

SPECTRUM OF THE WEIGHTED LAPLACE OPERATOR
IN UNBOUNDED DOMAINS

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Abstract. We investigate the spectral properties of the differential operator $-r^s \Delta$, $s \geq 0$ with the Dirichlet boundary condition in unbounded domains whose boundaries satisfy some geometrical condition. Considering this operator as a self-adjoint operator in the space with the norm $\|u\|_{L_{2,s}(\Omega)}^2 = \int_{\Omega} r^{-s} |u|^2 dx$, we study the structure of the spectrum with respect to the parameter s . Further we give an estimate of the rate of condensation of discrete spectra when it changes to continuous.

Keywords: Laplace operator, multiplicative perturbation, Dirichlet problem, Friedrichs extension, purely discrete spectra, purely continuous spectra

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1. INTRODUCTION AND MAIN RESULTS

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an unbounded domain whose closure does not contain the origin, with a boundary Γ . Let us consider the differential expression

$$(1) \quad lu = -r^s \Delta u, \quad r = |x|, \quad s \geq 0.$$

We shall treat the differential operator (1) in the Hilbert space $L_{2,s}(\Omega)$ with the norm $\|u\|_{L_{2,s}(\Omega)}^2 = \int_{\Omega} r^{-s} |u|^2 dx$. Let L be the self-adjoint Friedrichs extension in $L_{2,s}(\Omega)$ of the minimal operator generated by the differential expression (1). Then, L is a non-negative self-adjoint operator in $L_{2,s}(\Omega)$ that is an operator of the first boundary value problem for the differential expression (1).

We will study spectral properties of the operator L (location of spectrum on the real axis, density of spectrum on some sets, structure of the spectrum) with respect

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to the parameter s . For the operator (1) we know conditions on s and the domain Ω , guaranteeing the discreteness of the spectrum of the operator L ([1]). In particular, in [1] it was proved that the spectrum of the operator L is discrete at $s > 2$. On the other side, as follows from the results of [2], in the case $\Omega = \mathbb{R}^n$ with $0 \leq s \leq 2$ the spectrum of the operator with coefficients equal to (1) in the neighborhood of infinity is continuous and for $0 \leq s < 2$ fills the complete positive semi-axis.

Let $S_\eta = \Omega \cap \{r = \eta\}$, $\eta > 0$, and Σ_η be the set of points x belonging to the unit sphere Σ and satisfying $\eta x \in S_\eta$. In the sequel we will consider domains Ω such that

$$(2) \quad \Sigma_{\eta_1} \subset \Sigma_{\eta_2}, \quad \eta_1 < \eta_2,$$

(it is the star-shapeness condition for the set $\mathbb{R}^n \setminus \Omega$ with respect to the origin). Denote by $\hat{\lambda}(\eta)$ the modulus of the first eigenvalue of the Laplace-Beltrami operator in Σ_η with zero Dirichlet data on $\partial\Sigma_\eta$. By our supposition $\hat{\lambda}(\eta)$ is a decreasing nonnegative function on $[\inf_{x \in \Omega} r, +\infty)$. Denote $\Lambda = \lim_{\eta \rightarrow \infty} \hat{\lambda}(\eta)$. We will also suppose without loss of generality that $\ln r > 1$ in Ω .

Our first statement localizes the spectrum set $\sigma(L)$ of the operator L on the real axis.

Theorem 1. *The spectrum of L has the following properties:*

- i) if $0 \leq s < 2$, then $\sigma(L) = [0, +\infty)$;
- ii) if $s = 2$, then $\sigma(L) = [\frac{1}{4}(n-2)^2 + \Lambda, +\infty)$;
- iii) if $s > 2$, then $\sigma(L) \subset (\frac{1}{4}(n-2)^2 + \Lambda, +\infty)$.

The next statement declares that there exists a critical value of s for which the spectrum of L becomes discrete.

Theorem 2. *The spectrum of L has the following properties:*

- i) if $0 \leq s \leq 2$ and $\Gamma \in C^2$, then the spectrum of the operator L is continuous;
- ii) if $s > 2$, then the spectrum of the operator L is discrete.

It is natural to expect that the discrete spectrum condenses on the semi-axis $[\frac{1}{4}(n-2)^2 + \Lambda, +\infty)$ at $s \rightarrow 2+0$. In the next statement we establish an estimate of the rate of this condensation.

Theorem 3. *For any $\lambda \in [\frac{1}{4}(n-2)^2 + \Lambda, +\infty)$ there exist a constant $C > 0$ and a number $s_0 > 2$ such that for any $s \in (2, s_0]$ the following relation holds:*

$$(3) \quad \sigma(L) \cap (\lambda - \delta(s), \lambda + \delta(s)) \neq \emptyset,$$

where

$$\delta(s) = \hat{\lambda}(\ln \ln(1/(s-2))) - \Lambda + C \frac{\ln \ln \ln(1/(s-2))}{\ln \ln(1/(s-2))}.$$

The constant C depends on λ .

2. ENERGY SPACE AND DOMAIN

Let us define the space of functions

$$H_s^1(\Omega) = \{u: u \in L_{2,s}(\Omega) \cap H^1(\Omega_R), R > 0, u_{x_j} \in L_2(\Omega), j = 1, \dots, n\},$$

where $\Omega_R = \Omega \cap \{r < R\}$, with the norm $\|u\|_{H_s^1(\Omega)}^2 = \int_{\Omega} (|\nabla u|^2 + r^{-s}|u|^2) dx$. By $\mathring{H}_s^1(\Omega)$ denote the subspace of $H_s^1(\Omega)$ which is the closure of the set of functions $u \in H_s^1(\Omega)$ vanishing in a neighborhood of Γ . Consider the quadratic form $A[u] = \int_{\Omega} |\nabla u|^2 dx$ on the set of functions $\mathring{C}^\infty(\Omega) \subset L_{2,s}(\Omega)$.

Lemma 1. *The form $A[u]$ is closeable.*

Proof. Let $\{u_j\} \subset \mathring{C}^\infty(\Omega)$, $j = 1, 2, \dots$ be a sequence of functions such that $A[u_j - u_l] \rightarrow 0$, $j, l \rightarrow \infty$ and $\|u_j\|_{L_{2,s}(\Omega)} \rightarrow 0$, $j \rightarrow \infty$. Now, by $\|u_j\|_{H_s^1(\Omega)}^2 = A[u_j] + \|u_j\|_{L_{2,s}(\Omega)}^2$ we have that the sequence u_j is fundamental in the space $H_s^1(\Omega)$. By $\hat{u} \in H_s^1(\Omega)$ denote the limit function: $\lim_{j \rightarrow \infty} \|\hat{u} - u_j\|_{H_s^1(\Omega)} = 0$. Then $\lim_{j \rightarrow \infty} \|u_j - \hat{u}\|_{L_{2,s}(\Omega)} = 0$, i.e. $\|\hat{u}\|_{L_{2,s}(\Omega)} \leq \lim_{j \rightarrow \infty} \|u_j\|_{L_{2,s}(\Omega)} + \lim_{j \rightarrow \infty} \|\hat{u} - u_j\|_{L_{2,s}(\Omega)} = 0$. Hence, $\hat{u} = 0$ and $\lim_{j \rightarrow \infty} \|u_j\|_{L_{2,s}(\Omega)} = 0$. So, the possibility to close the form $A[u]$ is proved.

By Lemma 1 the energy space H_A of the operator L is the closure of the set of functions $\mathring{C}^\infty(\Omega)$ in the norm $\|u\|_{H_A^1(\Omega)}^2 = A[u] + \|u_j\|_{L_{2,s}(\Omega)}^2$.

Lemma 2. *The energy space of the operator L is*

$$(4) \quad H_A = \left\{ u: u \in \mathring{H}_s^1(\Omega), \int_{\Omega} r^{-2} \ln^{-2q} r |u|^2 dx < \infty \right\},$$

where $q = 0$ for $n \geq 3$ and $q = 1$ for $n = 2$.

Proof. It is sufficient to prove that for any function $u \in \mathring{H}_s^1(\Omega)$ such that $\int_{\Omega} r^{-2} \ln^{-2q} r |u|^2 dx < \infty$ and for any $\varepsilon > 0$ there exists a function $\tilde{u} \in \mathring{C}^\infty(\Omega)$, such that $A[u - \tilde{u}] < \varepsilon$.

First let us prove that for any function $u \in H_A$ the integral $\int_{\Omega} r^{-2} \ln^{-2q} r |u|^2 dx$ converges. We use the inequalities

$$(5) \quad \int_{\Omega_R} r^{-2} \ln^{-2} r |u|^2 dx \leq 4 \int_{\Omega_R} |u_r|^2 dx, \quad n = 2,$$

$$(6) \quad \int_{\Omega_R} r^{-2} |u|^2 dx \leq \frac{2}{(n-2)R} \int_{S_R} |u|^2 ds + \frac{4}{(n-2)^2} \int_{\Omega_R} |u_r|^2 dx, \quad n \geq 3,$$

which are valid for all $R > 1$ for functions $u \in H^1(\Omega_R)$ such that $u|_{\Gamma} = 0$.

For any function $u \in H_A$ there exists a sequence of functions $\{u_j\} \subset \mathring{C}^\infty(\Omega)$, $j = 1, 2, \dots$ such that $A[u - u_j] \rightarrow 0$, $\|u - u_j\|_{L_{2,s}(\Omega)} \rightarrow 0$, $j \rightarrow \infty$. Apply (5) (6) to u_j with sufficiently large R . Since the term on the right hand side containing $(u_j)_r$ is bounded for all j and R , we obtain $\int_{\Omega} r^{-2} \ln^{-2q} r |u_j|^2 dx \leq C_1$. Since $r^{-1} \ln^{-q} r u_j$ must converge in $L_2(\Omega)$ weakly to $r^{-1} \ln^{-q} r u$, we obtain

$$(7) \quad \int_{\Omega} r^{-2} \ln^{-2q} r |u|^2 dx \leq C_1.$$

Conversely, let us suppose that $u \in \mathring{H}_s^1(\Omega)$ and $\int_{\Omega} r^{-2} \ln^{-2q} r |u|^2 dx < \infty$. Let $n \geq 3$. Consider functions $\xi_m(x) = \eta(\ln(r/m) + 1)$, $m = 1, 2, \dots$, where $\eta(t) \in \mathring{C}^\infty([0, +\infty))$ is a nonnegative function satisfying the condition $0 \leq \eta \leq 1$ and such that $\eta = 1$ for $0 < t < 1$, $\eta = 0$ for $t > 2$. Hence $\xi_m = 1$, $x \in \Omega_m$, $\xi_m = 0$, $x \in \Omega \setminus \Omega_{me}$. We have an estimate

$$(8) \quad |\nabla \xi_m| = \left| \frac{d}{dr} \xi_m \right| = |\eta'(\ln(r/m) + 1)| r^{-1} \leq C_2 r^{-1}, \quad x \in \Omega.$$

The function $u \xi_m$ belongs to the space H_A . Let us prove that $\|u - u \xi_m\|_{H_s^1(\Omega)} \rightarrow 0$, $m \rightarrow \infty$. We get

$$(9) \quad \|u - u \xi_m\|_{H_s^1(\Omega)}^2 \leq 2(I_{1,m} + I_{2,m}),$$

where

$$I_{1,m} = \int_{\Omega \setminus \Omega_m} (|\nabla u|^2 + r^{-s} |u|^2) (1 - \xi_m)^2 dx, \quad I_{2,m} = \int_{\Omega_{me} \setminus \Omega_m} |u|^2 |\nabla \xi_m|^2 dx.$$

Since $u \in \mathring{H}_s^1(\Omega)$, we obtain $I_{1,m} \rightarrow 0$, $m \rightarrow \infty$. Furthermore, it follows from (7), (8) that

$$I_{2,m} \leq C_2^2 \int_{\Omega_{me} \setminus \Omega_m} r^{-2} |u|^2 dx \rightarrow 0, \quad m \rightarrow \infty.$$

Thus, $\|u - u\xi_m\|_{H_s^1(\Omega)} \rightarrow 0$, $m \rightarrow \infty$. Now, by virtue of $u\xi_m \in \mathring{H}_s^1(\Omega_{m\epsilon})$ there exist functions $\tilde{u}_m \in \mathring{C}^\infty(\Omega_{m\epsilon})$, such that $A[\tilde{u}_m - u\xi_m] \leq \|\tilde{u}_m - u\xi_m\|_{H_s^1(\Omega_{m\epsilon})}^2 \rightarrow 0$, $m \rightarrow \infty$. Consider the zero continuation of the function \tilde{u}_m to the set $\Omega \setminus \Omega_{m\epsilon}$ and denote the continued function also by \tilde{u}_m . Therefore $\tilde{u}_m \in \mathring{C}^\infty(\Omega)$ and $A[u - \tilde{u}_m] \leq 2(A[u - u\xi_m] + A[u\xi_m - \tilde{u}_m]) \rightarrow 0$, $m \rightarrow \infty$. The existence of a function $\tilde{u} \in \mathring{C}^\infty(\Omega)$ such that $A[u - \tilde{u}] < \epsilon$ in the case $n \geq 3$ is proved.

Let us consider the case $n = 2$. Put $\xi_m(x) = \eta(\ln(\ln r / \ln m))$ where the function η is the same as for $n \geq 3$. Then $\xi_m = 1$ for $x \in \Omega_{m\epsilon}$ and $\xi_m = 0$ for $x \in \Omega \setminus \Omega_{m\epsilon^2}$. For the function ξ_m we obtain

$$(10) \quad |\nabla \xi_m| = \left| \frac{d}{dr} \xi_m \right| = |\eta'(\ln(\ln r / \ln m))|(r \ln r)^{-1} \leq C_2(r \ln r)^{-1}, \quad x \in \Omega.$$

The estimate (9) with

$$I_{1,m} = \int_{\Omega \setminus \Omega_{m\epsilon}} (|\nabla u|^2 + r^{-s}|u|^2)(1 - \xi_m)^2 dx, \quad I_{2,m} = \int_{\Omega_{m\epsilon^2} \setminus \Omega_{m\epsilon}} |u|^2 |\nabla \xi_m|^2 dx$$

holds. As in the case $n \geq 3$, we obtain that $I_{1,m} \rightarrow 0$, $m \rightarrow \infty$. It follows from the estimate (7) with $q = 1$ and (10) that

$$I_{2,m} \leq C_2^2 \int_{\Omega_{m\epsilon^2} \setminus \Omega_{m\epsilon}} r^{-2} \ln^{-2} r |u|^2 dx \rightarrow 0, \quad m \rightarrow \infty.$$

Thus, $A[u - u\xi_m] \rightarrow 0$, $m \rightarrow \infty$. Now, we get the existence of a sequence $\tilde{u}_m \in \mathring{C}^\infty(\Omega)$, $\text{supp } \tilde{u}_m \subset \Omega_{m\epsilon^2}$ such that $A[\tilde{u}_m - u\xi_m] \leq \|\tilde{u}_m - u\xi_m\|_{H_s^1(\Omega_{m\epsilon^2})}^2 \rightarrow 0$, $m \rightarrow \infty$. Hence the existence of a function $\tilde{u} \in \mathring{C}^\infty(\Omega)$ such that $A[u - \tilde{u}] < \epsilon$ for $n = 2$ is proved. This completes the proof of Lemma 2.

Lemma 3. *The domain of the operator L is*

$$D(L) = \left\{ u: u \in \mathring{H}_s^1(\Omega) \cap H_{\text{loc}}^2(\Omega), lu \in L_{2,s}(\Omega), \int_{\Omega} r^{-2} \ln^{-2q} r |u|^2 dx < \infty \right\}.$$

In the case $\Gamma \in C^2$ the domain of the operator L is

$$D(L) = \left\{ u: u \in \mathring{H}_s^1(\Omega) \cap H^2(\Omega_R), R > 0, lu \in L_{2,s}(\Omega), \int_{\Omega} r^{-2} \ln^{-2q} r |u|^2 dx < \infty \right\}.$$

Proof. Applying interior estimates for the derivatives of solutions of elliptic equations ([3], p. 204, Lemma 7.1) to $u \in \mathring{H}_s^1(\Omega)$, $lu \in L_{2,s}(\Omega)$ and any domain

$\Omega' \Subset \Omega$, we get $u \in H^2(\Omega')$. If, furthermore, $\Gamma \in C^2$, applying the boundary estimates for derivatives of solutions of elliptic equations ([3], p. 224, Theorem 9.2), for all $R > 0$ we obtain $u \in H^2(\Omega_R)$. This completes the proof of Lemma 3.

3. LOCALIZATION OF SPECTRUM

It follows from the inequalities (5), (6) that for functions $u \in H_A$ we have the lower estimate

$$\begin{aligned} A[u] &= \int_{\Omega} |u_r|^2 dx + \int_{\Omega} r^{-2} |\nabla_{\Theta} u|^2 dx \\ &\geq \frac{(n-2-q)^2}{4} \int_{\Omega} r^{-2} \ln^{-2q} r |u|^2 dx + \int_{\Omega} r^{-2} |\nabla_{\Theta} u|^2 dx, \quad u(x) = u(r, \Theta). \end{aligned}$$

Since $\hat{\lambda}(r)$ is the modulus of the first eigenvalue of the Dirichlet problem for the Laplace-Beltrami operator in Σ_r , we get

$$(11) \quad \int_{\Sigma_r} |\nabla_{\Theta} u|^2 d\Theta \geq \hat{\lambda}(r) \int_{\Sigma_r} |u|^2 d\Theta \geq \Lambda \int_{\Sigma_r} |u|^2 d\Theta, \quad r > 0.$$

Therefore,

$$\begin{aligned} (12) \quad A[u] &\geq \frac{(n-2-q)^2}{4} \int_{\Omega} r^{-2} \ln^{-2q} r |u|^2 dx \\ &\quad + \Lambda \int_{\Omega} r^{-2} |u|^2 dx \geq \left(\frac{(n-2)^2}{4} + \Lambda \right) \int_{\Omega} r^{-2} |u|^2 dx, \end{aligned}$$

and for $s \geq 2$ we have an estimate $A[u] \geq \left(\frac{1}{4}(n-2)^2 + \Lambda \right) \|u\|_{L_{2,s}(\Omega)}^2$ and, consequently, $\sigma(L) \subset \left[\frac{1}{4}(n-2)^2 + \Lambda, +\infty \right)$, $s \geq 2$.

Let us prove that for $s > 2$ the number $\frac{1}{4}(n-2)^2 + \Lambda$ does not belong to the spectrum of the operator L . Assume the converse, let $s > 2$ and $\frac{1}{4}(n-2)^2 + \Lambda \in \sigma(L)$. We show that there exists a non-zero function $\hat{u} \in H_A$ such that

$$(13) \quad \int_{\Omega} |\nabla \hat{u}|^2 dx = \left(\frac{(n-2)^2}{4} + \Lambda \right) \int_{\Omega} r^{-s} |\hat{u}|^2 dx.$$

If $\frac{1}{4}(n-2)^2 + \Lambda$ is an eigenvalue of the operator L , the relation (13) holds for the corresponding eigenfunction \hat{u} . If $\frac{1}{4}(n-2)^2 + \Lambda$ is the continuous spectrum point, let us use I. M. Glazman lemma for quadratic forms ([5], Supplement 1, Lemma 3.1'), which is a modification of the corresponding operator statement ([4], Chapter 1, Section 1, Theorem 9^{bis}). By this lemma

$$(14) \quad N(\lambda - 0) = \sup_{\{F \subset H_A, A[u] < \lambda \|u\|_H^2, u \in F \setminus \{0\}\}} \dim F,$$

where H is the main space ($H = L_{2,s}(\Omega)$ in our case), F is a linear subspace of H_A , $N(\lambda) = \dim(E_\lambda H)$ where E_λ denotes the spectral projector of the spectral family corresponding to the self-adjoint operator L . As follows from this lemma, if λ is a continuous spectrum point, for any $\delta > 0$ the relation $N(\lambda + \delta) - N(\lambda - \delta) = \infty$ holds. Thus for any $\delta > 0$ there exists a function $u \in H_A$, $u \neq 0$, such that $A[u] \leq (\frac{1}{4}(n-2)^2 + \Lambda + \delta) \|u\|_{L_{2,s}(\Omega)}^2$.

Let us choose a sequence $\delta_j > 0$, $\delta_j \rightarrow 0$, $j \rightarrow \infty$. Then there exists a non-zero sequence $u_j \in H_A$ such that

$$(15) \quad \int_{\Omega} |\nabla u_j|^2 dx \leq \left(\frac{(n-2)^2}{4} + \Lambda + \delta_j \right) \int_{\Omega} r^{-s} |u_j|^2 dx.$$

Let $\|u_j\|_{L_{2,s}(\Omega)} = 1$. Then $\int_{\Omega} |\nabla u_j|^2 dx \leq \frac{1}{4}(n-2)^2 + \Lambda + \delta_j$ and, clearly, the inequalities (5), (6) imply

$$(16) \quad \int_{\Omega} r^{-2} \ln^{-2q} r |u_j|^2 dx \leq C_3.$$

From the sequence $\{u_j\}$ let us choose a subsequence which is weakly convergent in the space $L_{2,s}(\Omega)$ and show that it is pre-compact in $L_{2,s}(\Omega)$. In the same way as in Rellich's theorem about the compact imbedding of $\mathring{H}^1(\Omega')$ into $L_2(\Omega')$ in a bounded domain Ω' , we can prove that for any $R > 0$ the space $H_s^1(\Omega_R)$ imbeds compactly into $L_{2,s}(\Omega_R)$. Hence, there exists a function $\hat{u} \in L_{2,s}(\Omega)$ such that for any $R > 0$ we have $\lim_{j \rightarrow \infty} \|\hat{u} - u_j\|_{L_{2,s}(\Omega_R)} = 0$. It means that for any sequence $\{R_j\}$, $R_j \rightarrow \infty$, $j \rightarrow \infty$, it is possible to choose a subsequence $\{u_j\}$ (denoted also by $\{u_j\}$) such that $\|\hat{u} - u_j\|_{L_{2,s}(\Omega_{R_j})} < j^{-1}$. Therefore by (16) we have

$$\begin{aligned} \|\hat{u} - u_j\|_{L_{2,s}(\Omega)}^2 &= \|\hat{u} - u_j\|_{L_{2,s}(\Omega_{R_j})}^2 + \|\hat{u} - u_j\|_{L_{2,s}(\Omega \setminus \Omega_{R_j})}^2 \\ &< j^{-2} + 2(\|\hat{u}\|_{L_{2,s}(\Omega \setminus \Omega_{R_j})}^2 + \|u_j\|_{L_{2,s}(\Omega \setminus \Omega_{R_j})}^2) \\ &\leq j^{-2} + C_4 R_j^{2-s} \ln^{2q} R_j \rightarrow 0, \quad j \rightarrow \infty. \end{aligned}$$

So, the convergence of the sequence $\{u_j\}$ to \hat{u} in the space $L_{2,s}(\Omega)$ is proved. This, in particular, yields that $\|\hat{u}\|_{L_{2,s}(\Omega)} = 1$. It implies that

$$(17) \quad \int_{\Omega} |\nabla \hat{u}|^2 dx \leq \liminf_{j \rightarrow \infty} \int_{\Omega} |\nabla u_j|^2 dx = \left(\frac{(n-2)^2}{4} + \Lambda \right) \int_{\Omega} r^{-s} |\hat{u}|^2 dx.$$

The relation (13) is proved.

For the proof of the relation $\frac{1}{4}(n-2)^2 + \Lambda \notin \sigma(L)$ let us first consider the case $\frac{1}{4}(n-2)^2 + \Lambda > 0$. In this case

$$(18) \quad A[\hat{u}] = \left(\frac{(n-2)^2}{4} + \Lambda\right) \int_{\Omega} r^{-s} |\hat{u}|^2 dx < \left(\frac{(n-2)^2}{4} + \Lambda\right) \int_{\Omega} r^{-2} |\hat{u}|^2 dx,$$

which contradicts (12).

In the case $\frac{1}{4}(n-2)^2 + \Lambda = 0$ we have by (13) the equality $\int_{\Omega} |\nabla \hat{u}|^2 dx = 0$. Thus, $\nabla \hat{u} = 0$ and $\hat{u} = \text{const}$. But then $\hat{u} = 0$, which contradicts $\|\hat{u}\|_{L_{2,s}(\Omega)} = 1$. So, $\frac{1}{4}(n-2)^2 + \Lambda \notin \sigma(L)$ and the relation $\sigma(L) \subset (\frac{1}{4}(n-2)^2 + \Lambda, +\infty)$ holds true. Proof of the point iii) of Theorem 1 is now complete.

Finally, for $s > 2$ we have that any sequence bounded in the space H_A is precompact in $L_{2,s}(\Omega)$. By F. Rellich criterion ([5], Supplement 1, Par. 3), the spectrum of the operator L is discrete at $s > 2$. This completes the proof of the point ii) of Theorem 2.

4. DENSITY OF SPECTRUM ON THE SEMI-AXIS

First consider the case $0 \leq s < 2$. Let us use the relation (14). By this relation the number of points of the spectrum for the operator L in the interval $(\lambda - \delta, \lambda + \delta)$ with account of multiplicity is equal to the maximal dimension of the linear manifolds $F \subset H_A$ for which the following inequality is valid:

$$(19) \quad |A[u] - \lambda \|u\|_H^2| < \delta \|u\|_H^2, \quad u \neq 0.$$

In our case the relation (19) can be written as

$$(20) \quad \left| \int_{\Omega} (|\nabla u|^2 - \lambda r^{-s} |u|^2) dx \right| < \delta \int_{\Omega} r^{-s} |u|^2 dx.$$

Denote by $v_{\varrho}(\Theta) \in \mathring{H}^1(\Sigma_{\varrho})$, $\varrho > 0$, the first eigenfunction of the Laplace-Beltrami operator in the domain Σ_{ϱ} . Hence $\int_{\Sigma_{\varrho}} |\nabla_{\Theta} v_{\varrho}|^2 d\Theta = \hat{\lambda}(\varrho) \int_{\Sigma_{\varrho}} v_{\varrho}^2 d\Theta$. Let us continue the function v_{ϱ} by zero to the set Σ . Therefore $v_{\varrho} \in H^1(\Sigma)$ and $\int_{\Sigma} |\nabla_{\Theta} v_{\varrho}|^2 d\Theta = \hat{\lambda}(\varrho) \int_{\Sigma} v_{\varrho}^2 d\Theta$. We choose a nonzero real-valued function $\varphi(t) \in \mathring{C}^{\infty}(0, +\infty)$ such that $\text{supp } \varphi = [1, 2]$. Consider functions

$$(21) \quad u_{\varepsilon}(r, \Theta) = \sqrt{\varepsilon} r^{1-n/2} H_{\frac{n-2}{2-s}}^{(1)} \left(\frac{2\sqrt{\lambda}}{2-s} r^{1-s/2} \right) \varphi(\varepsilon r^{1-s/2}) v_{\varepsilon^{-2/(2-s)}}(\Theta),$$

$\varepsilon > 0, \lambda > 0,$

where $H_p^{(1)}(z)$ is a Hankel function. In this case we have $\text{supp } u_\varepsilon \subset \Omega$, $u_\varepsilon \in \mathring{H}_s^1(\Omega)$ and the inequality (20) has the form

$$(22) \quad \left| \int_{\mathbb{R}^n} (|\nabla u_\varepsilon|^2 - \lambda r^{-s} |u_\varepsilon|^2) dx \right| < \delta \int_{\mathbb{R}^n} r^{-s} |u_\varepsilon|^2 dx.$$

Consider the behavior of the left hand and right hand parts of the inequality (22) for $\varepsilon \rightarrow 0$. Let us note that $\inf_{x \in \text{supp } u_\varepsilon} r > \varepsilon^{-2/(2-s)} \rightarrow \infty$, $\varepsilon \rightarrow 0$. Now we use the relations for derivatives of Hankel functions and the asymptotic expansions of Hankel functions of large argument ([6], Chapter 9):

$$(23) \quad H_p^{(1)'}(z) = pz^{-1}H_p^{(1)}(z) - H_{p+1}^{(1)}(z),$$

$$(24) \quad H_p^{(1)}(z) = \sqrt{2(\pi z)^{-1}} \exp(i(z - \pi(p + \frac{1}{2})/2))(1 + O(|z|^{-1})), \quad |z| \rightarrow \infty.$$

By (24) we have

$$\int_{\mathbb{R}^n} r^{-s} |u_\varepsilon|^2 dx = \frac{\varepsilon(2-s)}{\pi\sqrt{\lambda}} \int_0^\infty (r^{-s/2} + f_1(r)) \varphi^2(\varepsilon r^{1-s/2}) dr \int_\Sigma v_{\varepsilon^{-2/(2-s)}}^2 d\Theta,$$

where $|f_1(r)| \leq C_5 r^{-1}$. Therefore,

$$\begin{aligned} \int_{\mathbb{R}^n} r^{-s} |u_\varepsilon|^2 dx &= \left(\frac{2\varepsilon}{\pi\sqrt{\lambda}} \int_0^\infty \varphi^2(\varepsilon z) dz + J_1(\varepsilon) \right) \int_\Sigma v_{\varepsilon^{-2/(2-s)}}^2 d\Theta \\ &= \left(\frac{2}{\pi\sqrt{\lambda}} \int_0^\infty \varphi^2(t) dt + J_1(\varepsilon) \right) \int_\Sigma v_{\varepsilon^{-2/(2-s)}}^2 d\Theta, \end{aligned}$$

where

$$|J_1| \leq \frac{(2-s)\varepsilon}{\pi\sqrt{\lambda}} \int_0^\infty |f_1(r)| \varphi^2(\varepsilon r^{1-s/2}) dr \leq \frac{2C_5\varepsilon}{\pi\sqrt{\lambda}} \int_0^\infty t^{-1} \varphi^2(t) dt.$$

Thus,

$$(25) \quad \int_{\mathbb{R}^n} r^{-s} |u_\varepsilon|^2 dx = (C_6 + O(\varepsilon)) \int_\Sigma v_{\varepsilon^{-2/(2-s)}}^2 d\Theta, \quad \varepsilon \rightarrow 0, \quad C_6 > 0.$$

Consider now the left hand side of the relation (22). It follows from (23)–(24) that

$$\begin{aligned} (26) \quad & \left| \int_{\mathbb{R}^n} (|\nabla u_\varepsilon|^2 - \lambda r^{-s} |u_\varepsilon|^2) dx \right| \\ &= \frac{\varepsilon(2-s)}{\pi\sqrt{\lambda}} \left| \int_0^\infty ((f_2(r) + \hat{\lambda}(\varepsilon^{-2/(2-s)})f_3(r))\varphi^2(\varepsilon r^{1-s/2}) \right. \\ & \quad + \varepsilon((2-s)(2-2n+s)(4r)^{-1} + f_4(r))\varphi(\varepsilon r^{1-s/2})\varphi'(\varepsilon r^{1-s/2}) \\ & \quad \left. + \varepsilon^2((2-s)^2 r^{-s/2}/4 + f_5(r))\varphi'^2(\varepsilon r^{1-s/2}) \right) dr \int_\Sigma v_{\varepsilon^{-2/(2-s)}}^2 d\Theta, \end{aligned}$$

where $f_j, j = 2, 3, 4, 5$, are functions satisfying the inequalities $|f_j| \leq C_7 r^{-1}, j = 2, 5$, $|f_j| \leq C_7 r^{s/2-2}, j = 3, 4$. Using equality (26), we get the estimate

$$(27) \quad \left| \int_{\mathbb{R}^n} (|\nabla u_\varepsilon|^2 - \lambda r^{-s} |u_\varepsilon|^2) dx \right| \leq C_8 \varepsilon \int_0^\infty r^{-1} (\varphi^2(\varepsilon r^{1-s/2}) + \varphi'^2(\varepsilon r^{1-s/2})) dr \\ = C_9 \varepsilon \int_0^\infty t^{-1} (\varphi^2(t) + \varphi'^2(t)) dt = C_{10} \varepsilon.$$

It follows from (25) and (27) that for any $\delta > 0$ there exists $\varepsilon > 0$ such that the function u_ε satisfies inequality (20). This implies $\sigma(L) \cap (\lambda - \delta, \lambda + \delta) \neq \emptyset$. Thus, $\sigma(L) = [0, \infty)$. Point i) of Theorem 1 is proved.

Let us investigate now the case $s = 2$. Consider the functions

$$(28) \quad u_\varepsilon(r, \Theta) = \sqrt{\varepsilon} r^{1-n/2} e^{i\sqrt{\lambda - (n-2)^2/4 - \Lambda} \ln r} \varphi(\varepsilon \ln r) v_{e^{1/\varepsilon}}(\Theta), \\ \lambda > \frac{1}{4}(n-2)^2 + \Lambda, \quad \varepsilon > 0,$$

where the function φ is the same as for $0 \leq s < 2$.

In this case we have $\text{supp } u_\varepsilon \subset \Omega$, $u_\varepsilon \in \mathring{H}_2^1(\Omega)$, and the inequality (20) can be written as

$$(29) \quad \left| \int_{\mathbb{R}^n} (|\nabla u_\varepsilon|^2 - \lambda r^{-2} |u_\varepsilon|^2) dx \right| < \delta \int_{\mathbb{R}^n} r^{-2} |u_\varepsilon|^2 dx.$$

Let us study the behavior of the left hand and right hand sides of the inequality (29) when $\varepsilon \rightarrow 0$. We have

$$(30) \quad \int_{\mathbb{R}^n} r^{-2} |u_\varepsilon|^2 dx \\ = \varepsilon \int_0^\infty r^{-1} \varphi^2(\varepsilon \ln r) dr \int_\Sigma v_{e^{1/\varepsilon}}^2 d\Theta \\ = \int_0^\infty \varphi^2(t) dt \int_\Sigma v_{e^{1/\varepsilon}}^2 d\Theta = C_{11} \int_\Sigma v_{e^{1/\varepsilon}}^2 d\Theta,$$

$$(31) \quad \left| \int_{\mathbb{R}^n} (|\nabla u_\varepsilon|^2 - \lambda r^{-2} |u_\varepsilon|^2) dx \right| \\ = \left| \varepsilon^2 \int_0^\infty r^{-1} ((2-n)\varphi(\varepsilon \ln r)\varphi'(\varepsilon \ln r) + \varepsilon\varphi'^2(\varepsilon \ln r)) dr \int_\Sigma v_{e^{1/\varepsilon}}^2 d\Theta \right. \\ \left. + \varepsilon \int_0^\infty r^{-1} \varphi^2(\varepsilon \ln r) dr \int_\Sigma (|\nabla_\Theta v_{e^{1/\varepsilon}}|^2 - \Lambda v_{e^{1/\varepsilon}}^2) d\Theta \right| \\ \leq \int_0^\infty (\varepsilon(n-2)|\varphi(t)||\varphi'(t)| \\ + \varepsilon^2 \varphi'^2(t) + (\hat{\lambda}(e^{1/\varepsilon}) - \Lambda)\varphi^2(t)) dt \int_\Sigma v_{e^{1/\varepsilon}}^2 d\Theta.$$

From the relations (30), (31) we get that for any $\delta > 0$ there exists $\varepsilon > 0$ such that the function u_ε satisfies the inequality (29). This implies that $\sigma(L) \cap (\lambda - \delta, \lambda + \delta) \neq \emptyset$. Thus, $\sigma(L) = [\frac{1}{4}(n-2)^2 + \Lambda, \infty)$. Point ii) of Theorem 1 is proved.

5. ON THE RATE OF CONDENSATION OF THE DISCRETE SPECTRUM

Let $s > 2$ and let the spectrum of the operator L be discrete. For any $\lambda \in (\frac{1}{4}(n-2)^2 + \Lambda, \infty)$ consider functions

$$(32) \quad u_s(r, \Theta) = r^{1-n/2} e^{i\sqrt{\lambda - (n-2)^2/4 - \Lambda} \ln r} \eta_s(r) v_{\ln \ln(1/(s-2))}(\Theta),$$

where $\eta_s(r) = r / \ln \ln(1/(s-2)) - 1$ for $\ln \ln(1/(s-2)) < r < 2 \ln \ln(1/(s-2))$, $\eta_s(r) = 1$ for $2 \ln \ln(1/(s-2)) < r < \ln(1/(s-2))$, $\eta_s(r) = 2 - r / \ln(1/(s-2))$ for $\ln(1/(s-2)) < r < 2 \ln(1/(s-2))$ and $\eta_s(r) = 0$ in the other cases, the function $v_\varrho(\Theta)$ being the same as in the proof of point ii) of Theorem 1. Let us continue the function v_ϱ by zero to Σ . As follows from (19), to prove the relation (3) it is sufficient for some $s_0 > 2$ and some constant $C > 0$ for all $2 < s < s_0$ establish inequality

$$(33) \quad \left| \int_{\mathbb{R}^n} (|\nabla u_s|^2 - \lambda r^{-s} |u_s|^2) dx \right| < \left(\hat{\lambda} \left(\ln \ln \frac{1}{s-2} \right) - \Lambda + C \frac{\ln \ln \ln(1/(s-2))}{\ln \ln(1/(s-2))} \right) \int_{\mathbb{R}^n} r^{-s} |u_s|^2 dx.$$

By (32) we have

$$(34) \quad \begin{aligned} & \int_{\mathbb{R}^n} r^{-s} |u_s|^2 dx \\ &= \int_0^\infty r^{1-s} \eta_s^2(r) dr \int_\Sigma v_{\ln \ln(1/(s-2))}^2 d\Theta \\ &= \left(\frac{\ln^{2-s} \ln(1/(s-2)) - \ln^{2-s}(1/(s-2))}{s-2} + O(1) \right) \int_\Sigma v_{\ln \ln(1/(s-2))}^2 d\Theta \\ &= \left(\ln \ln \frac{1}{s-2} + O\left(\ln \ln \ln \frac{1}{s-2} \right) \right) \int_\Sigma v_{\ln \ln(1/(s-2))}^2 d\Theta. \end{aligned}$$

Consider the behavior of the left hand side of the inequality (33) for $s \rightarrow 2 + 0$:

$$\begin{aligned}
 (35) \quad & \left| \int_{\mathbb{R}^n} (|\nabla u_s|^2 - \lambda r^{-s} |u_s|^2) dx \right| \\
 &= \left| \int_0^\infty (\eta_s'^2 + (2-n)r^{-1}\eta_s'\eta_s + (\lambda(r^{-1} - r^{1-s}) - \Lambda r^{-1})\eta_s^2) dr \right. \\
 &\quad \left. \times \int_\Sigma v_{\ln \ln(1/(s-2))}^2 d\Theta + \int_0^\infty r^{-1}\eta_s^2 dr \int_\Sigma |\nabla_\Theta v_{\ln \ln(1/(s-2))}|^2 d\Theta \right| \\
 &= \left| \int_0^\infty (\eta_s'^2 + (2-n)r^{-1}\eta_s'\eta_s \right. \\
 &\quad \left. + (\lambda(r^{-1} - r^{1-s}) + (\hat{\lambda}(\ln \ln \frac{1}{s-2}) - \Lambda)r^{-1})\eta_s^2) dr \right| \int_\Sigma v_{\ln \ln(1/(s-2))}^2 d\Theta \\
 &\leq \left(3(n-1) + \lambda \int_0^\infty (r^{-1} - r^{1-s})\eta_s^2 dr + (\hat{\lambda}(\ln \ln \frac{1}{s-2}) - \Lambda) \int_0^\infty r^{-1}\eta_s^2 dr \right) \\
 &\quad \times \int_\Sigma v_{\ln \ln(1/(s-2))}^2 d\Theta \\
 &< \left(\hat{\lambda}(\ln \ln \frac{1}{s-2}) - \Lambda + C_{12} \frac{\ln \ln \ln(1/(s-2))}{\ln \ln(1/(s-2))} \right) \\
 &\quad \times \ln \ln(1/(s-2)) \int_\Sigma v_{\ln \ln(1/(s-2))}^2 d\Theta, \quad C_{12} > 0.
 \end{aligned}$$

Hence, the inequality (33) follows from (34), (35). Proof of Theorem 3 is complete.

6. CONTINUITY OF SPECTRUM

Let us prove continuity of the spectrum of the operator L for $0 \leq s \leq 2$. Let $\lambda > 0$ and $u \in D(L)$ be non-zero functions, satisfying the equation $\Delta u + \lambda r^{-s}u = 0$ and vanishing on Γ (we consider the function u to be real-valued). By Lemma 3 for $\Gamma \in C^2$ we have the inclusion $u \in H^2(\Omega_R)$, $R > 0$. We multiply the equation by $2ru_r$ and integrate over the domain Ω_R . Thus we have the equality

$$\begin{aligned}
 (36) \quad & R \int_{S_R} \left(2u_r^2 - |\nabla u|^2 + \lambda R^{-s}u^2 + \frac{n-2}{R}uu_r \right) ds_x \\
 &\quad + \int_{\Gamma_R} (\nu, x) \left(\frac{\partial u}{\partial \nu} \right)^2 ds_x - \lambda(2-s) \int_{\Omega_R} r^{-s}u^2 dx = 0,
 \end{aligned}$$

where ν is the outward unit normal vector to Γ . From (36) we get the inequalities:

$$\begin{aligned}
 (37) \quad & \frac{R}{2} \int_{S_R} (nu_r^2 + (2\lambda + (n-2))R^{-s}u^2) ds_x \\
 & \geq R \int_{S_R} (u_r^2 + \lambda R^{-s}u^2 + \frac{n-2}{2}(u_r^2 + R^{-2}u^2)) ds_x \\
 & \geq R \int_{S_R} \left(u_r^2 + \lambda R^{-s}u^2 + \frac{n-2}{R}u_r u \right) ds_x \geq - \int_{\Gamma_R} (\nu, x) \left(\frac{\partial u}{\partial \nu} \right)^2 ds_x.
 \end{aligned}$$

Let us note that the star-shapeness condition for the set $\mathbb{R}^n \setminus \Omega$ for a smooth surface Γ means that $(\nu, x) \leq 0$, $x \in \Gamma$. The surface Γ is not a cone, so there exists a point $x_0 \in \Gamma$ such that $(\nu, x_0) < 0$. So, $u|_{\Gamma} = 0$, and then by the uniqueness theorem for the solution of the Cauchy problem for elliptic equations ([7]) there exists a neighborhood $U(x_0)$ such that $\int_{\Gamma \cap U(x_0)} (\nu, x) (\partial u / \partial \nu)^2 ds_x < 0$. Therefore, $\int_{S_R} (u_r^2 + R^{-s}u^2) ds_x \geq C_{13}R^{-1}$, $C_{13} > 0$, $R \geq R_0$ and $\|u\|_{H^1_s(\Omega)}^2 \geq \int_{\Omega \cap \{r \geq R_0\}} (u_r^2 + r^{-s}u^2) dx = \int_{R_0}^{\infty} dr \int_{S_R} (u_r^2 + r^{-s}u^2) ds_x \geq C_{13} \int_{R_0}^{\infty} r^{-1} dr = +\infty$, i.e. u is not an eigenfunction of the operator L . This completes the proof of point i) of Theorem 2.

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