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ON SINGULARLY PERTURBED ORDINARY DIFFERENTIAL EQUATIONS WITH MEASURE-VALUED LIMITS

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Abstract. The limit behaviour of solutions of a singularly perturbed system is examined in the case where the fast flow need not converge to a stationary point. The topological convergence as well as information about the distribution of the values of the solutions can be determined in the case that the support of the limit invariant measure of the fast flow is an asymptotically stable attractor.

Keywords: singular perturbations, invariant measures, slow and fast motions

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1. Introduction

We examine a singularly perturbed system of ordinary differential equations which involves coupled slow and fast motions of the form

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= f(x,y), \\ \varepsilon\frac{\mathrm{d}y}{\mathrm{d}t} &= g(x,y) \end{aligned} \tag{1.1}$$

with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. We assume throughout that $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are continuous functions. The initial value problem is determined by the initial conditions

$$x(0) = x_0, \quad y(0) = y_0. \tag{1.2}$$

The solution to (1.1) depends on the parameter $\varepsilon > 0$. The variables x and y are referred to as the slow state and the fast state, respectively. The form (1.1) covers a variety of examples, including the case where the slow dynamics is not present, and the case of time varying equations $f = f(x, y, t)$ and $g = g(x, y, t)$, this by adding the slow variable t and the equation $\mathrm{d}t/\mathrm{d}t = 1$.

The solution to (1.1)–(1.2) is denoted by

$$(x_\varepsilon(\cdot), y_\varepsilon(\cdot)). \tag{1.3}$$

We are interested in the limit behavior of the trajectory (1.2) as $\varepsilon \to 0$.

The standard approach examines conditions which guarantee that the solutions of (1.1) converge, as $\varepsilon \to 0$, to the solution of the differential-algebraic system (see (2.1) below), obtained when in (1.1) the value of the parameter is set as $\varepsilon = 0$; see O'Malley [9, Chapter 2, Section D], Tikhonov et al. [11, Chapter 7, Section 2], Wasow [13, Section 39]. In the next section we state a theorem along this approach and display an application of a relaxation oscillation type.

Setting $\varepsilon = 0$ in (1.1) yields the limit of solutions as $\varepsilon \to 0$ under restrictive conditions. In particular, a crucial condition is that for each fixed x, solutions of the differential equation $\mathrm{d}y/\mathrm{d}s = g(x, y)$ should converge, as $s \to \infty$, to a solution of the algebraic equation $0 = g(x, y)$. A number of interesting examples have been examined recently where this condition is not satisfied. An approach was developed where the stationary limit $y(x)$ is replaced by a probability measure, say $\mu(x)$, with $\mu(x)$ being an invariant measure of the equation $\mathrm{d}y/\mathrm{d}s = g(x, y)$. See Artstein and Vigodner [5], Artstein [1], [2], Artstein and Slemrod [3], [4]. In Section 3 we state a theorem pertaining to this situation.

The price one pays to cover the more general case of measure-valued limits is that the convergence to the limit is in a weaker sense; namely, one gets information about the limit distribution of the fast solutions only. In Section 4, we present new results which, under the condition that the support of the invariant measure is a topological attractor of the fast flow, complement the information about the statistics of the flow with information about the topological location of the flow.

The result is illustrated in Section 5 with a variation of the relaxation oscillation example, where the limit is a measure-valued map, and where the topological considerations help to determine the limit behavior of the solutions.

2. A classical result

In this section we state a result along classical lines concerning the convergence of solutions of (1.1) as $\varepsilon \to 0$. The abstract result is followed by an application.

Consider the differential-algebraic system obtained from (1.1) when ε is set to be equal to 0, namely

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= f(x,y), \\ 0 &= g(x,y) \end{aligned} \tag{2.1}$$

with the initial conditions displayed in (1.2). The terminology we use concerning attraction and stability is standard. Consult, for instance, Yoshizawa [14].

Theorem 2.1. *Assume*

(i) $y(\cdot)\colon C \to \mathbb{R}^m$ *is a given continuous function, where* C *is an open neighborhood of* x_0*, and such that* $g(x, y(x)) = 0$ *for* x *in* C.

(ii) *For each* $x \in C$*, the point* $y(x)$ *is a locally asymptotically stable equilibrium of the differential equation*

$$\frac{\mathrm{d}y}{\mathrm{d}s} = g(x,y), \tag{2.2}$$

where x *in* (2.2) *is regarded as a fixed parameter. Furthermore, the asymptotic stability is locally uniform in the sense that the set* $\{(x,y)\colon x \in C, y \in \mathrm{Bas}(y(x))\}$ *includes an open neighborhood of* $\{(x,y(x))\colon x \in C\}$*, where* $\mathrm{Bas}(y(x))$ *is the basin of attraction of* $y(x)$ *with respect to* (2.2).

(iii) *Solutions of* (2.2) *are uniquely determined by initial conditions.*

(iv) *The initial condition* y_0 *is in the basin of attraction of* $y(x_0)$ *with respect to equation* (2.2) *with the parameter* x_0.

(v) *The equation*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, y(x)), \quad x(0) = x_0, \tag{2.3}$$

has a unique solution as long as the solution stays in C*. Denote this solution by* $x_0(\cdot)$.

Then the following conclusions hold.

(a) *The slow part* $x_\varepsilon(\cdot)$ *of the solution* (1.3) *converges as* $\varepsilon \to 0$ *to* $x_0(\cdot)$*, uniformly on intervals of the form* $[0,T]$*, this as long as* $x_0(t)$ *stays in* C.

(b) *The fast part* $y_\varepsilon(\cdot)$ *in* (1.3) *converges as* $\varepsilon \to 0$ *to* $y(x_0(\cdot))$*, uniformly on intervals of the form* $[\delta, T]$ *for* $\delta > 0$*, this as long as* $x_0(t)$ *stays in* C.

(c) *On intervals* $[0, S]$ *with* $S > 0$ *fixed, the trajectories* $\bar{y}_\varepsilon(\cdot)$ *converge uniformly, as* $\varepsilon \to 0$, *to* $y_0(\cdot)$, *where* $\bar{y}_\varepsilon(s)$ *is derived from the fast part* $y_\varepsilon(t)$ *of* (1.3) *through the time change* $t = \varepsilon s$; *and* $y_0(\cdot)$ *is the solution of* (2.2) *with the parameter* $x = x_0$ *and with an initial condition* $y(0) = y_0$. *The limit as* $S \to \infty$ *of* $\lim_{\varepsilon \to 0} y_\varepsilon(\varepsilon S)$ *is equal to* $y(x_0)$.

The results in Theorem 2.1 follow classical lines, with, however, a slight improvement as the proofs available in the literature assume that the data f and g are continuously differentiable; see [13, Theorem 39.1] and [11, Theorem 7.4] (the differentiability is not stated explicitly in [13], but the proof relies on [10] which assumes it). Since Theorem 2.1 follows from Theorem 4.1 below as a particular case (see Remark 4.2), we provide here a telegraphic sketch only, of the main steps of the proof.

Proof of Theorem 2.1 (a brief sketch). A change of time scale $\varepsilon s = t$ followed by a standard continuous dependence argument and together with condition (iv), imply conclusion (c). By (iii) and (ii), for ε small, once the fast solution $y_\varepsilon(\cdot)$ in (1.3) reaches within a short time a small neighborhood of the manifold $(x, y(x))$, it stays there. In particular, $x_\varepsilon(\cdot)$ solves an equation which is a small perturbation of (2.3). Condition (v) together with a standard continuous dependence argument imply that $x_\varepsilon(t)$ is close to $x_0(t)$ uniformly on compact intervals. This verifies (a). The facts that $y_\varepsilon(t)$ is close to $y(x_\varepsilon(t))$ on compact intervals bounded away from $t = 0$, and that $x_0(t)$ and $x_\varepsilon(t)$ are close to each other for small ε, imply conclusion (b), and conclude the proof.

In the rest of this section we examine an example which illustrates the applicability of the theorem.

Example 2.2. Consider the system

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y, \\ \varepsilon \frac{\mathrm{d}y}{\mathrm{d}t} &= -x + y - y^3. \end{aligned} \tag{2.4}$$

When following the scheme suggested in Theorem 2.1, one should first detect the roots of the equation

$$0 = -x + y - y^3. \tag{2.5}$$

The graph of the solution is displayed in Figure 1. The next step is to locate those points in the graph which are locally asymptotically stable with respect to the

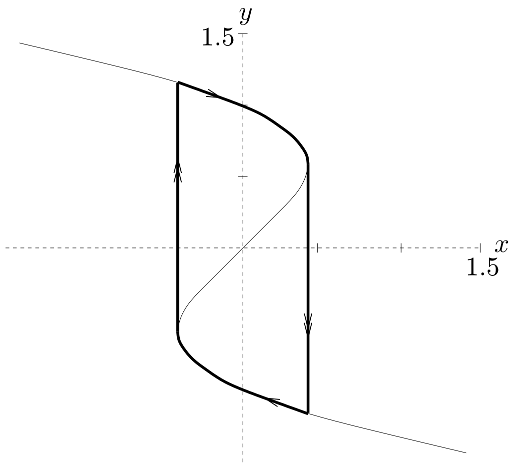

Figure 1

differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}s} = -x + y - y^3. \tag{2.6}$$

It is easy to see that each point (x, y) in the displayed graph such that $|y| > 3^{-\frac{1}{2}}$, possesses the local asymptotic stability property. Consider now an initial condition, say $(x(0), y(0)) = (-2, 0)$. The upper branch of the graph can be represented as a function $y(x)$ as required in Theorem 2.1, and all the conditions are satisfied for x satisfying $x < 3^{-\frac{3}{2}}2$. The conclusion is as follows. For small ε, on a short time interval the state $x = -2$ hardly changes, while the values $y_\varepsilon(t)$ converge to the value $y(-2) = -1.44225$. Following that short boundary layer interval, the solution stays close to the upper branch of the graph, following the pair $(x_0(t), y(x_0(t)))$ with $x_0(\cdot)$ being the solution of $\mathrm{d}x/\mathrm{d}t = y(x)$ with $x_0(0) = -2$ (which implies that $x(\cdot)$ is increasing). This description is valid until $x_0(t)$ reaches the value $3^{-\frac{3}{2}}2$.

In this specific example one can go beyond the point where $x_0(t)$ is equal to $3^{-\frac{3}{2}}2$. Indeed, right after that, the point $y_\varepsilon(t)$ enters the basin of attraction of the lower branch of the graph. The analogous analysis implies that in a very short interval the solution reaches a neighborhood of the lower branch, and then a slow dynamics following the lower branch occurs, with $x_\varepsilon(\cdot)$ decreasing, until $x_0(t) = -3^{-\frac{3}{2}}2$; and so on and so forth. Thus, the trajectory generates a well-known relaxation oscillation dynamics, as portrayed in bold in Figure 1 (the arrows point to the direction of the dynamics while a double arrow signifies the fast motion).

3. The case of measure-valued limits

In this section we state a result concerning the convergence of solutions of (1.1) as $\varepsilon \to 0$, when the fast flow need not converge to an equilibrium. A comparison with the result of the previous section follows.

We consider probability measure-valued maps. Denote by $\mathcal{P}(\mathbb{R}^m)$ the family of probability measures on $\mathbb{R}^m$. The measure-valued maps that we consider are maps of the form

$$\mu(\cdot)\colon\ [0,T] \to \mathcal{P}(\mathbb{R}^m) \tag{3.1}$$

which are measurable in the sense that $\mu(\cdot)(B)$ is a measurable real-valued function for each Borel set B in $\mathbb{R}^m$. Such maps are also referred to in literature as Young measures. A real valued function $h(\cdot)\colon\ [0,T] \to \mathbb{R}^m$ is interpreted as a measure-valued map when each value $h(t)$ is regarded as the Dirac measure supported on $\{h(t)\}$. We endow the space of Young measures with a convergence derived from the weak convergence of measures on $\mathcal{P}(\mathbb{R}^m)$ as follows. (We shall not refer to the weak convergence itself; a reference on this notion is, e.g., Billingsley [6].) A sequence $\mu_i(\cdot)$ converges to $\mu_0(\cdot)$ if

$$\int_0^T \int_{\mathbb{R}^m} \gamma(t,y)\mu_i(t)(\mathrm{d}y)\,\mathrm{d}t \to \int_0^T \int_{\mathbb{R}^m} \gamma(t,y)\mu_0(t)(\mathrm{d}y)\,\mathrm{d}t \tag{3.2}$$

for every $\gamma(t,y)\colon\ [0,T]\times\mathbb{R}^m \to \mathbb{R}$ which is bounded, measurable in t and continuous in y. We refer to this convergence as the narrow convergence or as convergence in the sense of Young measures. The convergence yields information about the distribution of the values. Indeed, if a sequence of $\mathbb{R}^m$-valued functions, say $h_i(\cdot)$, converges in the sense of Young measures to the measure valued map $\mu_0(\cdot)$, then for each subset A of positive measure in the interval $[0,T]$, the distribution of the values of $\{h_i(t)\colon\ t \in A\}$ is close to the distribution derived by integrating $\mu_0(\cdot)$ over A.

We also need the notion of an invariant measure of a differential equation. Let $y(\cdot, y_0)$ be the solution of the differential equation $\mathrm{d}y/\mathrm{d}s = g(y)$ with the initial condition $y(0) = y_0$, and assume that such a solution is unique. A probability measure on $\mathbb{R}^m$ is an invariant measure of the differential equation if for every Borel set $B \subseteq \mathbb{R}^m$, the equality $\mu(B) = \mu(\{y(s,y_0)\colon\ y_0 \in B\})$ holds for every s.

Theorem 3.1. *Assume*

(i) *On an interval* $[0,T]$, *the values* $(x_\varepsilon(t), y_\varepsilon(t))$ *of the solutions* (1.3) *of* (1.1)–(1.2) *for* $\varepsilon > 0$ *in a neighborhood of* 0 *are uniformly bounded in* $\mathbb{R}^n \times \mathbb{R}^m$, *say* $(x_\varepsilon(t), y_\varepsilon(t)) \in C \times D$ *with* C *an open set and* D *a closed set.*

(ii) *For each $x \in \mathbb{R}^n$, solutions of the differential equation (2.2) where x in (2.2) is regarded as a fixed parameter are uniquely determined by initial conditions.*

(iii) *For every $x \in C$, an invariant measure $\mu(x)$ of (2.2) with support in D exists, and it is unique.*

(iv) *The equation*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \int_{\mathbb{R}^m} f(x,y)\mu(x)(\mathrm{d}y) \tag{3.3}$$

with initial condition $x(0) = x_0$, has a unique solution on $[0,T]$. Denote this solution by $x_0(\cdot)$.

Then the following conclusions hold.

(a) *The trajectories $x_\varepsilon(\cdot)$ converge to $x_0(\cdot)$, as $\varepsilon \to 0$, uniformly on compact subsets $[0,T']$ of $[0,T]$ on which $x_0(t) \in C$.*

(b) *The trajectories $y_\varepsilon(\cdot)$ converge in the sense of Young measures, as $\varepsilon \to 0$, to $\mu(x_0(\cdot))$, as long as $x_0(t) \in C$.*

Results similar to those of Theorem 3.1 with complete proofs are presented in [5], [2], [3], [4]. Here we provide a sketch of the proof.

Proof of Theorem 3.1 (sketched). The uniform boundedness of $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$ implies the existence of a subsequence ε_i such that $x_{\varepsilon_i}(\cdot)$ converges uniformly, say to $\bar{x}_0(\cdot)$, and $y_{\varepsilon_i}(\cdot)$ converges in the Young measure sense to a probability measure-valued map, say $\bar{\mu}(\cdot)$. A classical continuous dependence argument implies that $\bar{x}_0(\cdot)$ is a solution of (3.3) with $\mu(x)$ replaced by $\bar{\mu}(t)$. Consider now the change of time scale $t = \varepsilon_i s$. Then $y_{\varepsilon_i}(\cdot)$ solves the equation $\mathrm{d}y/\mathrm{d}s = g(x_{\varepsilon_i}(\varepsilon s), y)$. On a small t interval the coefficient $x_{\varepsilon_i}(\varepsilon s)$ is almost constant, hence $y_{\varepsilon_i}(\cdot)$ is close on finite intervals to the solution with a constant parameter x. The s interval may, however, be large enough so that the distribution of the values $y_{\varepsilon_i}(s)$ yields an approximation to an invariant measure of the equation with the fixed parameter (along the lines of Kriloff and Bogoliuboff [8]). These arguments imply that almost everywhere, $\bar{\mu}(t)$ is an invariant measure of (2.2) with $x = x_0(t)$. The uniqueness assumed in (iii) implies that $\bar{\mu}(\cdot) = \bar{\mu}(x_0(\cdot))$, and hence $\bar{x}_0(\cdot)$ solves (3.3). It also implies that the convergence claims (a) and (b) hold for the subsequence determined by ε_i. The uniqueness of the invariant measure, and the compactness, namely, that converging subsequences can be extracted from any subsequence, imply that (a) and (b) hold.

Remark 3.2. One can get a meaningful result even without conditions (iii) and (iv). Namely, (a) and (b) then hold for a subsequence ε_i, with $\mu(t)$ being an invariant measure (rather than the invariant measure) of (2.2) in D. This claim follows from the proof.

R e m a r k 3.3. It is clear that rather than demanding that the solutions be included in a set $C \times D$ (see condition (i)), we could ask that the solution be included in a set of a form $\{(x,y)\colon\ x \in C, y \in D(x)\}$, as long as $D(x)$ is closed, and the graph of $D(\cdot)$ is the closure of an open set.

D i s c u s s i o n 3.4. The claims in Theorem 3.1 have been established under conditions milder than those of Theorem 2.1. In turn, the established convergence yields desired information about the limit distribution of the values of the solutions, but only partial information concerning the topological limits of the fast flow. Indeed, a sequence of functions may converge in the sense of Young measures without point wise convergence or topological convergence of the graphs. (A trajectory may converge in distribution to a fixed point, while topologically converging to a full cycle which contains the fixed point.) This is reflected in the lack, in Theorem 3.1, of an analog of the boundary layer claim (c) of Theorem 2.1. Indeed, the behavior of $y_\varepsilon(\cdot)$ on intervals $[0, \varepsilon S]$ does not affect the limit distribution. The applications and illustrations listed in references [5], [2], [3], [4] employ ad hoc arguments to derive better information about the topological behavior. In the next section we offer a general result in this direction.

4. A combined argument

In this section we combine the arguments of Theorems 2.1 and 3.1 into one set of conditions which yields information on both, the limit topology and the limit distribution of the solutions. To this end we need the following standard notions.

When $y \in \mathbb{R}^m$ and $K \subseteq \mathbb{R}^m$ we write $d(y,K) = \inf\{|y-z|\colon\ z \in K\}$. The Hausdorff distance between two compact sets K_1 and K_1 is $H(K_1,K_2) = \max\{d(z,K_1), d(y,K_2)\colon\ y \in K_1, z \in K_2\}$.

The compact set K in $\mathbb{R}^m$ is an asymptotically stable attractor of the differential equation $\mathrm{d}y/\mathrm{d}s = g(y)$ if: (1) for every $\eta > 0$ there exists a $\delta > 0$ such that if $y(\cdot)$ is a solution of the equation and $d(y(0),K) < \delta$, then $d(y(s),K) < \eta$ for all $s > 0$, and, (2) a number $b > 0$ exists such that whenever $y(\cdot)$ is a solution of the equation and $d(y(0),K) < b$ then $d(y(s),K) \to 0$ as $s \to \infty$. (See Ura [12] for a comprehensive study of asymptotically stable attractors.)

The support of a probability measure μ on $\mathbb{R}^m$ (namely the smallest closed set C such that $\mu(C) = 1$) is denoted by $\operatorname{supp}\mu$.

Theorem 4.1. *Assume*

(i) $\mu(\cdot)\colon\ C \to \mathcal{P}(\mathbb{R}^m)$ *is a given Young measure, where C is an open neighborhood of x_0, and such that for each $x \in C$ the set* $\operatorname{supp}\mu(x)$ *is compact, and* $\operatorname{supp}\mu(\cdot)$

is continuous in the x variable with respect to the Hausdorff distance. Furthermore, for each x in C the measure $\mu(x)$ is an invariant measure of (2.2)*, and it is the unique invariant measure with support included in* $\operatorname{supp}\mu(x)$*.*

(ii) *For each $x \in C$, the set $\operatorname{supp}\mu(x)$ is an asymptotically stable attractor of* (2.2) *where x is regarded as a fixed parameter. Furthermore, the asymptotic stability is locally uniform in the sense that the set $\{(x,y)\colon x \in C,\ y \in \operatorname{Bas}(\operatorname{supp}\mu(x))\}$ includes an open neighborhood of $\{(x,y)\colon x \in C,\ y \in \operatorname{supp}\mu(x))\}$, where* $\operatorname{Bas}(\operatorname{supp}\mu(x))$ *is the basin of attraction of the set* $\operatorname{supp}\mu(x)$ *with respect to* (2.2).

(iii) *Solutions of* (2.2) *are uniquely determined by initial conditions.*

(iv) *The initial condition y_0 is in the basin of attraction of* $\operatorname{supp}\mu(x_0)$ *with respect to the equation* (2.2) *with the parameter x_0.*

(v) *Equation* (3.3) *with the initial condition $x(0) = x_0$ has a unique solution as long as the solution is in C. Denote this solution by $x_0(\cdot)$.*

Then the following conclusions hold.

(a) *The slow part $x_\varepsilon(\cdot)$ of the solution* (1.3) *converges, as $\varepsilon \to 0$, to $x_0(\cdot)$, uniformly on intervals of the form $[0,T]$, this as long as $x_0(t)$ stays in C.*

(b) *The fast part $y_\varepsilon(\cdot)$ in* (1.3) *converges in the sense of Young measures, as $\varepsilon \to 0$, to $\mu(x_0(\cdot))$, on intervals of the form $[0,T]$, this as long as $x_0(t)$ stays in C.*

(c) *The distance $d(y_\varepsilon(t), \operatorname{supp}\mu(x_0(t)))$ converges to* 0*, as $\varepsilon \to 0$, uniformly on intervals of the form $[\delta,T]$ for $\delta > 0$, this as long as $x_0(t)$ stays in C.*

(d) *On intervals $[0,S]$ with $S > 0$ fixed, the trajectories $\bar{y}_\varepsilon(\cdot)$ converge uniformly, as $\varepsilon \to 0$, to $y_0(\cdot)$; here $\bar{y}_\varepsilon(s)$ is derived from the fast part $y_\varepsilon(t)$ of* (1.3) *through the time change $t = \varepsilon s$, and $y_0(\cdot)$ is the solution of* (2.2) *with the parameter $x = x_0$, and with initial condition $y(0) = y_0$. The limit as $S \to \infty$ of $\lim\limits_{\varepsilon\to 0} d(y_\varepsilon(\varepsilon S), \operatorname{supp}\mu(x_0))$ is equal to* 0*.*

Proof. The proof consists of a combination of arguments employed when Theorems 2.1 and 3.1 are being established. We start with claim (d). The change of time scales $\varepsilon s = t$ converts the singularly perturbed equation (1.1) on $[0,\varepsilon S]$ into a non-singularly perturbed one on $[0,S]$. Since by (iv) the solution $y_0(\cdot)$ stays bounded, it follows that for ε small and S fixed, the values $x_\varepsilon(t)$ for $t \in [0,\varepsilon S]$ converge uniformly to x_0 as $\varepsilon \to 0$. A standard continuous dependence argument implies that $\bar{y}_\varepsilon(\cdot)$ converges uniformly on any fixed interval $[0,S]$, as $\varepsilon \to 0$, to $y_0(\cdot)$. Now, the convergence of $\lim\limits_{\varepsilon\to 0} d(y_\varepsilon(\varepsilon S), \operatorname{supp}\mu(x_0))$ to 0 follows directly from (iv).

We now verify that if for small ε the value $y_\varepsilon(t)$ is close to $\operatorname{supp}\mu(x_\varepsilon(t))$, then $y_\varepsilon(\cdot)$ stays close to the graph of $\operatorname{supp}\mu(x)$. The exact statement is as follows.

C l a i m 1. Let $K \subset C$ be compact. For every $\bar{\eta} > 0$ there exist $\bar{\theta} > 0$ and ε_0 such that for $\varepsilon < \varepsilon_0$ if $d(y_\varepsilon(\delta), \operatorname{supp} \mu(x_\varepsilon(\delta))) < \bar{\theta}$ then $d(y_\varepsilon(t), \operatorname{supp} \mu(x_\varepsilon(t))) < \bar{\eta}$ for $t > \delta$, as long as $x_\varepsilon(t) \in K$.

To verify the claim we can assume that $\bar{\eta}$ is such that an $\bar{\eta}$-neighborhood of $\operatorname{supp} \mu(x)$ is in $\operatorname{Bas}(\operatorname{supp} \mu(x))$ for all $x \in K$. The existence of such an $\bar{\eta} > 0$ follows from (ii). For every $x \in K$ there exists a $\theta(x) > 0$ such that if $d(\bar{y}, \operatorname{supp} \mu(x)) < \theta(x)$ and $y_{(x)}(\cdot)$ is the solution of (2.2) satisfying $y_{(x)}(0) = \bar{y}$ then $d(y_{(x)}(s), \operatorname{supp} \mu(x)) < \bar{\eta}$ for $s \geqslant 0$. This follows from condition (ii). The compactness of K implies that $\theta(x)$ can be chosen independent of x; we choose $\bar{\theta}$ to be the independent value. If the claim is false, then there exists a sequence of $\varepsilon_i \to 0$ such that $d(y_{\varepsilon_i}(t_i), \operatorname{supp} \mu(x_{\varepsilon_i}(t_i))) = \bar{\theta}$ while $d(y_{\varepsilon_i}(t_i + \Delta_i), \operatorname{supp} \mu(x_{\varepsilon_i}(t_i + \Delta_i))) = \bar{\eta}$ for some t_i and Δ_i, while $x_{\varepsilon_i}(t) \in K$ and $\bar{\theta} \leqslant d(y_{\varepsilon_i}(t), \operatorname{supp} \mu(x_{\varepsilon_i}(t))) \leqslant \bar{\eta}$ for $t \in [t_i, t_i + \Delta_i]$. A change of variables $s = \varepsilon^{-1}(t - t_i)$ converts the fast equation in (1.1) into the form (2.2) with, however, a time varying parameter $x_\varepsilon(s)$. For short $(t - t_i)$-intervals this parameter does not vary much. We may assume that $x_{\varepsilon_i}(t_i)$ converges, say to $\bar{x} \in K$. Hence, as $\varepsilon_i \to 0$, the trajectories $y_{\varepsilon_i}(\cdot)$ for $s \in [0, \varepsilon^{-1}\Delta]$, converge uniformly on compact s-intervals to the solution $y_0(\cdot)$ of (2.2) with the parameter $\bar{x}$. Two possibilities may occur. First, that $\varepsilon_i \Delta_i \to \infty$. Then $\bar{\theta} \leqslant y_0(s) \leqslant \bar{\eta}$ for all $s \geqslant 0$, which contradicts condition (ii) of the theorem. Secondly, that $\varepsilon_i \Delta_i$ is finite. Then $d(y_0(s), \operatorname{supp} \mu(\bar{x})) = \bar{\eta}$ for some $s > 0$, which contradicts the choice of $\bar{\eta}$. The two alleged contradictions verify that Claim 1 is valid.

Together with property (d) which was verified earlier, Claim 1 completes the proof of property (c).

At this point notice that for every $\delta > 0$, if ε is small enough, the values $(x_\varepsilon(t), y_\varepsilon(t))$ of the solutions (1.3) of (1.1)–(1.2), for $t \in [\delta, T]$, remain, as long as $x_\varepsilon(t) \in C$, within an $\bar{\eta}$-neighborhood of the graph of $\operatorname{supp} \mu(\cdot)$, with a small positive $\bar{\eta}$. For a compact $K \subset C$ the $\bar{\eta}$-neighborhood can be chosen to be contained in the union of the basins of attraction of the corresponding $\operatorname{supp} \mu(x)$. Since $\bar{\eta}$ is arbitrarily small, we can apply Theorem 3.1 (in fact, the extension mentioned in Remark 3.3), and deduce the uniform convergence of $x_\varepsilon(\cdot)$ to the solution $x_0(\cdot)$ of (3.3) with the initial condition $x(0) = x_0$, and the convergence in the Young measures sense of $y_\varepsilon(\cdot)$ to $\mu(x_0(\cdot))$, this as long as $x_0(t) \in C$, as claimed in (a) and (b). This completes the proof. □

We wish to point out several consequences and extensions of the preceding result, as follows.

R e m a r k 4.2. Theorem 2.1 is a particular case of Theorem 4.1, since the equilibrium $y(x)$ is a particular case of an invariant measure, supported on $\{y(x)\}$, of the differential equation.

R e m a r k 4.3. The uniqueness of the invariant measure assumed in condition (i) implies a bit more than stated concerning the topological convergence, as follows. Let $t_0 > 0$ be in the domain of $x_0(\cdot)$ and let $\eta > 0$ be given. For any fixed $\tau > 0$ small enough, for small enough ε the set $\{y_\varepsilon(t)\colon\ t_0 - \tau \leqslant t \leqslant t_0 + \tau\}$ is within a Hausdorff distance η from $\operatorname{supp}\mu(x_0(t_0))$. Otherwise the arguments in Theorem 3.1 yield an invariant measure with a strictly smaller support.

R e m a r k 4.4. Rather than requiring that $\operatorname{supp}\mu(x)$ be an asymptotically stable attractor, we could demand the existence of an asymptotically stable attractor $K(x)$ of (2.2), which contains $\operatorname{supp}\mu(x)$ and such that $K(\cdot)$ is continuous with respect to the Hausdorff distance. The consequence then would be the topological convergence to $K(x_0(t))$, and the rest would stay unchanged (but Remark 4.3 would not be valid anymore).

R e m a r k 4.5. If the uniqueness of the invariant measure supported on $\operatorname{supp}\mu(x)$ is lifted, then a weaker consequence holds in full analogy to Remark 3.2. The consequences concerning the topological convergence remain then as in Theorem 4.1.

5. An example

We display a variant of Example 2.2 as an illustration demonstrating the applicability of Theorem 4.1.

E x a m p l e 5.1. Consider the system

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y_1, \\ \varepsilon\frac{\mathrm{d}y_1}{\mathrm{d}t} &= y_2, \\ \varepsilon\frac{\mathrm{d}y_2}{\mathrm{d}t} &= g(x, y_1, y_2) \end{aligned} \tag{5.1}$$

with x, y_1, y_2 scalars, and where $g(x, y_1, y_2)$ is designed as follows. For a fixed x, the system

$$\begin{aligned} \frac{\mathrm{d}y_1}{\mathrm{d}s} &= y_2, \\ \frac{\mathrm{d}y_2}{\mathrm{d}s} &= g(x, y_1, y_2) \end{aligned} \tag{5.2}$$

has stationary points of the form $(y_1, 0)$ with y_1 satisfying

$$0 = -x + y_1 - y_1^3 \tag{5.3}$$

(compare with the graph of the equilibria in Figure 1; for clarity of the drawing, the y_2-axis in Figure 2 originates at $(0, 1, 0)$). Furthermore, for x fixed, all the equilibria of (5.2) are unstable, and all other solutions converge to locally stable periodic orbits around either the upper or the lower branch of the equilibria curve (in particular, for $|x| > 3^{-\frac{3}{2}}2$ there is only one periodic limit cycle). The general structure of the equilibria and of the limit cycles is portrayed in Figure 2. Such a structure occurs in the following situation. Let $z_{\mathrm{up}}(x)$ and $z_{\mathrm{lo}}(x)$ denote the upper and, respectively, the lower branches of the equilibria determined by (5.3) (in particular, for $|x| > 3^{-\frac{3}{2}}2$ there is only one equilibrium). In a neighborhood of, say, $z_{\mathrm{up}}(x)$, the right hand side of (5.2) is determined by

$$g(x, y_1, y_2) = \alpha(x)(1 - A(x)(y_1 - z_{\mathrm{up}}(x))^2)y_2 - (y_1 - z_{\mathrm{up}}(x)) \tag{5.4}$$

with $A(x) \to \infty$ and $\alpha(x)A(x) \to 0$ as $x \to 3^{-\frac{3}{2}}2$. Indeed, then for a fixed $x < 3^{-\frac{3}{2}}2$, equation (5.2) is a van der Pol equation centered around $z_{\mathrm{up}}(x)$ with its limit cycle converging to a point as $x \to 3^{-\frac{3}{2}}2$. Compare with Boyce and DiPrima [7, page 417]. The same equation with $z_{\mathrm{lo}}(x)$ replacing $z_{\mathrm{up}}(x)$ and with the conditions on $A(x)$ and $\alpha(x)$ holding as $x \to -3^{-\frac{3}{2}}2$ would produce a limit cycle of (5.2) centered around $z_{\mathrm{lo}}(x)$ and vanishing as $x \to -3^{-\frac{3}{2}}2$. Gluing the two parts into a single global vector field is simple.

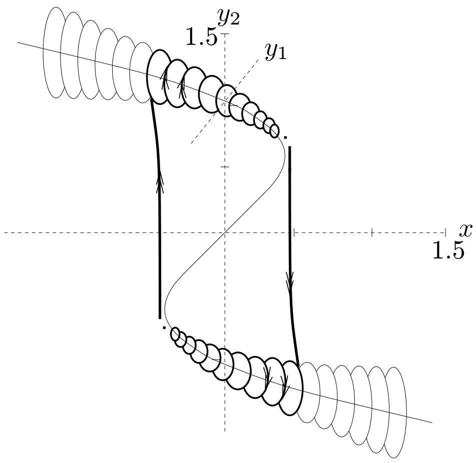

Figure 2

It is easy to see that each of the limit cycles around the points $z_{\mathrm{up}}(x)$ possesses a local asymptotic stability property. Equivalently, the support of the invariant measure induced by the dynamics on each limit cycle is an asymptotically stable attractor as required in Theorem 4.1. Consider now an initial condition, say

$(x(0), y_1(0), y_2(0)) = (-2, 0, 0)$. The invariant measures supported on the limit cycles associated with the upper branch of the graph can be represented as a function $\mu(x)$ as needed in Theorem 4.1, and all the conditions are satisfied for x satisfying $x < 3^{-\frac{3}{2}}2$. The conclusion is as follows. For small ε, the state $x = -2$ hardly changes in a short time interval, while the solution $y_\varepsilon(\cdot)$ converges to the limit cycle around $(y_1, y_2) = (-1.44225, 0)$. Following that short boundary layer interval, the solution continues its fast movement, following closely the limit cycles both topologically and statistically, this while in the x direction there is a slow movement following the x-equation in (5.1). This description is valid until $x_0(t)$ reaches the value $3^{-\frac{3}{2}}2$.

In this specific example one can go beyond the point where $x_0(t)$ is equal to $3^{-\frac{3}{2}}2$. Indeed, right after that, the point $y_\varepsilon(t)$ enters the basin of attraction of the lower branch of the graph. The analogous analysis implies that in a very short interval the solution reaches a neighborhood of the stable limit cycle around $(y_1, y_2) = (-1.44225, 0)$, and the fast dynamics continues along the limit cycles around the lower branch of the equilibria, while a slow down drift of x occurs, until $x_0(t) = -3^{-\frac{3}{2}}2$; and so on and so forth. Thus, the trajectory generates a relaxation oscillation dynamics where the slow motion is only in the x variable, while fast motion prevails in the (y_1, y_2) space, as portrayed in bold in Figure 2 (double arrow signifies fast motion).

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