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A non–Steady Variational Inequality of the Navier–Stokes–Boussinesq Type with Mixed Boundary Conditions

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Abstract

We prove the global in time existence of a weak solution to the variational inequality of the Navier–Stokes–Boussinesq type, simulating the flow of a viscous heat conductive fluid through the channel, with the so called natural boundary conditions on the outflow for velocity and temperature. The use of the variational inequality enables us to derive an energy–type estimate of the solution.

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Key words: Variational inequality, Navier–Stokes equation, Boussinesq approximation, mixed boundary conditions.

1 Introduction and notation

1.1. The Navier–Stokes–Boussinesq–type initial–boundary value problem. Let $T > 0$ and Ω be a bounded Lipschitzian domain in $\mathbb{R}^3$. Put $Q_T := \Omega \times (0, T)$. We study the mathematical model of a flow of an incompressible viscous heat–conductive fluid in domain Ω in the time interval $(0, T)$. The density ρ is assumed to vary only due to the varying temperature θ according to the law

$$\rho \;=\; \rho_{\rm ref}\left[1 - \alpha(\theta - \theta_{\rm ref})\right],$$

where $\rho_{\rm ref} > 0$ is a constant density corresponding to a chosen constant reference temperature $\theta_{\rm ref}$, and α is the coefficient of thermal expansion. Applying this formula, the acting volume force $\rho\boldsymbol{f}$ can be expressed as

$$\rho\boldsymbol{f} \;=\; \rho_{\rm ref}\left[1 - \alpha(\theta - \theta_{\rm ref})\right]\boldsymbol{f}. \tag{1.1}$$

The balance of momentum is described by the Navier–Stokes equation. Since the density usually differs very little from $\rho_{\rm ref}$ in the range of physically realistic temperatures, we may assume that the flow is incompressible and put $\rho := \rho_{\rm ref} = 1$ in all terms in the Navier–Stokes equations, except for the volume force. Thus, the Navier–Stokes equation takes the form

$$\partial_t\boldsymbol{u} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \;=\; -\nabla p + \left[1 - \alpha(\theta - \theta_{\rm ref})\right]\boldsymbol{f}, \tag{1.2}$$

where $\boldsymbol{u}$ is the velocity, p is the pressure and ν is the kinematic coefficient of viscosity. Similarly, assuming that the density is constant (equal to $\rho_{\rm ref} \equiv 1$), the internal energy in the fluid reduces to the heat, and neglecting the heat production due to viscosity (the Boussinesq approximation), we write the equation of balance of the internal energy in the form

$$\partial_t\theta - \beta\Delta\theta + (\boldsymbol{u}\cdot\nabla)\theta \;=\; g, \tag{1.3}$$

where g is the given specific intensity of sources of heat in Ω and β is the coefficient of heat conduction. All coefficients α, β and ν are supposed to be positive constants. The conservation of mass is described by the equation of continuity

$$\operatorname{div}\boldsymbol{u} \;=\; 0. \tag{1.4}$$

We assume that $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}$, where Γ_1 and Γ_2 are disjoint nonempty subsets of $\partial\Omega$, open in the 2D topology of $\partial\Omega$. Since the 2D measure of Γ_1 is positive, there exists $c_1 > 0$ such that the Friedrichs inequality

$$\| \,.\, \|_2 \;\leq\; c_1 \, \|\nabla .\, \|_2 \tag{1.5}$$

holds for all functions from $W^{1,2}(\Omega)$, whose trace on Γ_1 is equal to zero. (See [16, Theorem 1.1.9].) Set Γ_1 simulates the part of the boundary of Ω where the fluid flows into Ω or the part that coincides with a fixed wall, and it is therefore logical to use the Dirichlet boundary conditions

$$\boldsymbol{u} \;=\; \boldsymbol{u}^*, \quad \theta \;=\; \theta^* \qquad \text{on } \Gamma_1 \times (0,T) \tag{1.6}$$

for the velocity and the temperature. We assume that $\boldsymbol{u}^*$ and θ^* are given functions on $\Gamma_1 \times (0,T)$ that can be extended to $\Omega \times (0,T)$ so that the extended functions, which are for simplicity also denoted by $\boldsymbol{u}^*$ and θ^*, satisfy

$$\boldsymbol{u}^* \in L^\infty\big(0,T;\; \boldsymbol{W}^{1,2}(\Omega)\big) \cap W^{1,2}\big(0,T;\; \boldsymbol{W}^{-1,2}(\Omega)\big), \quad \operatorname{div} \boldsymbol{u}^* = 0 \;\text{ a.e. in } \Omega \times (0,T), \tag{1.7}$$

$$\theta^* \in L^\infty\big(0,T;\; W^{1,2}(\Omega)\big) \cap W^{1,2}\big(0,T;\; W^{-1,2}(\Omega)\big), \tag{1.8}$$

where $\boldsymbol{W}^{-1,2}(\Omega)$ (respectively $W^{-1,2}(\Omega)$) is the dual to $\boldsymbol{W}^{1,2}(\Omega)$ (respectively $W^{1,2}(\Omega)$). It follows from [15, Theorem I.3.1] that function $\boldsymbol{u}^*$, satisfying (1.7), is in $C^0\big([0,T];\; \boldsymbol{L}^2(\Omega)\big)$ and function θ^*, satisfying (1.8), is in $C^0\big([0,T];\; L^2(\Omega)\big)$. We assume that the fluid "essentially" flows out of Ω through Γ_2. (By "essentially" we mean that possible backward flows on Γ_2 cannot be excluded.) As the velocity and the temperature on the outflow cannot be expected to be known in advance, we use the so called "natural" boundary condition for the velocity and the pressure

$$-p\,\boldsymbol{n} + \nu \, \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \;=\; \boldsymbol{F} \qquad \text{on } \Gamma_2 \times (0,T) \tag{1.9}$$

and the Neumann boundary condition for the temperature:

$$\beta \, \frac{\partial \theta}{\partial \boldsymbol{n}} \;=\; G \qquad \text{on } \Gamma_2 \times (0,T). \tag{1.10}$$

The classical formulation of the considered initial–boundary value problem is completed by the initial conditions for the velocity and the temperature

$$\boldsymbol{u} \;=\; \boldsymbol{u}_0, \quad \theta \;=\; \theta_0 \qquad \text{in } \Omega \times \{0\}. \tag{1.11}$$

1.2. Aims of this paper and related previous results. The boundary condition (1.9) is a variant of the so called "do nothing" condition, because it naturally follows from the appropriate weak formulation, see [9]. (In fact, our condition (1.9) is more general, because the "do nothing" boundary condition considered in [9] has the zero right hand side.) The problem is that although we assume that the fluid enters Ω through Γ_1 and essentially leaves through Γ_2, condition (1.9) admits backward flows on Γ_2 that might possibly bring back to Ω an uncontrollable amount of kinetic energy. Thus, condition (1.9) does not enable us to derive an a priori estimate of velocity and temperature, analogous to the usual energy inequality, and consequently, the construction of a global in time weak solution fails. This is why we impose an additional restriction on the size of possible backward flows on Γ_2. However, then the velocity is from the beginning constructed in an arbitrarily large convex subset of a certain function space, and the Navier–Stokes equation (1.2) is therefore logically replaced by an appropriate variational inequality. We prove the global in time existence of a weak solution of the boundary–initial value problem, formulated by means of this inequality, and we also show that if the time derivative of the solution is in $L^2(0,T;\; \boldsymbol{V}^{-1})$ (the space $\boldsymbol{V}^{-1}$ is defined in subsection 1.3) then there exists an associated pressure p so that $\boldsymbol{u}$, p satisfy the Navier–Stokes equation in the sense of distributions in $\Omega \times (0,T)$. Moreover, if $\boldsymbol{u}$ and p are "smooth" then the variational inequality is reduced only to set $\Gamma_2 \times (0,T)$

and it substitutes the boundary condition (1.9). Finally, we also show that if the solution is in the interior of the aforementioned convex set then the boundary condition (1.9) is satisfied point-wise in $\Gamma_1 \times (0,T)$.

Our main result on the existence of a weak solution is formulated in Theorem 1 in Section 3. Note that the same theorem also holds if the heat conductivity of the fluid is not taken into account and one considers just the Navier–Stokes inequality (1.16), without equation (1.17), which represents the weak form of equation (1.3). The main reason why we study the coupled system (1.2), (1.3) for the main unknowns velocity and temperature in this paper, is that in addition to the "natural boundary condition" (1.9) for the velocity, we also use the "natural boundary condition" (1.10) for the temperature on the part Γ_2 of $\partial\Omega$, and we show that in contrast to (1.9), condition (1.10) does not have any negative effect on the energy estimates and there is therefore no need to replace equation (1.3) by an inequality.

The Navier–Stokes equations with the mixed boundary conditions (1.6) and (1.9) (for velocity) have been studied e.g. in papers [12], [13], [1], [2], where the authors prove the solvability under the assumptions that the given data are "small". (The flow of a heat–conductive fluid is also considered in [2].) A modification of condition (1.9) that enables one to derive an energy estimate was suggested in [3]. The Navier–Stokes equations with the boundary condition (1.9) modified in the sense of [3] have been studied in [4], [5] [6] and [17] in connection with flows through profile cascades. While the inflow to the cascade was assumed to be "sufficiently small" the first two papers, a finer treatment of the boundary condition has enabled the authors to improve the results to an "arbitrarily large" inflow and to prove the existence of weak solutions in papers [6] and [17]. Another approach was applied in [10] and [11], where the authors studied the steady Navier–Stokes problems with the mixed boundary conditions (1.6) and (1.9) (for velocity), and artificially restricted the size of a possible backward flow on Γ_2. The velocity was constructed in a "large" convex subset of a certain function space, and instead of the Navier–Stokes equation, the authors used an appropriate steady variational inequality of the Navier–Stokes type. A similar idea is also applied in this paper. Here, however, we consider the non–steady problem and the usual equations (1.2) and (1.4) are completed by the equation (1.3) of heat convection and conduction. Boundary conditions of different types on different parts of $\partial\Omega$ are applied not only to the velocity, but also to the temperature.

1.3. Notation, function spaces. Vector–functions and spaces of vector–functions are denoted by boldface letters.

- Let $1 < r < \infty$ and $k \in \{0\} \cup \mathbb{N}$. The norm of a scalar– or vector– or tensor–valued function, with components in the Lebesgue space $L^r(\Omega)$ (respectively the Sobolev space $W^{k,r}(\Omega)$) is denoted by $\|\,.\,\|_r$ (respectively $\|\,.\,\|_{k,r}$). The norm in $L^r(\Gamma_2)$ is denoted by $\|\,.\,\|_{r;\,\Gamma_2}$.
- Let $\boldsymbol{\mathcal{V}}$ be the space of infinitely differentiable divergence–free vector functions in Ω that have a compact support in $\Omega \cup \Gamma_2$. We denote by $\boldsymbol{H}$ the closure of $\boldsymbol{\mathcal{V}}$ in $\boldsymbol{L}^2(\Omega)$ and by $\boldsymbol{V}^1$ the closure of $\boldsymbol{\mathcal{V}}$ in $\boldsymbol{W}^{1,2}(\Omega)$. ($\boldsymbol{V}^1$ can also be characterized as a space of divergence–free functions from $\boldsymbol{W}^{1,2}(\Omega)$, whose trace on Γ_1 is zero.) The dual space to $\boldsymbol{V}^1$ is denoted by $\boldsymbol{V}^{-1}$. The norm in $\boldsymbol{V}^{-1}$, respectively the duality between elements of $\boldsymbol{V}^{-1}$ and $\boldsymbol{V}^1$, is denoted by $\|\,.\,\|_{-1,2}$, respectively by $\langle\,.\,,\,.\,\rangle$.
- Let $c_2 > 0$ and $a \in (2,4)$ be such numbers that

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_{\Gamma_2} (\boldsymbol{u}^* \cdot \boldsymbol{n})_-^a \, \mathrm{d}S \;<\; c_2. \tag{1.12}$$

  Here, $\boldsymbol{u}^*$ is the function from (1.6) and (1.7), $\boldsymbol{n}$ denotes the outer normal vector field and the subscript "$-$" denotes the negative part. (The negative part is taken "positively", i.e. if $c < 0$ then $c_- = -c$.) We define $\boldsymbol{K}^1$ to be the set of all functions $\boldsymbol{\phi} \in \boldsymbol{V}^1$ such that

$$\int_{\Gamma_2} \big[(\boldsymbol{u}^*(t) + \boldsymbol{\phi}) \cdot \boldsymbol{n}\big]_-^a \, \mathrm{d}S \;\leq\; c_2 \qquad \text{for a.a. } t \in (0,T). \tag{1.13}$$

  The numbers c_2, a and function $\boldsymbol{u}^*$ are fixed throughout the paper. Constant c_2 can be chosen to be arbitrarily large. Obviously, the larger is c_2, the larger is set $\boldsymbol{K}^1$.

- ○ We denote by $\mathscr{W}(0,T)$ the Banach space of of functions $\boldsymbol{w} \in L^2(0,T;\, \boldsymbol{V}^1)$ such that $\partial_t \boldsymbol{w} \in L^2(0,T;\, \boldsymbol{V}^{-1})$, equipped by the norm

$$\|\!|\boldsymbol{w}|\!\| \; := \; \left(\int_0^T \|\boldsymbol{w}\|_{1,2}^2 \, \mathrm{d}t + \int_0^T \|\partial_t \boldsymbol{w}\|_{-1,2}^2 \, \mathrm{d}t \right)^{1/2}.$$

Applying [15, Theorem I.3.1], one can deduce that each function $\boldsymbol{w}$ from $\mathscr{W}(0,T)$ is in $C^0\big([0,T];\, \boldsymbol{H}\big)$, too.
- ○ We denote by $\mathscr{K}(0,T) := \mathscr{W}(0,T) \cap L^2(0,T;\, \boldsymbol{K}^1)$.
- ○ Let $\mathcal{X}$ be the space of infinitely differentiable scalar functions in Ω that have a compact support in $\Omega \cup \Gamma_2$. We denote by X^1 the closure of $\mathcal{X}$ in $W^{1,2}(\Omega)$. The dual space to X^1 is denoted by X^{-1}. The norm in X^{-1}, respectively the duality between elements of X^{-1} and X^1, is also denoted by $\|\,.\,\|_{-1,2}$, respectively by $\langle\, .\, ,\, .\, \rangle$.

Lemma 1. *a)* $\boldsymbol{K}^1$ *is a closed convex subset of* $\boldsymbol{V}^1$*. There exists* $\epsilon_1 > 0$ *such that* $\boldsymbol{K}^1$ *contains the* ϵ_1*–neighborhood of* $\mathbf{0}$ *in* $\boldsymbol{V}^1$*.*

b) $\mathscr{K}(0,T)$ *is a closed convex subset of* $\mathscr{W}(0,T)$*.*

Proof. The fact that $\boldsymbol{K}^1$ is closed easily follows from the fact that the inequality in (1.13) is not strong. In order to prove the convexity of $\boldsymbol{K}^1$, assume that $\boldsymbol{\phi}_1,\, \boldsymbol{\phi}_2 \in \boldsymbol{K}^1$. Let $s \in (0,1)$. Then, applying Minkowski's inequality,

$$\begin{aligned}
&\left(\int_{\Gamma_2} \big[\big(\boldsymbol{u}^*(t) + s\boldsymbol{\phi}_1 + (1-s)\boldsymbol{\phi}_2\big) \cdot \boldsymbol{n}\big]_-^a \, \mathrm{d}S \right)^{\frac{1}{a}} \\
&\quad \leq \left(\int_{\Gamma_2} \big[s\big((\boldsymbol{u}^*(t) + \boldsymbol{\phi}_1) \cdot \boldsymbol{n}\big)_- + (1-s)\big((\boldsymbol{u}^*(t) + \boldsymbol{\phi}_2) \cdot \boldsymbol{n}\big)_- \big]^a \, \mathrm{d}S \right)^{\frac{1}{a}} \\
&\quad \leq s \left(\int_{\Gamma_2} \big[(\boldsymbol{u}^*(t) + \boldsymbol{\phi}_1) \cdot \boldsymbol{n}\big]_-^a \, \mathrm{d}S \right)^{\frac{1}{a}} + (1-s) \left(\int_{\Gamma_2} \big[(\boldsymbol{u}^*(t) + \boldsymbol{\phi}_2) \cdot \boldsymbol{n}\big]_-^a \, \mathrm{d}S \right)^{\frac{1}{a}} \\
&\quad \leq s\, c_2^{1/a} + (1-s)\, c_2^{1/a} \; = \; c_2^{1/a}.
\end{aligned}$$

This shows that $s\boldsymbol{\phi}_1 + (1-s)\boldsymbol{\phi}_2 \in \boldsymbol{K}^1$, too. The existence of $\epsilon_1 > 0$ such that $\boldsymbol{K}^1$ contains the ϵ_1–neighborhood of $\mathbf{0}$ follows from the fact that the inequality (1.12) is strong: let $\boldsymbol{\phi} \in \boldsymbol{V}^1$, $\|\boldsymbol{\phi}\|_{1,2} < \epsilon_1$. Then, applying again Minkowski's inequality, we obtain

$$\begin{aligned}
&\left(\int_{\Gamma_2} \big[(\boldsymbol{u}^*(t) + \boldsymbol{\phi}) \cdot \boldsymbol{n}\big]_-^a \, \mathrm{d}S \right)^{\frac{1}{a}} \; \leq \; \left(\int_{\Gamma_2} \big[(\boldsymbol{u}^*(t) \cdot \boldsymbol{n})_- + (\boldsymbol{\phi} \cdot \boldsymbol{n})_-\big]^a \, \mathrm{d}S \right)^{\frac{1}{a}} \\
&\quad \leq \left(\int_{\Gamma_2} (\boldsymbol{u}^*(t) \cdot \boldsymbol{n})_-^a \, \mathrm{d}S \right)^{\frac{1}{a}} + \left(\int_{\Gamma_2} (\boldsymbol{\phi} \cdot \boldsymbol{n})_-^a \, \mathrm{d}S \right)^{\frac{1}{a}} \\
&\quad \leq \left(\int_{\Gamma_2} (\boldsymbol{u}^*(t) \cdot \boldsymbol{n})_-^a \, \mathrm{d}S \right)^{\frac{1}{a}} + C(a)\, \|\boldsymbol{\phi}\|_{1,2} \; \leq \; \left(\int_{\Gamma_2} (\boldsymbol{u}^*(t) \cdot \boldsymbol{n})_-^a \, \mathrm{d}S \right)^{\frac{1}{a}} + C(a)\, \epsilon_1.
\end{aligned}$$

Since the first term on the right hand side is strongly less than $c_2^{1/a}$, the whole right hand side is less than or equal to $c_2^{1/a}$ if ϵ_1 is sufficiently small. Hence $\boldsymbol{\phi} \in \boldsymbol{K}^1$.

Statement b) is an easy consequence of a). □

1.4. A formal derivation of the variational inequality. Suppose that $\boldsymbol{u}$, θ and p is a "smooth" solution of the problem (1.2)–(1.4), (1.6), (1.9)–(1.11). Let $\boldsymbol{w}$ be a "smooth" function from $[0,T]$ to $\boldsymbol{K}^1$. Writing $\boldsymbol{u}$, respectively θ, in the form $\boldsymbol{u} = \boldsymbol{u}^* + \boldsymbol{v}$, respectively $\theta = \theta^* + \vartheta$, where $\boldsymbol{v}(t) \in \boldsymbol{V}^1$ and $\vartheta(t) \in X^1$ for $t \in (0,T)$, multiplying equation (1.2) by $\boldsymbol{w}-\boldsymbol{v}$, integrating over $\Omega\times(0,T)$ and applying the boundary conditions (1.6) and (1.9), we obtain

$$\begin{aligned}\int_0^T\!\!\int_\Omega \partial_t(\boldsymbol{u}^*+\boldsymbol{v})\cdot(\boldsymbol{w}-\boldsymbol{v})\;\mathrm{d}\boldsymbol{x}\,\mathrm{d}t+\int_0^T\!\!\int_\Omega(\boldsymbol{u}^*+\boldsymbol{v})\cdot\nabla(\boldsymbol{u}^*+\boldsymbol{v})\cdot(\boldsymbol{w}-\boldsymbol{v})\;\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ +\int_0^T\!\!\int_\Omega \nu\nabla(\boldsymbol{u}^*+\boldsymbol{v})\cdot\nabla(\boldsymbol{w}-\boldsymbol{v})\;\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ =\int_0^T\!\!\int_\Omega[1-\alpha(\theta^*+\vartheta-\theta_{\mathrm{ref}})]\,\boldsymbol{f}\cdot(\boldsymbol{w}-\boldsymbol{v})\;\mathrm{d}\boldsymbol{x}\,\mathrm{d}t+\int_0^T\!\!\int_{\Gamma_2}\boldsymbol{F}\cdot(\boldsymbol{w}-\boldsymbol{v})\;\mathrm{d}S\,\mathrm{d}t. \qquad (1.14)\end{aligned}$$

The first term on the left hand side satisfies

$$\begin{aligned}\int_0^T\!\!\int_\Omega \partial_t(\boldsymbol{u}^*+\boldsymbol{v})\cdot(\boldsymbol{w}-\boldsymbol{v})\;\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ =\int_0^T\!\!\int_\Omega \partial_t(\boldsymbol{u}^*+\boldsymbol{w})\cdot(\boldsymbol{w}-\boldsymbol{v})\;\mathrm{d}\boldsymbol{x}\,\mathrm{d}t+\int_0^T\!\!\int_\Omega \partial_t(\boldsymbol{v}-\boldsymbol{w})\cdot(\boldsymbol{w}-\boldsymbol{v})\;\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ =\int_0^T\!\!\int_\Omega \partial_t(\boldsymbol{u}^*+\boldsymbol{w})\cdot(\boldsymbol{w}-\boldsymbol{v})\;\mathrm{d}\boldsymbol{x}\,\mathrm{d}t-\frac{1}{2}\,\|\boldsymbol{w}(T)-\boldsymbol{v}(T)\|_2^2+\frac{1}{2}\,\|\boldsymbol{w}(0)-\boldsymbol{v}_0\|_2^2\\ \le\int_0^T\!\!\int_\Omega \partial_t(\boldsymbol{u}^*+\boldsymbol{w})\cdot(\boldsymbol{w}-\boldsymbol{v})\;\mathrm{d}\boldsymbol{x}\,\mathrm{d}t+\frac{1}{2}\,\|\boldsymbol{w}(0)-\boldsymbol{v}_0\|_2^2, \qquad (1.15)\end{aligned}$$

where $\boldsymbol{v}_0:=\boldsymbol{u}_0-\boldsymbol{u}^*(0)$. From now on, we consider $\boldsymbol{w}\in\mathscr{K}(0,T)$. Recall that $\partial_t\boldsymbol{u}^*$ is supposed to be in $L^2\big(0,T;\,\boldsymbol{W}^{-1,2}(\Omega)\big)$. This implies that $\partial_t(\boldsymbol{u}^*+\boldsymbol{w})$ is in $L^2(0,T;\,\boldsymbol{V}^{-1})$. Thus, we can write the integral of $\partial_t(\boldsymbol{u}^*+\boldsymbol{w})\cdot(\boldsymbol{w}-\boldsymbol{v})$ in Ω as the duality $\langle\partial_t(\boldsymbol{u}^*+\boldsymbol{w}),\boldsymbol{w}-\boldsymbol{v}\rangle$. Substituting from (1.15) to (1.14), we obtain

$$\begin{aligned}\int_0^T\big\langle\partial_t(\boldsymbol{u}^*+\boldsymbol{w}),\boldsymbol{w}-\boldsymbol{v}\big\rangle\;\mathrm{d}\boldsymbol{x}\,\mathrm{d}t+\int_0^T\!\!\int_\Omega(\boldsymbol{u}^*+\boldsymbol{v})\cdot\nabla(\boldsymbol{u}^*+\boldsymbol{v})\cdot(\boldsymbol{w}-\boldsymbol{v})\;\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ +\int_0^T\!\!\int_\Omega \nu\nabla(\boldsymbol{u}^*+\boldsymbol{v})\cdot\nabla(\boldsymbol{w}-\boldsymbol{v})\;\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\;\ge\;\int_0^T\!\!\int_\Omega[1-\alpha(\theta^*+\vartheta-\theta_{\mathrm{ref}})]\,\boldsymbol{f}\cdot(\boldsymbol{w}-\boldsymbol{v})\;\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ +\int_0^T\!\!\int_{\Gamma_2}\boldsymbol{F}\cdot(\boldsymbol{w}-\boldsymbol{v})\;\mathrm{d}S\,\mathrm{d}t-\frac{1}{2}\,\|\boldsymbol{w}(0)-\boldsymbol{v}_0\|_2^2. \qquad (1.16)\end{aligned}$$

The fact that (1.16) is an inequality, and not an equation, gives us a freedom to impose an additional condition on the solution $\boldsymbol{u}$: we require $\boldsymbol{v}(t)\equiv\boldsymbol{u}(t)-\boldsymbol{u}^*(t)$ to be in $\boldsymbol{K}^1$ for a.a. $t\in(0,T)$.

The weak form of equation (1.3) can be derived in a usual way: we write θ in the form $\theta=\theta^*+\vartheta$, multiply equation (1.3) by a test function $\psi\in C^\infty\big([0,T];\,X^1\big)$ such that $\psi(T)=0$, integrate in $\Omega\times(0,T)$, apply the integration by parts, apply the boundary condition (1.10) and write $\langle g,\psi\rangle$ instead of $\int_\Omega g\psi\;\mathrm{d}\boldsymbol{x}$. We get

$$\begin{aligned}-\int_0^T\!\!\int_\Omega(\theta^*+\vartheta)\,\partial_t\psi\;\mathrm{d}\boldsymbol{x}\,\mathrm{d}t+\int_0^T\!\!\int_\Omega(\boldsymbol{u}^*+\boldsymbol{v})\cdot\nabla(\theta^*+\vartheta)\,\psi\;\mathrm{d}\boldsymbol{x}\,\mathrm{d}t+\beta\int_0^T\!\!\int_\Omega\nabla(\theta^*+\vartheta)\cdot\nabla\psi\;\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ =\int_0^T\!\!\int_{\Gamma_2}G\psi\;\mathrm{d}S\,\mathrm{d}t+\int_0^T\langle g,\psi\rangle\;\mathrm{d}t+\int_\Omega(\theta^*(0)+\vartheta_0)\psi(0)\;\mathrm{d}\boldsymbol{x}, \qquad (1.17)\end{aligned}$$

where $\vartheta_0:=\theta_0-\theta^*(0)$. The functions $\boldsymbol{v}$ and ϑ are from now on considered to be the new unknowns.

1.5. The initial–boundary value problem $(\mathcal{P})$. *Given*

a) $\boldsymbol{u}^*$ *and* θ^**, satisfying (1.7) and (1.8),*

b) $\boldsymbol{u}_0\in\boldsymbol{L}^2(\Omega)$ *such that* $\operatorname{div}\boldsymbol{u}_0=0$ *in* Ω *in the sense of distributions,* $\theta_0\in L^2(\Omega)$*,*

c) $\boldsymbol{f}\in L^{\frac{4r_1}{6-r_1}}\big(0,T;\,\boldsymbol{L}^{s_1}(\Omega)\big)$ *for some* r_1 *and* s_1 *such that* $r_1^{-1}+s_1^{-1}=\frac{5}{6}$ *and* $2\le r_1<6$*,* *d)* $\boldsymbol{F} \in L^2\big(0,T;\, \boldsymbol{L}^{4/3}(\Gamma_2)\big)$, $g \in L^2(0,T;\, X^{-1})$ *and* $G \in L^2\big(0,T;\, L^{4/3}(\Gamma_2)\big)$.

We look for $\boldsymbol{v}$ *and* ϑ *such that* $\boldsymbol{v} \in L^\infty(0,T;\, \boldsymbol{H}) \cap L^2(0,T;\, \boldsymbol{K}^1)$, $\vartheta \in L^\infty\big(0,T;\, L^2(\Omega)\big) \cap L^2(0,T;\, X^1)$ *and the inequality (1.16) and the equation (1.17) are satisfied for all test functions* $\boldsymbol{w} \in \mathscr{K}(0,T)$ *and* $\psi \in C^\infty\big([0,T];\, X^1\big)$ *such that* $\psi(T) = 0$.

The assumptions on function $\boldsymbol{f}$ in item c) imply that the first integral on the right hand side of (1.16) converges. Similarly, the assumptions on $\boldsymbol{F}$, g and G in item d) guarantee the convergence of the second integral on the right hand side of (1.16) and the corresponding integrals in (1.17). If $(\boldsymbol{v}, \vartheta)$ is a solution of problem $(\mathcal{P})$ then, by analogy with the theory of the Navier–Stokes equations, a distribution p in $\Omega \times (0,T)$ is said to be an *associated pressure* if $\boldsymbol{u} \equiv \boldsymbol{u}^* + \boldsymbol{v}$, $\theta \equiv \theta^* + \vartheta$ and p satisfy equation (1.2) in the sense of distributions in $\Omega \times (0,T)$.

As it is usual in the theory of weak solutions of partial differential equations, one can show that if all the given data and the solution $(\boldsymbol{v}, \vartheta)$ are sufficiently smooth and integrable, then $\boldsymbol{u} \equiv \boldsymbol{u}^* + \boldsymbol{v}$ and $\theta \equiv \theta^* + \vartheta$ satisfy equations (1.3) and (1.4), boundary conditions (1.6) and (1.10) and initial conditions (1.11) in a strong sense in $\Omega \times (0,T)$ or on $\Gamma_1 \times (0,T)$ or on $\Gamma_2 \times (0,T)$ or in $\Omega \times \{0\}$. It is, however, not clear at the first sight whether the variational inequality (1.16) in some sense involves the momentum equation (1.2) and in which sense problem ($\mathcal{P}$) also involves the boundary condition (1.9). The next two lemmas concern these questions.

Lemma 2. *Let* $(\boldsymbol{v}, \vartheta)$ *be a solution of problem (*$\mathcal{P}$*). Let* $\partial_t \boldsymbol{v}$ *be, in addition, in* $L^2(0,T;\, \boldsymbol{V}^{-1})$ *and* $\overline{p}$ *be a given function from* $L^1(0,T)$. *Then an associated pressure* p *exists as a function from* $L^1\big(0,T;\, L^2(\Omega)\big)$*, such that* $\int_\Omega p(t)\, \mathrm{d}S = \overline{p}(t)$ *a.e. in* $(0,T)$.

Proof. Consider function $\boldsymbol{w}$ in (1.16) in the form $\boldsymbol{w} = \boldsymbol{v} + \chi(t)\boldsymbol{\phi}$, where $\chi \in C^1\big([0,T]\big)$, $\chi(0) = \chi(T) = 0$ and $\boldsymbol{\phi} \in \boldsymbol{W}^{1,2}_{0,\sigma}(\Omega)$ (the space of all divergence–free functions from $\boldsymbol{W}^{1,2}(\Omega)$ with the zero trace on $\partial\Omega$). Then (1.16) yields

$$\int_0^T \big\langle \partial_t(\boldsymbol{u} + \chi(t)\boldsymbol{\phi}), \chi(t)\boldsymbol{\phi} \big\rangle \, \mathrm{d}t + \int_0^T \chi(t) \int_\Omega \big[\boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{\phi} + \nu \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{\phi}\big] \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\ \geq \int_0^T \chi(t) \int_\Omega [1 - \alpha(\theta - \theta_{\mathrm{ref}})]\, \boldsymbol{f} \cdot \boldsymbol{\phi} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t. \tag{1.18}$$

Since

$$\int_0^T \big\langle \partial_t(\boldsymbol{u} + \chi(t)\boldsymbol{\phi}), \chi(t)\boldsymbol{\phi} \big\rangle \, \mathrm{d}t \;=\; \int_0^T \chi(t) \, \langle \partial_t \boldsymbol{u}, \boldsymbol{\phi} \rangle \, \mathrm{d}\boldsymbol{x},$$

(1.18) takes the form

$$\int_0^T \chi(t) \, \langle \partial_t \boldsymbol{u}, \boldsymbol{\phi} \rangle \, \mathrm{d}t + \int_0^T \chi(t) \int_\Omega \big[\boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{\phi} + \nu \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{\phi}\big] \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\ \geq \int_0^T \chi(t) \int_\Omega [1 - \alpha(\theta - \theta_{\mathrm{ref}})]\, \boldsymbol{f} \cdot \boldsymbol{\phi} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t. \tag{1.19}$$

Considering (1.19) with function $-\boldsymbol{\phi}$ instead of $\boldsymbol{\phi}$, we observe that (1.19) holds as an equation. Moreover, as it holds for all $\chi \in C^1\big[0,T]\big)$ such that $\chi(0) = \chi(T) = 0$, we deduce that the equation

$$\langle \partial_t \boldsymbol{u}, \boldsymbol{\phi} \rangle + \int_\Omega \big[\boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{\phi} + \nu \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{\phi}\big] \, \mathrm{d}\boldsymbol{x} \;=\; \int_\Omega [1 - \alpha(\theta - \theta_{\mathrm{ref}})]\, \boldsymbol{f} \cdot \boldsymbol{\phi} \, \mathrm{d}\boldsymbol{x}$$

holds at a.a. points $t \in (0,T)$. Integrating with respect to time from 0 to t, we get

$$\int_\Omega [\boldsymbol{u}(t) - \boldsymbol{u}_0] \cdot \boldsymbol{\phi} \, \mathrm{d}\boldsymbol{x} + \int_0^t \int_\Omega \big[\boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{\phi} + \nu \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{\phi}\big] \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\tau \;=\; \int_0^t \int_\Omega [1 - \alpha(\theta - \theta_{\mathrm{ref}})]\, \boldsymbol{f} \cdot \boldsymbol{\phi} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\tau.$$

Put

$$\mathfrak{F}_t(\boldsymbol{\psi}) := \int_\Omega [\boldsymbol{u}(t) - \boldsymbol{u}_0] \cdot \boldsymbol{\psi} \, \mathrm{d}\boldsymbol{x} + \int_0^t \int_\Omega \big\{ \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{\psi} + \nu \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{\psi} - [1 - \alpha(\theta - \theta_{\mathrm{ref}})] \boldsymbol{f} \cdot \boldsymbol{\psi} \big\} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\tau.$$

$\mathfrak{F}_t$ is a bounded linear functional on $\boldsymbol{W}_0^{1,2}(\Omega)$, that vanishes on $\boldsymbol{W}_{0,\sigma}^{1,2}(\Omega)$. Thus, due to [8, Corollary III.5.1], there exists $P(t) \in L^2(\Omega)$ such that $\mathfrak{F}_t(\psi) = \int_\Omega P(t) \operatorname{div} \boldsymbol{\psi} \, \mathrm{d}\boldsymbol{x}$. Substituting here for $\mathfrak{F}_t$, differentiating this identity with respect to t (in the sense of distributions) in $(0,T)$ and denoting $p := \partial_t P$, we deduce that $\boldsymbol{u}$, θ and p satisfy equation (1.2) in the sense of distributions in $\Omega \times (0,T)$.

It follows from the assumptions (1.7) and (1.8) (on functions $\boldsymbol{u}^*$ and θ^*) and the definition of the solution $(\boldsymbol{v}, \vartheta)$ (which includes the information on the integrability of $\boldsymbol{v}$ and ϑ) that $\Delta \boldsymbol{u} \in L^2\big(0,T;\, \boldsymbol{W}_0^{-1,2}(\Omega)\big)$ (where $\boldsymbol{W}_0^{-1,2}(\Omega)$ is the dual to $\boldsymbol{W}_0^{1,2}(\Omega)$), $\boldsymbol{u} \cdot \nabla \boldsymbol{u} \in L^{4/3}\big(0,T;\, \boldsymbol{W}_0^{-1,2}(\Omega)\big)$ and $[1 - \alpha(\theta - \theta_{\mathrm{ref}})] \boldsymbol{f}$ are in $L^1\big(0,T;\, \boldsymbol{W}_0^{-1,2}(\Omega)\big)$. The distributional derivative $\partial_t \boldsymbol{v}$ is supposed to be in $L^2(0,T;\, \boldsymbol{V}^{-1})$, which implies that $\partial_t \boldsymbol{u} \equiv \partial_t(\boldsymbol{u}^* + \boldsymbol{v})$ is in $L^2\big(0,T;\, \boldsymbol{W}_0^{-1,2}(\Omega)\big)$. Consequently, ∇p (the distributional gradient of p) belongs to $L^1\big(0,T;\, \boldsymbol{W}_0^{-1,2}(\Omega)\big)$ and p can be therefore chosen so that it belongs to $L^1\big(0,T;\, \boldsymbol{L}^2(\Omega)\big)$. Since p is unique up to an additive function of t, this function can be chosen so that p satisfies the condition $\int_\Omega p(t) \, \mathrm{d}S = \overline{p}(t)$ a.e. in $(0,T)$. □

Lemma 3. *a) Let $(\boldsymbol{v}, \vartheta)$ be a solution of problem ($\mathcal{P}$) and let all the terms $\partial_t \boldsymbol{u}$, $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$, $\Delta \boldsymbol{u}$ and $[1 - \alpha(\theta - \theta_{\mathrm{ref}})] \boldsymbol{f}$ (where $\boldsymbol{u} \equiv \boldsymbol{u}^* + \boldsymbol{v}$ and $\theta \equiv \theta^* + \vartheta$) belong to $L^2\big(0,T;\, \boldsymbol{L}^2(\Omega)\big)$. Then the associated pressure p exists as a function from $L^2\big(0,T;\, W^{1,2}(\Omega)\big)$ and the inequality*

$$\int_0^T \int_{\Gamma_2} \Big(\nu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} - p \boldsymbol{n} - \boldsymbol{F} \Big) \cdot (\boldsymbol{q} - \boldsymbol{v}) \, \mathrm{d}S \, \mathrm{d}t \ \geq \ 0 \tag{1.20}$$

holds for all $\boldsymbol{q} \in L^2(0,T;\, \boldsymbol{K}^1)$.

b) If, in addition to the assumptions in item a), $\boldsymbol{v}(t)$ lies uniformly in the interior of $\boldsymbol{K}^1$ in the sense that there exists $\epsilon_2 > 0$ such that $\boldsymbol{v}(t) + \boldsymbol{\phi} \in \boldsymbol{K}^1$ for all $\boldsymbol{\phi}$ from the ϵ_2–neighborhood of $\mathbf{0}$ in $\boldsymbol{V}^1$ and a.a. $t \in (0,T)$ then the boundary condition (1.9) is satisfied point-wise a.e. in $\Gamma_2 \times (0,T)$.

Proof. a) An associated pressure p exists due to Lemma 2. It follows from equation (1.2) and the assumptions in item a) that $\nabla p \in L^2\big(0,T;\, \boldsymbol{L}^2(\Omega)\big)$. Writing $\boldsymbol{u}$ instead of $\boldsymbol{u}^* + \boldsymbol{v}$ and θ instead of $\theta^* + \vartheta$ in (1.16) and expressing the first integral in (1.16) by means of the identities

$$\begin{aligned} \int_0^T \big\langle \partial_t(\boldsymbol{u}^* + \boldsymbol{w}), \boldsymbol{w} - \boldsymbol{v} \big\rangle \, \mathrm{d}t \ &= \int_0^T \big\langle \partial_t(\boldsymbol{u}^* + \boldsymbol{v}), \boldsymbol{w} - \boldsymbol{v} \big\rangle \, \mathrm{d}t + \int_0^T \big\langle \partial_t(\boldsymbol{w} - \boldsymbol{v}), \boldsymbol{w} - \boldsymbol{v} \big\rangle \, \mathrm{d}t \\ &= \int_0^T \int_\Omega \partial_t \boldsymbol{u} \cdot (\boldsymbol{w} - \boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \frac{1}{2} \, \|\boldsymbol{w}(T) - \boldsymbol{v}(T)\|_2^2 - \frac{1}{2} \, \|\boldsymbol{w}(0) - \boldsymbol{v}_0\|_2^2, \end{aligned}$$

we obtain

$$\begin{aligned} &\int_0^T \int_\Omega \partial_t \boldsymbol{u} \cdot (\boldsymbol{w} - \boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \int_0^T \int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot (\boldsymbol{w} - \boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\ &\qquad + \int_0^T \int_\Omega \nu \nabla \boldsymbol{u} \cdot \nabla(\boldsymbol{w} - \boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t - \int_0^T \int_\Omega [1 - \alpha(\theta - \theta_{\mathrm{ref}})] \boldsymbol{f} \cdot (\boldsymbol{w} - \boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\ &\qquad \geq \int_0^T \int_{\Gamma_2} \boldsymbol{F} \cdot (\boldsymbol{w} - \boldsymbol{v}) \, \mathrm{d}S \, \mathrm{d}t - \frac{1}{2} \, \|\boldsymbol{w}(T) - \boldsymbol{v}(T)\|_2^2. \end{aligned} \tag{1.21}$$

Let $\boldsymbol{q}$ be at first a function from the same class as $\boldsymbol{w}$, i.e. $\boldsymbol{q} \in L^2(0,T;\, \boldsymbol{K}^1)$, $\partial_t \boldsymbol{q} \in L^2(0,T;\, \boldsymbol{V}^{-1})$. Let $\xi \in (0,1)$. If we use $\boldsymbol{w}$ in (1.21) in the form $\boldsymbol{w} = \xi \boldsymbol{q} + (1 - \xi) \boldsymbol{v}$, divide the inequality by ξ and consider $\xi \to 0+$, we get

$$\int_0^T \int_\Omega \partial_t \boldsymbol{u} \cdot (\boldsymbol{q} - \boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \int_0^T \int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot (\boldsymbol{q} - \boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t$$

$$
\begin{aligned}
&+\int_0^T\!\!\int_\Omega \nu\nabla\boldsymbol{u}\cdot\nabla(\boldsymbol{q}-\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t-\int_0^T\!\!\int_\Omega [1-\alpha(\theta-\theta_{\mathrm{ref}})]\,\boldsymbol{f}\cdot(\boldsymbol{q}-\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
\geq\ &\int_0^T\!\!\int_{\Gamma_2}\boldsymbol{F}\cdot(\boldsymbol{q}-\boldsymbol{v})\,\mathrm{d}S\,\mathrm{d}t.
\end{aligned}
$$

Applying the integration by parts to the third integral on the left hand side, we obtain

$$
\begin{aligned}
&\int_0^T\!\!\int_\Omega \big\{\partial_t\boldsymbol{u}-\nu\Delta\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}-\big[1-\alpha(\theta-\theta_{\mathrm{ref}})\big]\,\boldsymbol{f}\big\}\cdot(\boldsymbol{q}-\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t+\int_0^T\!\!\int_{\Gamma_2}\nu\,\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}\cdot(\boldsymbol{q}-\boldsymbol{v})\,\mathrm{d}S\,\mathrm{d}t\\
&\qquad-\int_0^T\!\!\int_{\Gamma_2}\boldsymbol{F}\cdot(\boldsymbol{q}-\boldsymbol{v})\,\mathrm{d}S\,\mathrm{d}t\ \geq\ 0,\\
&-\int_0^T\!\!\int_\Omega \nabla p\cdot(\boldsymbol{q}-\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t+\int_0^T\!\!\int_{\Gamma_2}\nu\,\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}\cdot(\boldsymbol{q}-\boldsymbol{v})\,\mathrm{d}S\,\mathrm{d}t-\int_0^T\!\!\int_{\Gamma_2}\boldsymbol{F}\cdot(\boldsymbol{q}-\boldsymbol{v})\,\mathrm{d}S\,\mathrm{d}t\ \geq\ 0.
\end{aligned}
$$

Integrating by parts in the first integral, we get (1.20). Since the set of functions $\boldsymbol{q}\in L^2(0,T;\,\boldsymbol{K}^1)$, such that $\partial_t\boldsymbol{q}\in L^2(0,T;\,\boldsymbol{V}^{-1})$, is dense in $L^2(0,T;\,\boldsymbol{K}^1)$, (1.20) holds for all $\boldsymbol{q}\in L^2(0,T;\,\boldsymbol{K}^1)$.

b) Let $\boldsymbol{h}\in L^2(0,T;\,\boldsymbol{V}^1)$ such that $\partial_t\boldsymbol{h}\in L^2(0,T;\,\boldsymbol{V}^{-1})$. Then there exists $\delta>0$ such that both $\boldsymbol{q}=\boldsymbol{v}+\delta\boldsymbol{h}$ and $\boldsymbol{q}=\boldsymbol{v}-\delta\boldsymbol{h}$ are admissible test functions in (1.20). Using these test functions and dividing (1.20) by δ, we obtain the equation

$$
\int_0^T\!\!\int_{\Gamma_2}\Big(\nu\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}-p\boldsymbol{n}-\boldsymbol{F}\Big)\cdot\boldsymbol{h}\,\mathrm{d}S\,\mathrm{d}t\ =\ 0.
$$

Since $\nu\partial\boldsymbol{u}/\partial\boldsymbol{n}-p\boldsymbol{n}-\boldsymbol{F}\in L^2\big(0,T;\,\boldsymbol{L}^{4/3}(\Gamma_2)\big)$ and the set of traces of the functions $\boldsymbol{h}$ on Γ_2 is dense in $L^2\big(0,T;\,\boldsymbol{L}^4(\Gamma_2)\big)$, we deduce that condition (1.9) holds a.e. in $\Gamma_2\times(0,T)$. □

2 Approximations and their estimates

2.1. Auxiliary notions and considerations. Let us choose $\kappa\in\mathbb{R}$ such that $\kappa>\frac{3}{2}-2/a$ and $\frac{1}{2}+1/a<\kappa<1$. Denote by $\boldsymbol{V}^\kappa$ a closure of $\boldsymbol{\mathcal{V}}$ in $\boldsymbol{W}^{\kappa,2}(\Omega)$, which is the interpolation space $[\boldsymbol{L}^2(\Omega),\boldsymbol{W}^{1,2}(\Omega)]_\kappa$. Then $\boldsymbol{V}^1$ is compactly imbedded to $\boldsymbol{V}^\kappa$ and there exists a continuous operator of traces from $\boldsymbol{V}^\kappa$ to $\boldsymbol{L}^{4/(3-2\kappa)}(\partial\Omega)$. Particularly, due to the choice of κ, the operator of traces maps continuously $\boldsymbol{V}^\kappa$ to $\boldsymbol{L}^a(\Gamma_2)$ and to $\boldsymbol{L}^{2a/(a-1)}(\Gamma_2)$. Define $\boldsymbol{K}^\kappa$ to be the set of all functions $\boldsymbol{\phi}\in\boldsymbol{V}^\kappa$ that satisfy inequality (1.13). By analogy with $\boldsymbol{K}^1$, set $\boldsymbol{K}^\kappa$ is convex and closed in $\boldsymbol{V}^\kappa$. Denote by P_1 (respectively P_κ) the projector in $\boldsymbol{V}^1$ (respectively in $\boldsymbol{V}^\kappa$), which assigns to each element of $\boldsymbol{V}^1$ (respectively in $\boldsymbol{V}^\kappa$) the nearest element in $\boldsymbol{K}^1$ (respectively in $\boldsymbol{K}^\kappa$). Due to the convexity of $\boldsymbol{K}^1$ (respectively $\boldsymbol{K}^\kappa$), P_1 (respectively P_κ) is a continuous mapping of $\boldsymbol{V}^1$ (respectively $\boldsymbol{V}^\kappa$) into itself.

We denote by $(\,.\,,\,.\,)_{1,2}$ the scalar product in $\boldsymbol{W}^{1,2}(\Omega)$. For $\boldsymbol{\phi}\in\boldsymbol{V}^1$, we put $\boldsymbol{\Psi}(\boldsymbol{\phi}):=\boldsymbol{\phi}-P_1(\boldsymbol{\phi})$.

Lemma 4. *Operator $\boldsymbol{\Psi}$ is monotone and satisfies the inequalities*

$$
\big(\boldsymbol{\Psi}(\boldsymbol{\phi}),\boldsymbol{\phi}\big)_{1,2}\geq\|\boldsymbol{\Psi}(\boldsymbol{\phi})\|_{1,2}^2\,,\qquad \big(\boldsymbol{\Psi}(\boldsymbol{\phi}),\boldsymbol{\phi}\big)_{1,2}\geq\epsilon_1\,\|\boldsymbol{\Psi}(\boldsymbol{\phi})\|_{1,2} \tag{2.1}
$$

for all $\boldsymbol{\phi}\in\boldsymbol{V}^1$, where ϵ_1 is the number given by Lemma 1.

Proof. If $\boldsymbol{\phi}_1,\,\boldsymbol{\phi}_2\in\boldsymbol{V}^1$ then, due to the convexity of $\boldsymbol{K}^1$, $\|P_1(\boldsymbol{\phi}_1)-P_1(\boldsymbol{\phi}_1)\|_{1,2}\leq\|\boldsymbol{\phi}_1-\boldsymbol{\phi}_2\|_{1,2}$. Hence

$$
\begin{aligned}
\big(\boldsymbol{\Psi}(\boldsymbol{\phi}_1)-\boldsymbol{\Psi}(\boldsymbol{\phi}_2),\boldsymbol{\phi}_1-\boldsymbol{\phi}_2\big)_{1,2}\ &=\ \|\boldsymbol{\phi}_1-\boldsymbol{\phi}_1\|_{1,2}^2-\big(P_1(\boldsymbol{\phi}_1)-P_1(\boldsymbol{\phi}_2),\boldsymbol{\phi}_1-\boldsymbol{\phi}_2\big)_{1,2}\\
&\geq\ \|\boldsymbol{\phi}_1-\boldsymbol{\phi}_2\|_{1,2}^2-\|P_1(\boldsymbol{\phi}_1)-P_1(\boldsymbol{\phi}_2)\|_{1,2}\,\|\boldsymbol{\phi}_1-\boldsymbol{\phi}_2\|_{1,2}\ \geq\ 0.
\end{aligned}
$$

This proves the monotonicity of $\boldsymbol{\Psi}$. Furthermore,

$$\begin{aligned}\big(\boldsymbol{\Psi}(\boldsymbol{\phi}),\boldsymbol{\phi}\big)_{1,2} \;&=\; \big(\boldsymbol{\phi}-P_1(\boldsymbol{\phi}),\boldsymbol{\phi}-P_1(\boldsymbol{\phi})\big)_{1,2}+\big(\boldsymbol{\phi}-P_1(\boldsymbol{\phi}),P_1(\boldsymbol{\phi})\big)_{1,2} \\ &\geq\; \big(\boldsymbol{\phi}-P_1(\boldsymbol{\phi}),\boldsymbol{\phi}-P_1(\boldsymbol{\phi})\big)_{1,2} \;=\; \|\boldsymbol{\Psi}(\boldsymbol{\phi})\|_{1,2}^2.\end{aligned}$$

(We have used the inequality $\big(\boldsymbol{\phi}-P_1(\boldsymbol{\phi}),P_1(\boldsymbol{\phi})\big)_{1,2}\geq 0$, which holds because $\boldsymbol{K}^1$ is convex.) Obviously, the second inequality in (2.1) holds if $\boldsymbol{\Psi}(\boldsymbol{\phi})=\boldsymbol{0}$. Thus, assume that $\boldsymbol{\Psi}(\boldsymbol{\phi})\neq\boldsymbol{0}$ and put $\boldsymbol{h}:=\epsilon_1\,\boldsymbol{\Psi}(\boldsymbol{\phi})/\|\boldsymbol{\Psi}(\boldsymbol{\phi})\|_{1,2}$. Then

$$\big(\boldsymbol{\Psi}(\boldsymbol{\phi}),\boldsymbol{\phi}\big)_{1,2} \;=\; \big(\boldsymbol{\phi}-P_1(\boldsymbol{\phi}),\boldsymbol{\phi}-P_1(\boldsymbol{\phi})\big)_{1,2}+\big(\boldsymbol{\phi}-P_1(\boldsymbol{\phi}),P_1(\boldsymbol{\phi})-\boldsymbol{h}\big)_{1,2}+\big(\boldsymbol{\phi}-P_1(\boldsymbol{\phi}),\boldsymbol{h}\big)_{1,2}.$$

The first term on the right hand side is nonnegative. The second term is also nonnegative, because $\boldsymbol{h}\in\boldsymbol{K}^1$ and $\boldsymbol{K}^1$ is convex. Thus, substituting for $\boldsymbol{h}$, we get

$$\big(\boldsymbol{\Psi}(\boldsymbol{\phi}),\boldsymbol{\phi}\big)_{1,2} \;\geq\; \epsilon_1\left(\boldsymbol{\Psi}(\boldsymbol{\phi}),\,\frac{\boldsymbol{\Psi}(\boldsymbol{\phi})}{\|\boldsymbol{\Psi}(\boldsymbol{\phi})\|_{1,2}}\right)_{1,2} \;=\; \epsilon_1\,\|\boldsymbol{\Psi}(\boldsymbol{\phi})\|_{1,2}. \qquad \square$$

Put $\boldsymbol{V}^2:=\boldsymbol{V}^1\cap\boldsymbol{W}^{2,2}(\Omega)$. $\boldsymbol{V}^2$ is a Hilbert space with the scalar product $(\,.\,,\,.\,)_{2,2}$, identical with the scalar product in $\boldsymbol{W}^{2,2}(\Omega)$. Let $\boldsymbol{e}_1,\boldsymbol{e}_2\dots$ be a basis in $\boldsymbol{V}^2$, orthonormal in $\boldsymbol{H}$. It follows from the density of $\boldsymbol{V}^2$ in $\boldsymbol{H}$ and from the continuous imbedding $\boldsymbol{V}^2\hookrightarrow\hookrightarrow\boldsymbol{H}$ that the functions $\boldsymbol{e}_1,\boldsymbol{e}_2,\dots$ also form a basis in $\boldsymbol{H}$. Let $\zeta_1,\zeta_2,\dots$ be a basis in $W^{2,2}(\Omega)$.

1.2. Construction of approximations. Let $n\in\mathbb{N}$. We look for the coefficients $a_k^{(n)}$, $b_k^{(n)}\in C^1([0,T])$, $(k=1,2,\dots,n)$ such that the functions

$$\boldsymbol{v}^{(n)}:=\sum_{k=1}^n a_k^{(n)}\,\boldsymbol{e}_k,\qquad \vartheta^{(n)}:=\sum_{k=1}^n b_k^{(n)}\,\zeta_k \tag{2.2}$$

satisfy the initial conditions

$$\boldsymbol{v}^{(n)}(0)=\sum_{k=1}^n\big(\boldsymbol{v}_0,\boldsymbol{e}_k\big)_2\,\boldsymbol{e}_k,\qquad \vartheta^{(n)}(0)=\sum_{k=1}^n\big(\vartheta_0,\zeta_k\big)_2\,\zeta_k \tag{2.3}$$

(where $(\,.\,,\,.\,)_2$ denotes the scalar product in $\boldsymbol{L}^2(\Omega)$ or in $L^2(\Omega)$) and the integral equations

$$\begin{aligned}&\int_\Omega\big[\partial_t\boldsymbol{v}^{(n)}+(\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)}))\cdot\nabla(\boldsymbol{u}^*+\boldsymbol{v}^{(n)})\big]\cdot\boldsymbol{e}_k\,\mathrm{d}\boldsymbol{x}+\nu\int_\Omega\nabla(\boldsymbol{u}^*+\boldsymbol{v}^{(n)}):\nabla\boldsymbol{e}_k\,\mathrm{d}\boldsymbol{x}+\langle\partial_t\boldsymbol{u}^*,\boldsymbol{e}_k\rangle\\ &\quad-\int_\Omega[1-\alpha(\theta^*+\vartheta^{(n)}-\theta_{\mathrm{ref}})]\,\boldsymbol{f}\cdot\boldsymbol{e}_k\,\mathrm{d}\boldsymbol{x}+n\,\big(\boldsymbol{\Psi}(\boldsymbol{v}^{(n)}),\boldsymbol{e}_k\big)_{1,2}\;=\;\int_{\Gamma_2}\boldsymbol{F}\cdot\boldsymbol{e}_k\,\mathrm{d}S,\end{aligned} \tag{2.4}$$

$$\begin{aligned}&\int_\Omega\big[\partial_t\vartheta^{(n)}+(\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)}))\cdot\nabla(\theta^*+\vartheta^{(n)})\big]\,\zeta_k\,\mathrm{d}\boldsymbol{x}+\beta\int_\Omega\nabla(\theta^*+\vartheta^{(n)})\cdot\nabla\zeta_k\,\mathrm{d}\boldsymbol{x}+\langle\partial_t\theta^*,\zeta_k\rangle\\ &\quad=\;\langle g,\zeta_k\rangle+\int_{\Gamma_2}G\zeta_k\,\mathrm{d}S\end{aligned} \tag{2.5}$$

hold for all $k=1,\dots,n$. Substituting here from (2.2), we obtain a system of $2n$ ordinary differential equations for the unknown coefficients $a_k^{(n)}$, $b_k^{(n)}$ $(k=1,\dots,n)$. The system is completed by the initial conditions

$$a_k^{(n)}(0)=\big(\boldsymbol{v}_0,\boldsymbol{e}_k\big)_2,\qquad b_k^{(n)}(0)=\big(\vartheta_0,\zeta_k\big)_2. \tag{2.6}$$

The local solvability of the system follows from Caratheodory's theorem. In order to prove the global solvability on the time interval $(0,T)$, we derive global estimates of $a_k^{(n)}$ and $b_k^{(n)}$ $(k=1,\dots,n)$.

2.3. A priori estimates. Multiplying the k–th equation in (2.4) by $a_k^{(n)}$ and summing over k from 1 to n, we obtain

$$\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|\boldsymbol{v}^{(n)}\|_2^2+\nu\,\|\nabla\boldsymbol{v}^{(n)}\|_2^2+n\,\big(\boldsymbol{\Psi}(\boldsymbol{v}^{(n)}),\boldsymbol{v}^{(n)}\big)_{1,2}\;=\;-\int_\Omega(\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)}))\cdot\nabla(\boldsymbol{u}^*+\boldsymbol{v}^{(n)})\cdot\boldsymbol{v}^{(n)}\,\mathrm{d}\boldsymbol{x}$$
$$-\,\nu\int_\Omega\nabla\boldsymbol{u}^*:\nabla\boldsymbol{v}^{(n)}\,\mathrm{d}\boldsymbol{x}-\big\langle\partial_t\boldsymbol{u}^*,\boldsymbol{v}^{(n)}\big\rangle+\int_\Omega[1-\alpha(\theta^*+\vartheta^{(n)}-\theta_{\mathrm{ref}})]\,\boldsymbol{f}\cdot\boldsymbol{v}^{(n)}\,\mathrm{d}\boldsymbol{x}+\int_{\Gamma_2}\boldsymbol{F}\cdot\boldsymbol{v}^{(n)}\,\mathrm{d}S. \tag{2.7}$$

Similarly, multiplying the k–th equation in (2.5) by $b_k^{(n)}$ and summing for $k=1,\dots,n$, we get

$$\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|\vartheta^{(n)}\|_2^2+\beta\,\|\nabla\vartheta^{(n)}\|_2^2\;=\;-\int_\Omega P_\kappa(\boldsymbol{u}^*+\boldsymbol{v}^{(n)})\cdot\nabla(\theta^*+\vartheta^{(n)})\,\vartheta^{(n)}\,\mathrm{d}\boldsymbol{x}$$
$$-\beta\int_\Omega\nabla\theta^*\cdot\nabla\vartheta^{(n)}\,\mathrm{d}\boldsymbol{x}-\langle\partial_t\theta^*,\vartheta^{(n)}\rangle+\langle g,\vartheta^{(n)}\rangle+\int_{\Gamma_2}G\vartheta^{(n)}\,\mathrm{d}S. \tag{2.8}$$

The first integral on the right hand side of (2.7) equals

$$\begin{aligned}
&-\int_\Omega(\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)}))\cdot\nabla\boldsymbol{v}^{(n)}\cdot\boldsymbol{v}^{(n)}\,\mathrm{d}\boldsymbol{x}-\int_\Omega(\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)}))\cdot\nabla\boldsymbol{u}^*\cdot\boldsymbol{v}^{(n)}\,\mathrm{d}\boldsymbol{x}\\
&=\;-\frac{1}{2}\int_{\Gamma_2}(\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)})))\cdot\boldsymbol{n}\,|\boldsymbol{v}^{(n)}|^2\,\mathrm{d}S-\int_\Omega(\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)})))\cdot\nabla\boldsymbol{u}^*\cdot\boldsymbol{v}^{(n)}\,\mathrm{d}\boldsymbol{x}\\
&\le\;\frac{1}{2}\left(\int_{\Gamma_2}\big[(\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)}))\cdot\boldsymbol{n}\big]_-^a\,\mathrm{d}S\right)^{\frac{1}{a}}\left(\int_{\Gamma_2}|\boldsymbol{v}^{(n)}|^{\frac{2a}{a-1}}\,\mathrm{d}S\right)^{\frac{a-1}{a}}-\int_\Omega(\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)}))\cdot\nabla\boldsymbol{u}^*\cdot\boldsymbol{v}^{(n)}\,\mathrm{d}\boldsymbol{x}\\
&\le\;\frac{1}{2}\,c_2^{1/a}\,C\|\boldsymbol{v}^{(n)}\|_{\kappa,2}^2+\big\|\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)})\big\|_{r_2}\,\|\nabla\boldsymbol{u}^*\|_2\,\|\boldsymbol{v}^{(n)}\|_{s_2},
\end{aligned}$$

where $r_2^{-1}+s_2^{-1}=\frac{1}{2}$. If r_2 is chosen so that $3<r_2<6/(3-2\kappa)$ (which is <6) then $\boldsymbol{V}^\kappa\hookrightarrow\hookrightarrow\boldsymbol{L}^{r_2}(\Omega)$ and $\boldsymbol{V}^1\hookrightarrow\hookrightarrow\boldsymbol{L}^{s_2}(\Omega)$. Moreover, using also the inequalities $\|\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)})\|_{r_2}\le\|\boldsymbol{u}^*\|_{r_2}+\|\boldsymbol{v}^{(n)}\|_{r_2}\le C+\|\boldsymbol{v}^{(n)}\|_{r_2}$, we observe that the right hand side of the last inequality is

$$\le\;\delta\,\|\nabla\boldsymbol{v}^{(n)}\|_2^2+C(\delta)\,\|\boldsymbol{v}^{(n)}\|_2^2+C\,\|\boldsymbol{v}^{(n)}\|_{s_2}+C\,\|\boldsymbol{v}^{(n)}\|_{r_2}\,\|\boldsymbol{v}^{(n)}\|_{s_2}.$$

Interpolating the norms $\|\boldsymbol{v}^{(n)}\|_{r_2}$ and $\|\boldsymbol{v}^{(n)}\|_{s_2}$ between $\|\boldsymbol{v}^{(n)}\|_2$ and $\|\nabla\boldsymbol{v}^{(n)}\|_2$, we further obtain:

$$\dots\;\le\;3\delta\,\|\nabla\boldsymbol{v}^{(n)}\|_2^2+C(\delta)\,\|\boldsymbol{v}^{(n)}\|_2^2+C(\delta). \tag{2.9}$$

The second term on the right hand side of (2.7) satisfies

$$\left|\nu\int_\Omega\nabla\boldsymbol{u}^*:\nabla\boldsymbol{v}^{(n)}\,\mathrm{d}\boldsymbol{x}\right|\;\le\;\delta\,\|\nabla\boldsymbol{v}^{(n)}\|_2^2+C(\delta)\,\|\nabla\boldsymbol{u}^*\|_2^2. \tag{2.10}$$

The third term on the right hand side of (2.7) can be estimated by means of (1.5) and (1.7) as follows:

$$\big|\big\langle\partial_t\boldsymbol{u}^*,\boldsymbol{v}^{(n)}\big\rangle\big|\;\le\;\|\partial_t\boldsymbol{u}^*\|_{-1,2}\,\|\boldsymbol{v}^{(n)}\|_{1,2}\;\le\;\delta\,\|\nabla\boldsymbol{v}^{(n)}\|_2^2+C(\delta)\,\|\partial_t\boldsymbol{u}^*\|_2^2. \tag{2.11}$$

The fourth term on the right hand side of (2.7) can be estimated by means of the assumptions on the integrability of function $\boldsymbol{f}$, Young's inequality, the continuous imbedding of $\boldsymbol{W}^{1,2}(\Omega)$ to $\boldsymbol{L}^6(\Omega)$ and Friedrichs' inequality:

$$\begin{aligned}
&\left|\int_\Omega[1-\alpha(\theta^*+\vartheta^{(n)}-\theta_{\mathrm{ref}})]\,\boldsymbol{f}\cdot\boldsymbol{v}^{(n)}\,\mathrm{d}\boldsymbol{x}\right|\le\;\|1-\alpha(\theta^*+\vartheta^{(n)}-\theta_{\mathrm{ref}})\|_{r_1}\,\|\boldsymbol{f}\|_{s_1}\,\|\boldsymbol{v}^{(n)}\|_6\\
&\le\;\|1-\alpha(\theta^*+\vartheta^{(n)}-\theta_{\mathrm{ref}})\|_2^{\frac{6-r_1}{2r_1}}\,\|1-\alpha(\theta^*+\vartheta^{(n)}-\theta_{\mathrm{ref}})\|_6^{\frac{3r_1-6}{2r_1}}\,\|\boldsymbol{f}\|_{s_1}\,\|\boldsymbol{v}^{(n)}\|_6\\
&\le\;C\,\|1-\alpha(\theta^*+\vartheta^{(n)}-\theta_{\mathrm{ref}})\|_2^{\frac{6-r_1}{2r_1}}\,\|1-\alpha(\theta^*+\vartheta^{(n)}-\theta_{\mathrm{ref}})\|_{1,2}^{\frac{3r_1-6}{2r_1}}\,\|\boldsymbol{f}\|_{s_1}\,\|\boldsymbol{v}^{(n)}\|_6
\end{aligned}$$

$$\le\ C\,\|1-\alpha(\theta^*+\vartheta^{(n)}-\theta_{\mathrm{ref}})\|_2^{\frac{6-r_1}{2r_1}}\ \big(\|\nabla\vartheta^{(n)}\|_2^{\frac{3r_1-6}{2r_1}}+\|1-\alpha(\theta^*-\theta_{\mathrm{ref}})\|_{1,2}^{\frac{3r_1-6}{2r_1}}\big)\,\|\boldsymbol{f}\|_{s_1}\,\|\nabla\boldsymbol{v}^{(n)}\|_2$$

$$\le\ \delta\,\|\nabla\boldsymbol{v}^{(n)}\|_2^2+\delta\,\|\nabla\vartheta^{(n)}\|_2^2+C(\delta)+C(\delta)\,\|1-\alpha(\theta^*+\vartheta^{(n)}-\theta_{\mathrm{ref}})\|_2^2\,\|\boldsymbol{f}\|_{s_1}^{\frac{4r_1}{6-r_1}}. \tag{2.12}$$

Finally, the last term on the right hand side of (2.7) can be estimated by means of the continuity of the operator of traces from $\boldsymbol{W}^{1,2}(\Omega)$ to $\boldsymbol{L}^4(\partial\Omega)$ and Friedrich's inequality:

$$\left|\int_{\Gamma_2}\boldsymbol{F}\cdot\boldsymbol{v}^{(n)}\,\mathrm{d}S\right|\ \le\ \|\boldsymbol{F}\|_{4/3;\,\Gamma_2}\,\|\boldsymbol{v}^{(n)}\|_{4;\,\Gamma_2}\ \le\ C\,\|\boldsymbol{F}\|_{4/3;\,\Gamma_2}\,\|\boldsymbol{v}^{(n)}\|_{1,2}\ \le\ C\,\|\boldsymbol{F}\|_{4/3;\,\Gamma_2}\,\|\nabla\boldsymbol{v}^{(n)}\|_2$$

$$\le\ \delta\,\|\nabla\boldsymbol{v}^{(n)}\|_2^2+C(\delta)\,\|\boldsymbol{F}\|_{4/3;\,\Gamma_2}^2. \tag{2.13}$$

Substituting now from (2.10)–(2.13) to (2.7), we obtain

$$\frac12\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|\boldsymbol{v}^{(n)}\|_2^2+\nu\,\|\nabla\boldsymbol{v}^{(n)}\|_2^2+n\,\big(\boldsymbol{\Psi}(\boldsymbol{u}^*+\boldsymbol{v}^{(n)}),\,\boldsymbol{v}^{(n)}\big)_{1,2}\ \le\ 7\delta\,\|\nabla\boldsymbol{v}^{(n)}\|_2^2+\delta c_3\,\|\nabla\vartheta^{(n)}\|_2^2$$
$$+\,C(\delta)\,\|\boldsymbol{v}^{(n)}\|_2^2+C(\delta)\,\|1-\alpha(\theta^*+\vartheta^{(n)}-\theta_{\mathrm{ref}})\|_2^2\,\|\boldsymbol{f}\|_{s_1}^{\frac{4r_1}{6-r_1}}+C(\delta)\,\|\nabla\boldsymbol{u}^*\|_2^2$$
$$+\,C(\delta)\,\|\theta^*\|_{1,2}^2+C(\delta)+C(\delta)\,\|\boldsymbol{F}\|_{4/3;\,\Gamma_2}^2. \tag{2.14}$$

The first term on the right hand side of (2.8) can be estimated by analogy with (2.9):

$$-\int_\Omega\big[(\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)}))\cdot\nabla(\theta^*+\vartheta^{(n)})\big]\,\vartheta^{(n)}\,\mathrm{d}\boldsymbol{x}$$
$$=\ -\int_\Omega\big[(\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)}))\cdot\nabla\vartheta^{(n)}\big]\,\vartheta^{(n)}\,\mathrm{d}\boldsymbol{x}-\int_\Omega\big[(\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)}))\cdot\nabla\theta^*\big]\,\vartheta^{(n)}\,\mathrm{d}\boldsymbol{x}$$
$$=\ -\frac12\int_{\Gamma_2}(\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)}))\cdot\boldsymbol{n}\,|\vartheta^{(n)}|^2\,\mathrm{d}S-\int_\Omega\big[(\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)}))\cdot\nabla\theta^*\big]\,\vartheta^{(n)}\,\mathrm{d}\boldsymbol{x}$$
$$\le\ \frac12\left(\int_{\Gamma_2}\big[(\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)}))\cdot\boldsymbol{n}\big]_-^a\,\mathrm{d}S\right)^{\frac1a}\left(\int_{\Gamma_2}|\vartheta^{(n)}|^{\frac{2a}{a-1}}\,\mathrm{d}S\right)^{\frac{a-1}{a}}-\int_\Omega\big[(\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)}))\cdot\nabla\theta^*\big]\,\vartheta^{(n)}\,\mathrm{d}\boldsymbol{x}$$
$$\le\ \frac12\,c_2^{1/a}\,C\|\vartheta^{(n)}\|_{\kappa,2}^2+\big\|\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)})\big\|_{r_2}\,\|\nabla\theta^*\|_2\,\|\vartheta^{(n)}\|_{s_2}$$
$$\le\ \frac12\,c_2^{1/a}\,C\|\vartheta^{(n)}\|_{\kappa,2}^2+C\,\big\|\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)})\big\|_{r_2}\,\|\nabla\theta^*\|_2\,\|\vartheta^{(n)}\|_{s_2}$$
$$\le\ \delta\,\|\nabla\vartheta^{(n)}\|_2^2+C(\delta)\,\|\vartheta^{(n)}\|_2^2+C\,\big(\|\boldsymbol{u}^*\|_{r_2}+\|\boldsymbol{v}^{(n)}\|_{r_2}\big)\,\|\nabla\theta^*\|_2\,\|\vartheta^{(n)}\|_{s_2}$$
$$\le\ \delta\,\|\nabla\vartheta^{(n)}\|_2^2+C(\delta)\,\|\vartheta^{(n)}\|_2^2+C\,\|\vartheta^{(n)}\|_{s_2}+C\,\|\boldsymbol{v}^{(n)}\|_{r_2}\,\|\vartheta^{(n)}\|_{s_2}$$
$$\le\ 3\delta\,\|\nabla\vartheta^{(n)}\|_2^2+\delta\,\|\nabla\boldsymbol{v}^{(n)}\|_2^2+C(\delta)\,\|\vartheta^{(n)}\|_2^2+C(\delta)\,\|\boldsymbol{v}^{(n)}\|_2^2+C(\delta). \tag{2.15}$$

The estimates of the second, third and fourth term on the right hand side of (2.8) are standard:

$$\left|\beta\int_\Omega\nabla\theta^*\cdot\nabla\vartheta^{(n)}\,\mathrm{d}\boldsymbol{x}\right|\ \le\ \delta\,\|\nabla\vartheta^{(n)}\|_2^2+C(\delta)\,\|\nabla\theta^*\|_2^2, \tag{2.16}$$

$$\big|\langle g,\vartheta^{(n)}\rangle\big|\ \le\ \|g\|_{-1,2}\,\|\vartheta^{(n)}\|_{1,2}\ \le\ C\,\|g\|_{-1,2}\,\|\nabla\vartheta^{(n)}\|_2\ \le\ \|\nabla\vartheta^{(n)}\|_2^2+C(\delta)\,\|g\|_{-1,2}^2, \tag{2.17}$$

$$\left|\int_{\Gamma_2}G\vartheta^{(n)}\,\mathrm{d}S\right|\ \le\ \|G\|_{4/3;\,\Gamma_2}\,\|\vartheta^{(n)}\|_{4;\,\Gamma_2}\ \le\ C\,\|G\|_{4/3;\,\Gamma_2}\,\|\vartheta^{(n)}\|_{1,2}\ \le\ C\,\|G\|_{4/3;\,\Gamma_2}\,\|\nabla\vartheta^{(n)}\|_2$$
$$\le\ \delta\,\|\nabla\vartheta^{(n)}\|_2^2+C(\delta)\,\|G\|_{4/3;\,\Gamma_2}^2. \tag{2.18}$$

Substituting from (2.15)–(2.18) to (2.8), we obtain

$$\frac12\,\frac{\mathrm{d}}{\mathrm{d}t}\,\big\|\vartheta^{(n)}\big\|_2^2+\beta\,\|\nabla\vartheta^{(n)}\|_2^2\ \le\ 5\delta\,\|\nabla\vartheta^{(n)}\|_2^2+\delta\,\|\nabla\boldsymbol{v}^{(n)}\|_2^2+C(\delta)\,\|\vartheta^{(n)}\|_2^2+C(\delta)\,\|\boldsymbol{v}^{(n)}\|_2^2$$

$$+C(\delta)\,\|\nabla\theta^*\|_2^2 + \,C(\delta)\,\|g\|_{-1,2}^2 + C(\delta)\,\|G\|_{4/3;\,\Gamma_2}^2. \tag{2.19}$$

Choosing $\delta > 0$ so small that $8\delta \le \frac{1}{2}\nu$ and $(5+c_3)\delta \le \frac{1}{2}\beta$, summing (2.14) and (2.19), and using the assumptions (1.7) and (1.8) on $\boldsymbol{u}^*$ and θ^*, we get

$$\begin{aligned}
&\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\bigl(\|\boldsymbol{v}^{(n)}\|_2^2 + \|\vartheta^{(n)}\|_2^2\bigr) + \frac{\nu}{2}\,\|\nabla\boldsymbol{v}^{(n)}\|_2^2 + \frac{\beta}{2}\,\|\nabla\vartheta^{(n)}\|_2^2 + n\,\bigl(\boldsymbol{\Psi}(\boldsymbol{v}^{(n)}), \boldsymbol{v}^{(n)}\bigr)_{1,2} \\
&\quad\le\; C\,\|\boldsymbol{v}^{(n)}\|_2^2 + C\,\bigl(1+\|\boldsymbol{f}\|_{s_1}^{\frac{4r_1}{6-r_1}}\bigr)\,\|\vartheta^{(n)}\|_2^2 + C\,\bigl(1+\|\boldsymbol{f}\|_{s_1}^{\frac{4r_1}{6-r_1}}\bigr) \\
&\qquad + C(\delta)\,\|\boldsymbol{F}\|_{4/3;\,\Gamma_2}^2 + C(\delta)\,\|g\|_{-1,2}^2 + C(\delta)\,\|G\|_{4/3;\,\Gamma_2}^2 + C(\delta).
\end{aligned} \tag{2.20}$$

Integrating finally this inequality on the time interval $(0,T)$, we derive the estimates

$$\|\boldsymbol{v}^{(n)}(t)\|_2 + \|\vartheta^{(n)}(t)\|_2 \;\le\; c_4 \qquad \text{for all } t\in(0,T) \text{ and } n\in\mathbb{N}, \tag{2.21}$$

$$\int_0^T \bigl(\|\nabla\boldsymbol{v}^{(n)}\|_2^2 + \|\nabla\vartheta^{(n)}\|_2^2\bigr)\,\mathrm{d}t \;\le\; c_5 \qquad \text{for all } n\in\mathbb{N}, \tag{2.22}$$

$$\int_0^T n\,\bigl(\boldsymbol{\Psi}(\boldsymbol{v}^{(n)}), \boldsymbol{v}^{(n)}\bigr)_{1,2}\,\mathrm{d}t \;\le\; c_6 \qquad \text{for all } n\in\mathbb{N}. \tag{2.23}$$

The upper bounds c_4, c_5 and c_6 depend on the functions $\boldsymbol{u}^*$, θ^*, $\boldsymbol{f}$, g, $\boldsymbol{F}$, G, $\boldsymbol{v}_0$ and θ_0 and also on the coefficients ν and β and on the constants c_1 in Friedrichs' inequality (1.5) and c_2 in the definition of the convex set $\boldsymbol{K}^1$ (see (1.13)). They are, however, independent of n. Note that the inequalities (2.1) and (2.23) yield

$$\int_0^T n\,\|\boldsymbol{\Psi}(\boldsymbol{v}^{(n)})\|_{1,2}^2\,\mathrm{d}t \le c_6, \qquad \int_0^T n\,\|\boldsymbol{\Psi}(\boldsymbol{v}^{(n)})\|_{1,2}\,\mathrm{d}t \le \frac{c_6}{\epsilon_1}. \tag{2.24}$$

2.4. Existence of approximations. Estimates (2.21) imply that

$$\sum_{k=1}^{n}\bigl({a_k^{(n)}}^2(t) + {b_k^{(n)}}^2(t)\bigr) \;\le\; c_4 \qquad \text{for all } t\in(0,T) \text{ and } n\in\mathbb{N}.$$

From this, one can deduce that the system of ordinary differential equations for the unknowns $a_k^{(n)}$ and $b_k^{(n)}$ ($k=1,\dots,n$) (which we obtain if we substitute $\boldsymbol{v}^{(n)}$ and $\vartheta^{(n)}$ in the forms (2.3) to (2.4) and (2.5)) is uniquely solvable on the whole time interval $(0,T)$. Consequently, functions $\boldsymbol{v}^{(n)}$ and $\vartheta^{(n)}$ (defined by formulas (2.3)) also exist on the whole interval $(0,T)$ and satisfy estimates (2.21)–(2.23).

2.5. An estimate of a fractional derivative. In order to pass to the limit for $n\to\infty$ in weak forms of (2.4) and (2.5), we also need an information on a strong convergence of the sequence $\{\boldsymbol{v}^{(n)}\}$. For this purpose, we derive an estimate of a fractional derivative of $\boldsymbol{v}^{(n)}$ with respect to t. Choose $r\in(0,\frac{1}{2})$ and put

$$\begin{aligned}
\boldsymbol{\mathcal{H}}^r \;&:=\; \bigl\{\boldsymbol{v}\in L^2(0,T;\,\boldsymbol{V}^1);\; |\tau|^r\,\widehat{\boldsymbol{v}}(\tau)\in L^2\bigl(-\infty,\infty;\,\boldsymbol{V}^{-2}(\Omega)\bigr)\bigr\}, \\
\|\boldsymbol{v}\|_{\boldsymbol{\mathcal{H}}^r}^2 \;&:=\; \int_0^T \|\boldsymbol{v}(t)\|_{1,2}^2\,\mathrm{d}t + \int_{-\infty}^{\infty}|\tau|^{2r}\,\|\widehat{\boldsymbol{v}}(\tau)\|_{-2,2}^2\,\mathrm{d}\tau,
\end{aligned}$$

where $\boldsymbol{V}^{-2}$ denotes the dual space to $\boldsymbol{V}^2$ and $\widehat{\boldsymbol{v}}$ is the Fourier transform of $\boldsymbol{v}$ in variable t. (In order to calculate the Fourier transform, we extend $\boldsymbol{v}(t)$ by zero for $t\in(-\infty,0)\cup(T,\infty)$.) Recall that $\boldsymbol{V}^2 := \boldsymbol{V}^1\cap\boldsymbol{W}^{2,2}(\Omega)$. We denote by $\|\,.\,\|_{-2,2}$ the norm in $\boldsymbol{V}^{-2}$. Our next objective is to show that the sequence $\{\boldsymbol{v}^{(n)}\}$ is bounded in $\boldsymbol{\mathcal{H}}^r$. Let $\boldsymbol{h}\in\boldsymbol{V}^2$. Then there exist coefficients $\alpha_1,\alpha_2,\dots$ such that $\boldsymbol{h}=\sum_{k=1}^{\infty}\alpha_k\,\boldsymbol{e}_k$. Denote $\boldsymbol{h}^{(n)} := \sum_{k=1}^{n}\alpha_k\,\boldsymbol{e}_k$. Multiplying equation (2.4) by $\boldsymbol{h}^{(n)}$ and applying the Fourier transform, we get

$$\int_0^T \mathrm{e}^{-2\pi\mathrm{i}\tau t}\int_\Omega \bigl[\partial_t\boldsymbol{v}^{(n)} + (\boldsymbol{u}^* + P_\kappa(\boldsymbol{v}^{(n)}))\cdot\nabla(\boldsymbol{u}^*+\boldsymbol{v}^{(n)})\bigr]\cdot\boldsymbol{h}^{(n)}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$
\begin{aligned}
&+ \nu \int_0^T \mathrm{e}^{-2\pi\mathrm{i}\tau t} \int_\Omega \nabla(\boldsymbol{u}^* + \boldsymbol{v}^{(n)}) : \nabla \boldsymbol{h}^{(n)} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^T \langle \partial_t \boldsymbol{u}^*, \boldsymbol{h}^{(n)} \rangle \,\mathrm{d}t \\
&- \int_0^T \mathrm{e}^{-2\pi\mathrm{i}\tau t} \int_\Omega [1 - \alpha(\theta^* + \vartheta^{(n)} - \theta_{\mathrm{ref}})] \boldsymbol{f} \cdot \boldsymbol{h}^{(n)} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + n \int_0^T \mathrm{e}^{-2\pi\mathrm{i}\tau t} \big(\boldsymbol{\Psi}(\boldsymbol{v}^{(n)}), \boldsymbol{h}^{(n)}\big)_{1,2} \,\mathrm{d}t \\
= &\int_0^T \mathrm{e}^{-2\pi\mathrm{i}\tau t} \int_{\Gamma_2} \boldsymbol{F} \cdot \boldsymbol{h}^{(n)} \,\mathrm{d}S\,\mathrm{d}t. \qquad (2.25)
\end{aligned}
$$

Applying the integration by parts to the first term on the left hand side, we get

$$
\begin{aligned}
&\int_0^T \mathrm{e}^{-2\pi\mathrm{i}\tau t} \int_\Omega \partial_t \boldsymbol{v}^{(n)} \cdot \boldsymbol{h}^{(n)} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\quad = \mathrm{e}^{-2\pi\mathrm{i}\tau T} \int_\Omega \boldsymbol{v}^{(n)}(T) \cdot \boldsymbol{h}^{(n)} \,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{v}^{(n)}(0) \cdot \boldsymbol{h}^{(n)} \,\mathrm{d}\boldsymbol{x} + 2\pi\mathrm{i}\tau \int_\Omega \widehat{\boldsymbol{v}}^{(n)}(\tau) \cdot \boldsymbol{h}^{(n)} \,\mathrm{d}\boldsymbol{x}. \qquad (2.26)
\end{aligned}
$$

(Here, $\widehat{\boldsymbol{v}}^{(n)}$ denotes the Fourier transform of $\boldsymbol{v}^{(n)}$.) We claim that there exists $c_7 > 0$ such that all other terms in (2.25) can be estimated (in the absolute value) by $c_7 \, \|\boldsymbol{h}^{(n)}\|_{2,2}$. We show it on the example of the last term on the left hand side of (2.25):

$$
\begin{aligned}
\left| n \int_0^T \mathrm{e}^{-2\pi\mathrm{i}\tau t} \big(\boldsymbol{\Psi}(\boldsymbol{v}^{(n)}), \boldsymbol{h}^{(n)}\big)_{1,2} \,\mathrm{d}t \right| \;&\leq\; C\, n\, \|\boldsymbol{h}^{(n)}\|_{1,2} \int_0^T \|\boldsymbol{\Psi}(\boldsymbol{v}^{(n)})\|_{1,2} \,\mathrm{d}t \\
&\leq\; C\, \|\boldsymbol{h}^{(n)}\|_{2,2}\, \frac{n}{\epsilon_1} \int_0^T \big(\boldsymbol{\Psi}(\boldsymbol{v}^{(n)}), \boldsymbol{v}^{(n)}\big)_{1,2} \,\mathrm{d}t \;\leq\; \frac{C}{\epsilon_1}\, \|\boldsymbol{h}^{(n)}\|_{2,2}\, c_6.
\end{aligned}
$$

(We have used (2.1) and (2.23).) The other terms in (2.25) can be treated similarly, and this also holds on the first two terms on the right hand side of (2.26). Thus, the last term on the right hand side of (2.26) satisfies

$$
\left| 2\pi\mathrm{i}\tau \int_\Omega \widehat{\boldsymbol{v}}^{(n)}(\tau) \cdot \boldsymbol{h}^{(n)} \,\mathrm{d}\boldsymbol{x} \right| \;=\; \left| 2\pi\mathrm{i}\tau \int_\Omega \widehat{\boldsymbol{v}}^{(n)}(\tau) \cdot \boldsymbol{h} \,\mathrm{d}\boldsymbol{x} \right| \;\leq\; c_7\, \|\boldsymbol{h}^{(n)}\|_{2,2}.
$$

There exists $c_8 > 0$, independent of n and $\boldsymbol{h}$, such that $\|\boldsymbol{h}^{(n)}\|_{2,2} \leq c_8\, \|\boldsymbol{h}\|_{2,2}$. This simple inequality can be proven by means of the Banach–Steinhaus theorem: denote by $P^{(n)}$ the projector in $\boldsymbol{V}^2$, that assigns to each $\boldsymbol{h} \in \boldsymbol{V}^2$ the function $\boldsymbol{h}^{(n)}$. Since the sequence $\{P^{(n)}\boldsymbol{h}\}$ is bounded in $\boldsymbol{V}^2$ for each $\boldsymbol{h} \in \boldsymbol{V}^2$, $\{P^{(n)}\}$ is a bounded sequence in $\mathcal{L}(\boldsymbol{V}^2)$ (the space of bounded linear operators in $\boldsymbol{V}^2$). Consequently, $\|P^{(n)}\boldsymbol{h}\|_{2,2}$ is less than or equal to $C\, \|\boldsymbol{h}\|_{2,2}$, where C is independent of n. The inequality $\big|2\pi\mathrm{i}\tau \big(\widehat{\boldsymbol{v}}^{(n)}(\tau), \boldsymbol{h}\big)_2\big| \leq c_7 c_8\, \|\boldsymbol{h}\|_{2,2}$ (for all $\boldsymbol{h} \in \boldsymbol{V}^2$) implies that $\|\widehat{\boldsymbol{v}}^{(n)}(\tau)\|_{-2,2} \leq c_7 c_8 / 2\pi|\tau|$. Thus, there exists $c_9 > 0$, independent of n, such that

$$
\begin{aligned}
\|\boldsymbol{v}^{(n)}\|^2_{\boldsymbol{\mathcal{H}}^r} \;&=\; \int_0^T \|\boldsymbol{v}^{(n)}(t)\|^2_{1,2} \,\mathrm{d}t + \left(\int_{-\infty}^{-1} + \int_{-1}^{1} + \int_1^{\infty} \right) |\tau|^{2r}\, \|\widehat{\boldsymbol{v}}^{(n)}(\tau)\|^2_{-2,2} \,\mathrm{d}\tau \\
&\leq\; C + C \left(\int_{-\infty}^{-1} + \int_1^{\infty} \right) |\tau|^{2r-2} \,\mathrm{d}\tau + \int_{-1}^{1} \|\widehat{\boldsymbol{v}}^{(n)}(\tau)\|^2_{-2,2} \,\mathrm{d}\tau \\
&\leq\; C + \int_{-\infty}^{\infty} \|\widehat{\boldsymbol{v}}^{(n)}(\tau)\|^2_2 \,\mathrm{d}\tau \;\leq\; C + \int_0^T \|\boldsymbol{v}^{(n)}(t)\|^2_2 \,\mathrm{d}t \;\leq\; c_9. \qquad (2.27)
\end{aligned}
$$

3 The limit procedure for $n \to \infty$

3.1. Convergence of the approximations. It follows from (2.21), (2.22) and (2.27) that there exist $\boldsymbol{v} \in L^\infty(0,T;\, \boldsymbol{H}) \cap \boldsymbol{\mathcal{H}}^r$, $\vartheta \in L^\infty\big(0,T;\, L^2(\Omega)\big) \cap L^2(0,T;\, X^1)$ and subsequences of $\{\boldsymbol{v}^{(n)}\}$ and $\{\vartheta^{(n)}\}$ (which we again denote by $\{\boldsymbol{v}^{(n)}\}$ and $\{\vartheta^{(n)}\}$) such that

$$
\boldsymbol{v}^{(n)} \;\longrightarrow\; \boldsymbol{v} \qquad \text{weakly in } \boldsymbol{\mathcal{H}}^r \text{ and weakly–}* \text{ in } L^\infty(0,T;\, \boldsymbol{H}), \qquad (3.1)
$$

$$\vartheta^{(n)} \longrightarrow \vartheta \qquad \text{weakly in } L^2(0,T;\, X^1) \text{ and weakly–}* \text{ in } L^\infty\big(0,T;\, L^2(\Omega)\big). \tag{3.2}$$

Due to the compact imbedding $\boldsymbol{\mathcal{H}}^r \hookrightarrow\hookrightarrow L^2(0,T;\, \boldsymbol{V}^\kappa)$ (see e.g. [14, Chap. I.5.2]), we also have

$$\boldsymbol{v}^{(n)} \longrightarrow \boldsymbol{v} \qquad \text{strongly in } L^2(0,T;\, \boldsymbol{V}^\kappa). \tag{3.3}$$

The convergence (3.3) further yields

$$\boldsymbol{v}^{(n)} \longrightarrow \boldsymbol{v} \qquad \text{strongly in } L^2(0,T;\, \boldsymbol{H}), \tag{3.4}$$

$$\boldsymbol{v}^{(n)} \longrightarrow \boldsymbol{v} \qquad \text{strongly in } L^2\big(0,T;\, \boldsymbol{L}^a(\Gamma_2)\big), \tag{3.5}$$

$$\boldsymbol{v}^{(n)} \longrightarrow \boldsymbol{v} \qquad \text{strongly in } L^2\big(0,T;\, \boldsymbol{L}^{2a/(a-1)}(\Gamma_2)\big). \tag{3.6}$$

(Recall that number a has been introduced in subsection 1.3.) The first inequality in (2.24) implies that

$$\boldsymbol{\Psi}(\boldsymbol{v}^{(n)}) \longrightarrow \boldsymbol{0} \qquad \text{strongly in } L^2(0,T;\, \boldsymbol{W}^{1,2}(\Omega))\big). \tag{3.7}$$

3.2. The inclusion $\boldsymbol{v}(t) \in \boldsymbol{K}^1$. Due to the monotonicity of operator $\boldsymbol{\Psi}$ in $\boldsymbol{W}^{1,2}$, we have

$$\int_0^T \big(\boldsymbol{\Psi}(\boldsymbol{v}^{(n)}) - \boldsymbol{\Psi}(\boldsymbol{z}),\, \boldsymbol{v}^{(n)} - \boldsymbol{z}\big)_{1,2}\, \mathrm{d}t \;\geq\; 0 \tag{3.8}$$

for all $n \in \mathbb{N}$ and $\boldsymbol{z} \in \boldsymbol{V}^1$. Using (3.1) and (3.7), we get

$$\lim_{n\to\infty} \int_0^T \big(\boldsymbol{\Psi}(\boldsymbol{v}^{(n)}),\, \boldsymbol{v}^{(n)} - \boldsymbol{z}\big)_{1,2}\, \mathrm{d}t \;=\; 0,$$

$$\lim_{n\to\infty} \int_0^T \big(\boldsymbol{\Psi}(\boldsymbol{z}),\, \boldsymbol{v}^{(n)}\big)_{1,2}\, \mathrm{d}t \;=\; \int_0^T \big(\boldsymbol{\Psi}(\boldsymbol{z}),\, \boldsymbol{v}\big)_{1,2}\, \mathrm{d}t.$$

Thus, passing to the limit for $n \to \infty$ in (3.8), we obtain $\int_0^T \big(\boldsymbol{\Psi}(\boldsymbol{z}),\, \boldsymbol{v}-\boldsymbol{z}\big)_{1,2}\, \mathrm{d}t \leq 0$. Put $\boldsymbol{z} = \boldsymbol{v} - \xi\, \boldsymbol{\Psi}(\boldsymbol{v})$ where $\xi > 0$. Dividing the inequality by ξ and passing to the limit for $\xi \to 0+$, we get $\int_0^T \big(\boldsymbol{\Psi}(\boldsymbol{v}),\, \boldsymbol{\Psi}(\boldsymbol{v})\big)_{1,2}\, \mathrm{d}t \leq 0$, which means that $\boldsymbol{\Psi}(\boldsymbol{v}(t)) = \boldsymbol{0}$ for a.a. $t \in (0,T)$. This implies that $\boldsymbol{v}(t) \in \boldsymbol{K}^1$ for a.a. $t \in (0,T)$.

3.3. Passage to the limit (for $n \to \infty$) in equation (2.4) for $\boldsymbol{w}$ in the class $\mathscr{K}_m(0,T)$. Recall that the functions $\boldsymbol{e}_k$ ($k = 1, 2, \dots$) form a basis in $\boldsymbol{V}^2$, orthonormal in $\boldsymbol{H}$. For $m \in \mathbb{N}$, we denote by $\mathscr{W}_m(0,T)$ the set of functions $\boldsymbol{w} \in \mathscr{W}(0,T)$ that have a finite expansion $\boldsymbol{w}(t) = \sum_{k=1}^m \mu_k(t)\, \boldsymbol{e}_k$, and we put $\mathscr{K}_m(0,T) := \mathscr{W}_m(0,T) \cap L^2(0,T;\, \boldsymbol{K}^1)$.

We want to show that the functions $\boldsymbol{u} \equiv \boldsymbol{u}^* + \boldsymbol{v}$ and $\theta \equiv \theta^* + \vartheta$ satisfy the inequality (1.16) and the equation (1.17). Assume at first that the test function $\boldsymbol{w}$ in (1.16) is chosen from set $\mathscr{K}_m(0,T)$ and $n > m$. Recall that $\boldsymbol{v}^{(n)}$ has the expansion (2.2). Let us multiply equation (2.4) by $\mu_k - a_k^{(n)}$ if $k \leq m$ and by $-a_k^{(n)}$ if $m < k \leq n$ and sum the equations for $k = 1, \dots, n$. We obtain

$$\begin{aligned} &\int_\Omega \big[\partial_t \boldsymbol{v}^{(n)} + (\boldsymbol{u}^* + P_\kappa(\boldsymbol{v}^{(n)})) \cdot \nabla(\boldsymbol{u}^* + \boldsymbol{v}^{(n)})\big] \cdot (\boldsymbol{w} - \boldsymbol{v}^{(n)})\, \mathrm{d}\boldsymbol{x} + \nu \int_\Omega \nabla(\boldsymbol{u}^* + \boldsymbol{v}^{(n)}) : \nabla(\boldsymbol{w} - \boldsymbol{v}^{(n)})\, \mathrm{d}\boldsymbol{x} \\ &\qquad + \langle \partial_t \boldsymbol{u}^*, \boldsymbol{w} - \boldsymbol{v}^{(n)} \rangle - \int_\Omega [1 - \alpha(\theta^* + \vartheta^{(n)} - \theta_{\mathrm{ref}})]\, \boldsymbol{f} \cdot (\boldsymbol{w} - \boldsymbol{v}^{(n)})\, \mathrm{d}\boldsymbol{x} + n\, \big(\boldsymbol{\Psi}(\boldsymbol{v}^{(n)}),\, \boldsymbol{w} - \boldsymbol{v}^{(n)}\big)_{1,2} \\ &\quad = \int_{\Gamma_2} \boldsymbol{F} \cdot (\boldsymbol{w} - \boldsymbol{v}^{(n)})\, \mathrm{d}S. \end{aligned} \tag{3.9}$$

Further, we integrate this equation with respect to time on $(0,T)$. The two terms that contain the derivatives with respect to t yield

$$\int_0^T \int_\Omega \big[\partial_t \boldsymbol{v}^{(n)} \cdot (\boldsymbol{w} - \boldsymbol{v}^{(n)}) + \langle \partial_t \boldsymbol{u}^*, \boldsymbol{w} - \boldsymbol{v}^{(n)} \rangle\big]\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t$$

$$
\begin{aligned}
&= \int_\Omega \langle \partial_t \boldsymbol{u}^* + \partial_t \boldsymbol{w}, \boldsymbol{w} - \boldsymbol{v}^{(n)} \rangle \, \mathrm{d}t - \int_0^T \int_\Omega (\partial_t \boldsymbol{w} - \partial_t \boldsymbol{v}^{(n)}) \cdot (\boldsymbol{w} - \boldsymbol{v}^{(n)}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\
&= \int_\Omega \langle \partial_t \boldsymbol{u}^* + \partial_t \boldsymbol{w}, \boldsymbol{w} - \boldsymbol{v}^{(n)} \rangle \, \mathrm{d}t - \frac{1}{2} \big\| \boldsymbol{w}(T) - \boldsymbol{v}^{(n)}(T) \big\|_2^2 + \frac{1}{2} \big\| \boldsymbol{w}(0) - \boldsymbol{v}^{(n)}(0) \big\|_2^2 \\
&\leq \int_\Omega \langle \partial_t \boldsymbol{u}^* + \partial_t \boldsymbol{w}, \boldsymbol{w} - \boldsymbol{v}^{(n)} \rangle \, \mathrm{d}t + \frac{1}{2} \big\| \boldsymbol{w}(0) - \boldsymbol{v}^{(n)}(0) \big\|_2^2. \qquad (3.10)
\end{aligned}
$$

The integral of $n\big(\boldsymbol{\Psi}(\boldsymbol{v}^{(n)}), \boldsymbol{w} - \boldsymbol{v}^{(n)}\big)_{1,2}$ can estimated by means of the monotonicity of operator $\boldsymbol{\Psi}$ and the identity $\boldsymbol{\Psi}(\boldsymbol{w})(t)) = \boldsymbol{0}$ as follows:

$$
\int_0^T n\big(\boldsymbol{\Psi}(\boldsymbol{v}^{(n)}), \boldsymbol{w} - \boldsymbol{v}^{(n)}\big)_{1,2} \, \mathrm{d}t \;=\; -n \int_0^T \big(\boldsymbol{\Psi}(\boldsymbol{w}) - \boldsymbol{\Psi}(\boldsymbol{v}^{(n)}), \boldsymbol{w} - \boldsymbol{v}^{(n)}\big)_{1,2} \, \mathrm{d}t \;\leq\; 0. \qquad (3.11)
$$

Thus, (3.9), (3.10) and (3.11) yield

$$
\begin{aligned}
&\int_0^T \langle \partial_t \boldsymbol{u}^* + \partial_t \boldsymbol{w}, \boldsymbol{w} - \boldsymbol{v}^{(n)} \rangle \, \mathrm{d}t + \int_0^T \int_\Omega (\boldsymbol{u}^* + P_\kappa(\boldsymbol{v}^{(n)})) \cdot \nabla(\boldsymbol{u}^* + \boldsymbol{v}^{(n)}) \cdot (\boldsymbol{w} - \boldsymbol{v}^{(n)}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\
&\quad + \nu \int_0^T \int_\Omega \nabla(\boldsymbol{u}^* + \boldsymbol{v}^{(n)}) : \nabla(\boldsymbol{w} - \boldsymbol{v}^{(n)}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t - \int_0^T \int_\Omega [1 - \alpha(\theta^* + \vartheta^{(n)} - \theta_{\mathrm{ref}})] \, \boldsymbol{f} \cdot (\boldsymbol{w} - \boldsymbol{v}^{(n)}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\
&\quad - \int_0^T \int_{\Gamma_2} \boldsymbol{F} \cdot (\boldsymbol{w} - \boldsymbol{v}^{(n)}) \, \mathrm{d}S \, \mathrm{d}t \;\geq\; -\frac{1}{2} \big\| \boldsymbol{w}(0) - \boldsymbol{v}^{(n)}(0) \big\|_2^2. \qquad (3.12)
\end{aligned}
$$

The next step is the passage to the limit for $n \to \infty$ in (3.12). Here, we apply all types of convergence (3.1)–(3.7). Since the limit passage in some terms is the same or analogous to the proof of the global in time existence of weak solutions of the Navier–Stokes equations with the no slip boundary condition (see e.g. [7]) or [14], we focus only on the two “most difficult” nonlinear terms: a) the inequality

$$
\liminf_{n \to \infty} \left(-\nu \int_0^T \int_\Omega \nabla \boldsymbol{v}^{(n)} : \nabla \boldsymbol{v}^{(n)} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \right) \;\leq\; -\nu \int_0^T \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \qquad (3.13)
$$

holds due to (3.1). b) The second integral on the left hand side of (3.12) can be treated as follows:

$$
\begin{aligned}
&\int_0^T \int_\Omega (\boldsymbol{u}^* + P_\kappa(\boldsymbol{v}^{(n)})) \cdot \nabla(\boldsymbol{u}^* + \boldsymbol{v}^{(n)}) \cdot (\boldsymbol{w} - \boldsymbol{v}^{(n)}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\
&\quad = \int_0^T \int_\Omega (\boldsymbol{u}^* + P_\kappa(\boldsymbol{v}^{(n)})) \cdot \nabla(\boldsymbol{u}^* + \boldsymbol{v}^{(n)}) \cdot (\boldsymbol{u}^* + \boldsymbol{w}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\
&\quad\quad - \int_0^T \int_\Omega (\boldsymbol{u}^* + P_\kappa(\boldsymbol{v}^{(n)})) \cdot \nabla(\boldsymbol{u}^* + \boldsymbol{v}^{(n)}) \cdot (\boldsymbol{u}^* + \boldsymbol{v}^{(n)}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\
&\quad = \int_0^T \int_\Omega (\boldsymbol{u}^* + P_\kappa(\boldsymbol{v}^{(n)})) \cdot \nabla(\boldsymbol{u}^* + \boldsymbol{v}^{(n)}) \cdot (\boldsymbol{u}^* + \boldsymbol{w}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\
&\quad\quad - \frac{1}{2} \int_0^T \int_{\Gamma_1} (\boldsymbol{u}^* \cdot \boldsymbol{n}) \, |\boldsymbol{u}^*|^2 \, \mathrm{d}S \, \mathrm{d}t - \frac{1}{2} \int_0^T \int_{\Gamma_2} \big[\big(\boldsymbol{u}^* + P_\kappa(\boldsymbol{v}^{(n)})\big) \cdot \boldsymbol{n} \big] \, |\boldsymbol{u}^* + \boldsymbol{v}^{(n)}|^2 \, \mathrm{d}S \, \mathrm{d}t
\end{aligned}
$$

Due to (3.5) and (3.6),

$$
\int_{\Gamma_2} \big[(\boldsymbol{u}^* + P_\kappa(\boldsymbol{v}^{(n)})) \cdot \boldsymbol{n} \big] \, |\boldsymbol{u}^* + \boldsymbol{v}^{(n)}|^2 \, \mathrm{d}S \;\longrightarrow\; \int_{\Gamma_2} \big[(\boldsymbol{u}^* + \boldsymbol{v}) \cdot \boldsymbol{n} \big] \, |\boldsymbol{u}^* + \boldsymbol{v}|^2 \, \mathrm{d}S
$$

at a.a. points $t \in (0, T)$. Thus, applying Fatou’s lemma on the interval $(0, T)$, we get

$$
\liminf_{n \to \infty} \left(-\frac{1}{2} \int_0^T \int_{\Gamma_2} \big[(\boldsymbol{u}^* + P_\kappa(\boldsymbol{v}^{(n)})) \cdot \boldsymbol{n} \big] \, |\boldsymbol{u}^* + \boldsymbol{v}^{(n)}|^2 \, \mathrm{d}S \, \mathrm{d}t \right)
$$

$$\le\ -\frac{1}{2}\int_0^T\int_{\Gamma_2}\big[(\boldsymbol{u}^*+\boldsymbol{v})\cdot\boldsymbol{n}\big]\ |\boldsymbol{u}^*+\boldsymbol{v}|^2\ \mathrm{d}S\,\mathrm{d}t.$$

The convergence (3.3) and the properties of projector P_κ imply that $P_\kappa(\boldsymbol{v}^{(n)})\to\boldsymbol{v}$ in $L^2(0,T;\ \boldsymbol{V}^\kappa)$. Hence $P_\kappa(\boldsymbol{v}^{(n)})\to\boldsymbol{v}$ in $L^2\big(0,T;\ \boldsymbol{L}^4(\Omega)\big)$, too. Consequently, $\big(\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)})\big)\otimes(\boldsymbol{u}^*+\boldsymbol{w})\to(\boldsymbol{u}^*+\boldsymbol{v})\otimes(\boldsymbol{u}^*+\boldsymbol{w})$ (for $n\to\infty$) in $L^2\big(0,T;\ L^2(\Omega)^{3\times3}\big)$. This and (3.1) yield

$$\int_0^T\int_\Omega(\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)}))\cdot\nabla(\boldsymbol{u}^*+\boldsymbol{v}^{(n)})\cdot(\boldsymbol{u}^*+\boldsymbol{w})\ \mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ \longrightarrow\ \int_0^T\int_\Omega(\boldsymbol{u}^*+\boldsymbol{v})\cdot\nabla(\boldsymbol{u}^*+\boldsymbol{v})\cdot(\boldsymbol{u}^*+\boldsymbol{w})\ \mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

for $n\to\infty$. Thus, we have

$$\begin{aligned}\liminf_{n\to\infty}\ &\int_0^T\int_\Omega(\boldsymbol{u}^*+P_\kappa(\boldsymbol{v}^{(n)}))\cdot\nabla(\boldsymbol{u}^*+\boldsymbol{v}^{(n)})\cdot(\boldsymbol{w}-\boldsymbol{v}^{(n)})\ \mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ &\le\ -\frac{1}{2}\int_0^T\int_{\Gamma_1}(\boldsymbol{u}^*\cdot\boldsymbol{n})\,|\boldsymbol{u}^*|^2\ \mathrm{d}S\,\mathrm{d}t-\frac{1}{2}\int_0^T\int_{\Gamma_2}\big[(\boldsymbol{u}^*+\boldsymbol{v})\cdot\boldsymbol{n}\big]\ |\boldsymbol{u}^*+\boldsymbol{v}|^2\ \mathrm{d}S\,\mathrm{d}t\\ &\quad+\int_0^T\int_\Omega(\boldsymbol{u}^*+\boldsymbol{v})\cdot\nabla(\boldsymbol{u}^*+\boldsymbol{v})\cdot(\boldsymbol{u}^*+\boldsymbol{w})\ \mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ &=\ \int_0^T\int_\Omega(\boldsymbol{u}^*+\boldsymbol{v})\cdot\nabla(\boldsymbol{u}^*+\boldsymbol{v})\cdot(\boldsymbol{w}-\boldsymbol{v})\ \mathrm{d}\boldsymbol{x}\,\mathrm{d}t.\end{aligned}\tag{3.14}$$

The limit passage in all other terms in (3.12) is simpler than (3.13) and (3.14). Thus, $\boldsymbol{v}$ and ϑ satisfy inequality (1.16) for all test functions $\boldsymbol{w}\in\mathscr{K}_m(0,T)$. Since m was an arbitrary number from $\mathbb{N}$, $\boldsymbol{v}$ and ϑ satisfy (1.16) for all $\boldsymbol{w}\in\bigcup_{m=1}^\infty\mathscr{K}_m(0,T)$.

3.4. The validity of equation (1.16) for $\boldsymbol{w}\in\mathscr{K}(0,T)$. We still need to show that (1.16) is satisfied for all $\boldsymbol{w}\in\mathscr{K}(0,T)$. For this purpose, it is sufficient to show that $\bigcup_{m=1}^\infty\mathscr{K}_m(0,T)$ is dense in $\mathscr{K}(0,T)$ in the norm $|||\,.\,|||$. Obviously, $\bigcup_{m=1}^\infty\mathscr{W}_m(0,T)$ is dense in $\mathscr{W}(0,T)$. Set $\mathscr{K}(0,T)$ is closed in $\mathscr{W}(0,T)$, with the property that $\mathscr{K}(0,T)$ is equal to the closure of its interior. Hence $\big(\bigcup_{m=1}^\infty\mathscr{W}_m(0,T)\big)\cap\mathscr{K}(0,T)$ (which coincides with $\bigcup_{m=1}^\infty\mathscr{K}_m(0,T)$) is dense in $\mathscr{W}(0,T)\cap\mathscr{K}(0,T)$ (which coincides with $\mathscr{K}(0,T)$).

3.5. Passage to the limit (for $n\to\infty$) in equation (2.5). The way one can obtain equation (1.17) from (2.5) is standard and we do not therefore describe it here. We only mention that by analogy with the test function $\boldsymbol{w}$ in (1.16), we at first consider the test function ψ in (1.17) in a finite–dimensional subspace of $C^\infty\big([0,T];\ X^1\big)$, and use (3.1), (3.2), (3.4) to show that the limit functions $\boldsymbol{v}$ and ϑ satisfy (1.17). Then we use similar arguments as in subsection 3.4 and show that (1.17) holds for ψ in the whole class of test functions considered in the definition of problem ($\mathcal{P}$) in subsection 1.5, i.e. for all $\psi\in C^\infty\big([0,T];\ X^1\big)$ such that $\psi(T)=0$.

We have proven the theorem:

Theorem 1. *There exists a solution $\mathbf{v}$, ϑ of problem ($\mathcal{P}$).*

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