15) has the form

$$\limsup_{k\to\infty} \int_k^{\lambda k - l\tau} (\lambda k - l\tau - s)q(s)\,\mathrm{d}s > \frac{1-p}{1-p^{l+1}}, \tag{16}$$

where l is an arbitrary nonnegative integer. Simple computation shows that (16) is equivalent to

$$\ln\Big(\frac{1}{\lambda}\Big) + \lambda - 1 > \frac{1-p}{1-p^{l+1}}. \tag{17}$$

Due to the fact that l is an arbitrary nonnegative constant, (17) is satisfied if

$$\ln\Big(\frac{1}{\lambda}\Big) + \lambda > 2 - p. \tag{18}$$

Thus condition (18) guarantees that all unbounded solutions of Eq. (15) are oscillatory. It can be easily verified that (18) holds for example for $p = 1/2$ and $\lambda = 2$.

Our results can be easily extended to the neutral equation

$$(x(t) - px(t-\tau))'' - q_1(t)x(\sigma_1(t)) - q_2(t)x(\sigma_2(t)) = 0, \tag{19}$$

where (H1) holds and moreover we assume that

(H4) $q_1, q_2, \sigma_1\sigma_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\lim\limits_{t\to\infty} \sigma_1(t) = \infty$, $\sigma_1(t) \leqslant t$, $\sigma_2(t) \geqslant t$;

(H5) $\sigma_1\sigma_2$ are nondecreasing.

Theorem 3. *Let there exist an integer number* $n \geqslant 0$ *such that*

$$\limsup_{k\to\infty} \int_{\sigma_1(k)}^{k} (s-\sigma_1(k))q_1(s)\,\mathrm{d}s > \frac{1-p}{1-p^{n+1}}.$$

Let there exist an integer number $l \geqslant 0$ *such that for all* k *large enough,* $\sigma_2(k) > k+l\tau$ *and*

$$\limsup_{k\to\infty} \int_{k}^{\sigma_2(k)-l\tau} (\sigma_2(k)-l\tau-s)q_2(s)\,\mathrm{d}s > \frac{1-p}{1-p^{l+1}}.$$

Then Eq. (19) *is oscillatory.*

Proof. Without loss of generality we may assume that $x(t)$ is an eventually positive solution of Eq. (19) on (t_0,∞). Setting $z(t)=x(t)-px(t-\tau)$, we have

$$z''(t)=q_1(t)x(\sigma_1(t))+q_2(t)x(\sigma_2(t))>0.$$

Therefore we are again led to Case (A) (with subcases (A1) and (A2)) and Case (B).

Case (A1). Considering the inequality

$$z''(t)\geqslant q_1(t)x(\sigma_1(t))$$

and using (4), one gets

$$z''(t)\geqslant q_1(t)z(\sigma_1(t))+pq_1(t)x(\sigma_1(t)-\tau).$$

Repeating this process, we get (5) with q,σ replaced by q_1,σ_1, respectively. The rest of the proof of this part runs similarly to the corresponding proof of Theorem 1 with $\alpha_k=k$.

Case (A2). To show that this subcase is infeasible we start from the inequality (6) with q,σ replaced by q_2, σ_2, respectively. Now we only follow all corresponding steps of proof of Theorem 1 with $\beta_k=k$.

The Case (B) can be excluded exactly as in Theorem 1. □

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