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THE ASCOLI PROPERTY FOR FUNCTION SPACES AND THE WEAK TOPOLOGY OF BANACH AND FRÉCHET SPACES

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Abstract. Following [3] we say that a Tychonoff space X is an Ascoli space if every compact subset $\mathcal{K}$ of $C_k(X)$ is evenly continuous; this notion is closely related to the classical Ascoli theorem. Every $k_{\mathbb{R}}$-space, hence any k-space, is Ascoli.

Let X be a metrizable space. We prove that the space $C_k(X)$ is Ascoli iff $C_k(X)$ is a $k_{\mathbb{R}}$-space iff X is locally compact. Moreover, $C_k(X)$ endowed with the weak topology is Ascoli iff X is countable and discrete.

Using some basic concepts from probability theory and measure-theoretic properties of ℓ_1, we show that the following assertions are equivalent for a Banach space E: (i) E does not contain isomorphic copy of ℓ_1, (ii) every real-valued sequentially continuous map on the unit ball B_w with the weak topology is continuous, (iii) B_w is a $k_{\mathbb{R}}$-space, (iv) B_w is an Ascoli space.

We prove also that a Fréchet lcs F does not contain isomorphic copy of ℓ_1 iff each closed and convex bounded subset of F is Ascoli in the weak topology. However we show that a Banach space E in the weak topology is Ascoli iff E is finite-dimensional. We supplement the last result by showing that a Fréchet lcs F which is a quojection is Ascoli in the weak topology iff either F is finite dimensional or F is isomorphic to the product $\mathbb{K}^{\mathbb{N}}$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

1. Introduction

Several topological properties of function spaces have been intensively studied for many years, see for instance [1, 17, 20] and references therein. In particular, various topological properties generalizing metrizability attracted a lot of attention. Let us mention, for example, Fréchet–Urysohn property, sequentiality, k-space property and $k_{\mathbb{R}}$-space property (all relevant definitions are given in Section 2 below). It is well known that

$$\text{metric} \Longrightarrow \begin{matrix}\text{Fréchet–}\\ \text{Urysohn}\end{matrix} \Longrightarrow \text{sequential} \Longrightarrow k\text{-space} \Longrightarrow k_{\mathbb{R}}\text{-space},$$

and none of these implications is reversible (see [9, 21]).

For topological spaces X and Y, we denote by $C_k(X,Y)$ the space $C(X,Y)$ of all continuous functions from X into Y endowed with the compact-open topology. For $\mathbb{I} = [0,1]$, Pol [28] proved the following remarkable result

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Theorem 1.1 ([28]). *Let X be a first countable paracompact space. Then the space $C_k(X,\mathbb{I})$ is a k-space if and only if $X = L \cup D$ is the topological sum of a locally compact Lindelöf space L and a discrete space D.*

Theorem 1.1 easily implies the following result noticed in [13], see also [19], where McCoy proved that for a first-countable paracompact X the space $C_k(X)$ is a k-space if and only if X is hemicompact.

Corollary 1.2. *For a metric space X, the space $C_k(X)$ is a k-space if and only if $C_k(X)$ is a Polish space if and only if X is a Polish locally compact space.*

Note also that by a result of Pytkeev [31], for a topological space X the space $C_k(X)$ is a k-space if and only if it is Fréchet–Urysohn. For a metrizable space X and the doubleton $\mathbf{2} = \{0,1\}$, topological properties of the space $C_k(X,\mathbf{2})$ are thoroughly studied in [13].

For a topological space X, denote by $\psi : X \times C_k(X) \to \mathbb{R}$, $\psi(x,f) := f(x)$, the evaluation map. Recall that a subset $\mathcal{K}$ of $C_k(X)$ is *evenly continuous* if the restriction of ψ onto $X \times \mathcal{K}$ is jointly continuous, i.e. for any $x \in X$, each $f \in \mathcal{K}$ and every neighborhood $O_{f(x)} \subset Y$ of $f(x)$ there exist neighborhoods $U_f \subset \mathcal{K}$ of f and $O_x \subset X$ of x such that $U_f(O_x) := \{g(y) : g \in U_f,\ y \in O_x\} \subset O_{f(x)}$.

Following [3], a Tychonoff (Hausdorff) space X is called an *Ascoli space* if each compact subset $\mathcal{K}$ of $C_k(X)$ is evenly continuous. In other words, X is Ascoli if and only if the compact-open topology of $C_k(X)$ is Ascoli in the sense of [20, p.45].

It is easy to see that a space X is Ascoli if and only if the canonical valuation map $X \hookrightarrow C_k(C_k(X))$ is an embedding, see [3]. By Ascoli's theorem [9, 3.4.20], each k-space is Ascoli. Moreover, Noble [23] proved that any $k_{\mathbb{R}}$-space is Ascoli. We have the following implication

$$k_{\mathbb{R}}\text{-space} \Rightarrow \text{Ascoli},$$

and this implication is not reversible ([2]).

The aforementioned results motivate the following general question.

Question 1.3. *For which spaces X and Y the space $C_k(X,Y)$ is Ascoli?*

Below we present the following partial answer to this question.

Theorem 1.4. *For a metrizable space X, $C_k(X)$ is Ascoli if and only if $C_k(X)$ is a $k_{\mathbb{R}}$-space if and only if X is locally compact.*

Corson [7] started a systematic study of various topological properties of the weak topology of Banach spaces. The famous Kaplansky Theorem states that a normed space E in the weak topology has countable tightness; for further results see [8, 14]. Schlüchtermann and Wheeler [33] showed that an infinite-dimensional Banach space is never a k-space in the weak topology. We strengthen this result as follows.

Theorem 1.5. *A Banach space E in the weak topology is Ascoli if and only if E is finite-dimensional.*

Below we generalize Theorem 1.5 to an interesting class of Fréchet locally convex spaces, i.e. metrizable and complete locally convex space (lcs). We say that a Fréchet lcs E is a *quojection* if it is isomorphic to the projective limit of a sequence of Banach spaces with surjective linking maps or, equivalently, if every quotient of E which admits a continuous norm is a Banach space, see [4]. Obviously a countable product of Banach spaces is a quojection. Moscatelli [22] gave examples of quojections which are not isomorphic to countable products of Banach spaces.

Theorem 1.6. *Let a Fréchet lcs E be a quojection. Then E in the weak topology is Ascoli if and only if E is either finite-dimensional or is isomorphic to the product $\mathbb{K}^{\mathbb{N}}$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.*

Since every Fréchet lcs $C_k(X)$ is a quojection, see the survey [5], Theorem 1.6 yields the following

Corollary 1.7. *For a Fréchet lcs $C_k(X)$, the space $C_k(X)$ in the weak topology is Ascoli if and only if X is countable and discrete.*

Let E be a Banach space; denote by B_w the closed unit ball $B = B_E$ in E endowed with the weak topology of E. Schlüchtermann and Wheeler [33] showed that some topological properties of B_w are closely related to the isomorphic structure of E:

Theorem 1.8 ([33])**.** *The following conditions for a Banach space E are equivalent:* (a) *B_w is Fréchet–Urysohn;* (b) *B_w is sequential;* (c) *B_w is a k-space;* (d) *E contains no isomorphic copy of ℓ_1.*

Therefore it seems to be natural to verify whether there exists a Banach space E containing a copy of ℓ_1 and such that B_w is Ascoli or a $k_{\mathbb{R}}$-space. We answer such a question in the negative, by proving the following extension of Theorem 1.8.

Theorem 1.9. *Let E be a Banach space and B_w its closed unit ball with the weak topology. Then the following assertions are equivalent:*

(i) *B_w is an Ascoli space;*
(ii) *B_w is a $k_{\mathbb{R}}$-space;*
(iii) *every sequentially continuous real-valued map on B_w is continuous;*
(iv) *E does not contain a copy of ℓ_1.*

The proof of (i)⇒(iv) in Theorem 1.9, given in Proposition 4.5 below, uses basic properties of stochastically independent measurable functions. We also present a result related to Theorem 1.9 (ii), namely for Banach spaces containing an isomorphic copy of ℓ_1 we provide, in a sense, a canonical example of a sequentially continuous but not continuous function on B_w. Our construction builds on measure-theoretic properties of ℓ_1-sequences of continuous functions, see Example 5.2 below.

For Fréchet lcs we supplement Theorem 1.8 by proving the following theorem.

Theorem 1.10. *For a Fréchet lcs E the following conditions are equivalent:*

(i) *E contains no isomorphic copy of ℓ_1;*
(ii) *each closed and convex bounded subset of E is Ascoli in the weak topology.*

Theorems 1.9–1.10 heavily depend on our result stating that the closed unit ball B of ℓ_1 in the weak topology is not an Ascoli space, see Proposition 4.1 below.

2. The Ascoli property for function spaces. Proof of Theorem 1.4

We start from the definitions of the following well-known notions. A topological space X is called

- *Fréchet-Urysohn* if for any cluster point $a \in X$ of a subset $A \subset X$ there is a sequence $\{a_n\}_{n\in\mathbb{N}} \subset A$ which converges to a;
- *sequential* if for each non-closed subset $A \subset X$ there is a sequence $\{a_n\}_{n\in\mathbb{N}} \subset A$ converging to some point $a \in \bar{A} \setminus A$;
- a *k-space* if for each non-closed subset $A \subset X$ there is a compact subset $K \subset X$ such that $A \cap K$ is not closed in K;
- a *$k_{\mathbb{R}}$-space* if a real-valued function f on X is continuous if and only if its restriction $f|_K$ to any compact subset K of X is continuous.

Recall that the family of subsets

$$[C;\epsilon] := \{f \in C_k(X) : |f(x)| < \epsilon \ \forall x \in C\},$$

where C is a compact subset of X and $\epsilon > 0$, forms a basis of open neighborhoods at the zero function $\mathbf{0} \in C_k(X)$. Below we give a simple sufficient condition on a space X not to be Ascoli.

Proposition 2.1. *Assume a Tychonoff space X admits a family $\mathcal{U} = \{U_i : i \in I\}$ of open subsets of X, a subset $A = \{a_i : i \in I\} \subset X$ and a point $z \in X$ such that*

(i) *$a_i \in U_i$ for every $i \in I$;*
(ii) *$\big|\{i \in I : C \cap U_i \neq \emptyset\}\big| < \infty$ for each compact subset C of X;*
(iii) *z is a cluster point of A.*

Then X is not an Ascoli space.

Proof. For every $i \in I$, take a continuous function $f_i : X \to [0,1]$ such that $f_i(a_i) = 1$ and $f_i(X \setminus U_i) = \{0\}$. Set $\mathcal{K} := \{f_i : i \in I\} \cup \{\mathbf{0}\}$.

We claim that $\mathcal{K}$ is a compact subset of $C_k(X)$ and $\mathbf{0}$ is a unique cluster point of $\mathcal{K}$. Indeed, let C be a compact subset of X and $\epsilon > 0$. By (ii), the set $J := \{i \in I : C \cap U_i \neq \emptyset\}$ is finite. So, if $i \notin J$, then $f_i(C) = \{0\}$. Hence $f_i \in [C;\epsilon]$ for every $i \in I \setminus J$. This means that $\mathcal{K}$ is a compact set with the unique cluster point $\mathbf{0}$.

We show that $\mathcal{K}$ is not evenly continuous considering $\mathbf{0}$, z and $O = (-1/2, 1/2)$. By the claim, any neighborhood $U_{\mathbf{0}} \subset \mathcal{K}$ of $\mathbf{0}$ contains almost all functions f_i, and, by (iii), any neighborhood O_z of z contains infinitely many points a_i. So, there is $m \in I$ such that $f_m \in U_{\mathbf{0}}$ and $a_m \in O_z$. Since $f_m(a_m) = 1$, we obtain that $U_{\mathbf{0}}(O_z) \not\subset O$. Hence $\mathcal{K}$ is not evenly continuous. Thus X is not Ascoli. □

The next corollary follows also from Proposition 5.11(1) of [3].

Corollary 2.2. *Let X be a Tychonoff space with a unique cluster point z and such that every compact subspace of X is finite. Then X is not an Ascoli space.*

Proof. Since every $x \in X$, $x \neq z$, is isolated, we set $I = A = X \setminus \{z\}$ and $U_x = \{x\}$ for $x \in A$. Now Proposition 2.1 applies. □

The proof of the next proposition is a modification of the proof of the assertion in Section 5 of [28].

Proposition 2.3. *Let X be a first countable paracompact space. If X is not locally compact, then $C_k(X)$ contains a countable family $\mathcal{U} = \{U_s\}_{s\in\mathbb{N}}$ of open subsets in $C_k(X)$ and a countable subset $A = \{a_s\}_{s\in\mathbb{N}} \subset C_k(X,\mathbb{I})$ such that*

(i) *$a_s \in U_s$ for every $s \in \mathbb{N}$;*
(ii) *if $K \subset C_k(X)$ is compact, the set $\{s : U_s \cap K \neq \emptyset\}$ is finite;*
(iii) *the zero function $\mathbf{0}$ is a cluster point of A.*

In particular, the spaces $C_k(X)$ and $C_k(X,\mathbb{I})$ are not Ascoli.

Proof. Suppose for a contradiction that X is not locally compact and let $x_0 \in X$ be a point which does not have compact neighborhood. Take open bases $\{V'_i\}_{i\in\mathbb{N}}$ and $\{W_i\}_{i\in\mathbb{N}}$ at x_0 such that

$$V'_i \supset \overline{W_i} \supset W_i \supset \overline{V'_{i+1}}, \quad \forall i \in \mathbb{N}.$$

Set $P'_i := \overline{V'_i} \setminus V'_{i+1}$, $\forall i \in \mathbb{N}$. Since none of the sets V'_i is compact, there exists a sequence $k_1 < k_2 < \dots$ such that P'_{k_i} is not compact and $k_{i+1} > k_i + 1$. Set $P_i = P'_{k_i}$ and $V_i = V'_{k_i}$. Then $\{P_i\}_{i\in\mathbb{N}}$ is a sequence of closed, non-compact subsets of X, $\{V_i\}_{i\in\mathbb{N}}$ is a decreasing open base at x_0 and

$$P_i \subset \overline{V_i} \setminus \overline{W_{k_i+1}} \text{ and } \overline{V_{i+1}} \subset W_{k_i+1}. \tag{2.1}$$

Fix arbitrarily $i \in \mathbb{N}$. Since P_i is not compact, by [9, 3.1.23], there is a one-to-one sequence $\{x_{j,i}\}_{j\in\mathbb{N}} \subset P_i$ which is discrete and closed in X. Now the paracompactness of X and (2.1) imply that there exists an open sequence $\{V_{j,i}\}_{j\in\mathbb{N}}$ such that

$$x_{j,i} \in V_{j,i}, \text{ and } V_{j,i} \cap \overline{V_{i+1}} = \emptyset, \forall j \in \mathbb{N}, \text{ and } \{V_{j,i}\}_{j\in\mathbb{N}} \text{ is discrete in } X. \tag{2.2}$$

For every $p, q \in \mathbb{N}$ such that $1 \le p < q$, choose continuous functions $f_{q,p} : X \to [0,1]$ such that

$$f_{q,p}(x_{q,p}) = 1, \ f_{q,p}(x_{q,q}) = 0, \ f_{q,p}(x_0) = 1/p \text{ and } f_{q,p}(x) \le 1/p \text{ for } x \notin V_{q,p}. \tag{2.3}$$

Set $A := \{f_{q,p} : 1 \le p < q < \infty\}$ and $\mathcal{U} = \{U_{q,p} : 1 \le p < q < \infty\}$, where $U_{q,p}$ is the set of all functions $h \in C_k(X)$ satisfying the inequalities

$$\big|h(x_{q,p}) - 1\big| < \frac{1}{4^{p+q}}, \quad \Big|h(x_0) - \frac{1}{p}\Big| < \frac{1}{4^{p+q}}, \quad \big|h(x_{q,q})\big| < \frac{1}{4^{p+q}}. \tag{2.4}$$

Let us show that A and $\mathcal{U}$ are as desired. Clearly, (i) holds. Let us prove (ii).

Fix a compact subset K of $C_k(X)$. Let us first observe that

$$\text{there exists } p_0 \in \mathbb{N} \text{ such that if } p \ge p_0 \text{ and } q > p, \text{ then } U_{q,p} \cap K = \emptyset. \tag{2.5}$$

Indeed, otherwise we would find sequences $p_1 < q_1 < p_2 < q_2 < \dots$ and $h_{q_i,p_i} \in U_{q_i,p_i} \cap K$. Set

$$Z_1 := \{x_{q_i,p_i} : i \in \mathbb{N}\} \cup \{x_0\}.$$

From (2.1) it follows that Z_1 is compact, and thus, by the Ascoli Theorem [9, 3.4.20], there exists $r > 10$ such that if $z', z'' \in Z_1 \cap \overline{V_r}$ and $f \in K$, then $|f(z') - f(z'')| < 1/3$. But since $10 < r \le p_r < q_r$ we obtain $x_0, x_{q_r,p_r} \in Z_1 \cap \overline{V_r}$. Hence, by (2.4), we have

$$\big|h_{q_r,p_r}(x_{q_r,p_r}) - h_{q_r,p_r}(x_0)\big| > \left(1 - \frac{1}{4^{20}}\right) - \left(\frac{1}{p_r} + \frac{1}{4^{20}}\right) > \frac{1}{3}.$$

Since $h_{q_r,p_r} \in K$, we get a contradiction.

We shall now prove that

$$\text{there exists } q_0 \in \mathbb{N} \text{ such that if } q \geq q_0 \text{ and } 1 \leq p < p_0, \text{ then } U_{q,p} \cap K = \emptyset, \tag{2.6}$$

where p_0 is defined in (2.5). Indeed, set

$$Z_2 := \{x_{j,i} : 1 \leq j \leq i < \infty\} \cup \{x_0\}.$$

Then Z_2 is compact by (2.1). Again by the Ascoli Theorem, it follows that there exists $q_0 \in \mathbb{N}$ such that for $z', z'' \in Z_2 \cap \overline{V_{q_0}}$ and $f \in K$ we have $|f(z') - f(z'')| < 1/4p_0$. The q_0 chosen in this way satisfies (2.6), since otherwise there would exist $q \geq q_0$ and $1 \leq p < p_0$ such that $U_{q,p} \cap K \neq \emptyset$. Fix $h_{q,p} \in U_{q,p} \cap K$. Then $x_0, x_{q,q} \in Z_2 \cap \overline{V_{q_0}}$, and by (2.3) and (2.4), we obtain

$$\left|h_{q,p}(x_{q,q}) - h_{q,p}(x_0)\right| > \left(\frac{1}{p} - \frac{1}{4^{p+q}}\right) - \frac{1}{4^{p+q}} > \frac{1}{3p} > \frac{1}{4p_0},$$

which gives a contradiction. Now (2.5) and (2.6) immediately imply (ii).

Now we prove (iii). Fix arbitrarily a compact subset $Z \subset X$ and $\epsilon > 0$. Choose p_0 such that $1/p_0 < \epsilon$. By (2.2), we can find $j_0 \in \mathbb{N}$ such that $Z \cap V_{j,p_0} = \emptyset$ for every $j \geq j_0$. Take $q_0 = p_0 + j_0$. Then $f_{q_0,p_0} \in A$, and for $z \in Z$ we have $z \notin V_{q_0,p_0}$, and thus, in accordance with (2.3), $f_{q_0,p_0}(z) \leq 1/p_0 < \epsilon$. Thus $f_{q_0,p_0} \in [Z;\epsilon]$.

Finally, the spaces $C_k(X)$ and $C_k(X,\mathbb{I})$ are not Ascoli by Proposition 2.1. ☐

The next corollary proved by R. Pol solves Problem 6.8 in [3].

Corollary 2.4 ([29])**.** *For a separable metrizable space X, $C_k(X)$ is Ascoli if and only if X is locally compact.*

Proof. If $C_k(X)$ is Ascoli, then X is locally compact by Proposition 2.3. Conversely, if X is a separable metrizable locally compact space, then $C_k(X)$ is even a Polish space. ☐

Recall that a family $\mathcal{N}$ of subsets of a topological space X is called a *network* in X if, whenever $x \in U$ with U open in X, then $x \in N \subset U$ for some $N \in \mathcal{N}$. A space X is called a *σ-space* if it is regular and has a σ-locally finite network. Any metrizable space is a σ-space by the Nagata-Smirnov Metrization Theorem.

Now Theorem 1.4 follows from the following theorem in which the equivalence of (i) and (ii) is well-known.

Theorem 2.5. *Let X be a first-countable paracompact σ-space. Then the following assertions are equivalent:*

(i) *X is a locally compact metrizable space;*
(ii) *$X = \bigoplus_{i\in\kappa} X_i$, where all X_i are separable metrizable locally compact spaces;*
(iii) *$C_k(X)$ is a $k_{\mathbb{R}}$-space;*
(iv) *$C_k(X)$ is an Ascoli space;*
(v) *$C_k(X,\mathbb{I})$ is a $k_{\mathbb{R}}$-space;*
(vi) *$C_k(X,\mathbb{I})$ is an Ascoli space.*

In cases (i)–(vi), the spaces $C_k(X)$ and $C_k(X,\mathbb{I})$ are the products of families of Polish spaces.

Proof. (i)⇒(ii) follows from [9, 5.1.27].

(ii)⇒(iii),(v): If $X = \bigoplus_{i\in\kappa} X_i$, then

$$C_k(X) = \prod_{i\in\kappa} C_k(X_i) \quad \text{and} \quad C_k(X,\mathbb{I}) = \prod_{i\in\kappa} C_k(X_i,\mathbb{I}),$$

where all the spaces $C_k(X_i)$ and $C_k(X_i,\mathbb{I})$ are Polish (see Corollary 1.2). So $C_k(X)$ and $C_k(X,\mathbb{I})$ are $k_{\mathbb{R}}$-spaces by [24, Theorem 5.6].

(iii)⇒(iv) and (v)⇒(vi) follow from [23]. The implications (iv)⇒(i) and (vi)⇒(i) follow from Proposition 2.3 and the fact that any locally compact σ-space is metrizable by [25]. □

Note that Theorem 2.5 holds true for first-countable stratifiable spaces since any stratifiable space is a paracompact σ-space (see Theorems 5.7 and 5.9 in [16]).

3. Proofs of Theorems 1.5 and 1.6

Following Arhangel'skii [1, II.2], we say that a topological space X has *countable fan tightness at a point* $x\in X$ if for each sets $A_n \subset X$, $n\in\mathbb{N}$, with $x\in\bigcap_{n\in\mathbb{N}}\overline{A_n}$ there are finite sets $F_n\subset A_n$, $n\in\mathbb{N}$, such that $x\in\overline{\cup_{n\in\mathbb{N}} F_n}$; X has *countable fan tightness* if X has countable fan tightness at each point $x\in X$. Clearly, if X has countable fan tightness, then X also has countable tightness.

For a topological space X we denote by $C_p(X)$ the space $C(X)$ endowed with the topology of poitwise convergence.

For a lcs E, denote by E' the dual space of E. The space E endowed with the weak topology $\sigma(E,E')$ is denoted by E_w. The closure of a subset $A\subset E$ in $\sigma(E,E')$ we denote by $\overline{A}^{w}$. If E is a metrizable lcs, then $X := (E',\sigma(E',E))$ is σ-compact by the Alaoglu–Bourbaki Theorem. Since E_w embeds into $C_p(X)$, Theorem II.2.2 of [1] immediately implies the following result noticed in [14].

Fact 3.1 ([14]). *If E is a metrizable lcs, then E_w has countable fan tightness.*

Denote the unit sphere of a normed space E by S_E. Theorem 1.5 immediately follows from the next proposition.

Proposition 3.2. *Let E be a normed space. Then E with the weak topology is Ascoli if and only if E is finite-dimensional.*

Proof. We show that E_w is not Ascoli for any infinite-dimensional normed space E.

For every $n\in\mathbb{N}$, let A_n be a countable subset of nS such that $0\in\overline{A_n}^{w}$ (see [10, Exercise 3.46] and Fact 3.1). Now Fact 3.1 implies that there are finite sets $F_n\subset A_n$, $n\in\mathbb{N}$, such that $0\in\overline{\cup_{n\in\mathbb{N}} F_n}$. Set $A := \bigcup_{n\in\mathbb{N}} F_n$. Using the Hahn–Banach Theorem, for every $n\in\mathbb{N}$ and each $a\in F_n$ take a weakly open neighborhood U_a of a such that

$$U_a \cap \left(n-\frac{1}{2}\right) B = \emptyset. \tag{3.1}$$

Let us show that the family $\mathcal{U} = \{U_a : a\in A\}$, the set A and the zero 0 satisfy conditions (i)–(iii) of Proposition 2.1. Clearly, (i) and (iii) hold. To check (ii), let C

be a compact subset of E_w. Then $C \subset mB$ for some $m \in \mathbb{N}$, and (3.1) implies that the set

$$\{a \in A : U_a \cap C \neq \emptyset\} \subset \bigcup_{n \leq m} F_n$$

is finite. Finally, Proposition 2.1 implies that E_w is not Ascoli. □ □

We need also the following

Proposition 3.3. *Let $p : X \to Y$ be an open continuous map of a topological space X onto a regular space Y. If X is Ascoli, then Y is also an Ascoli space.*

Proof. Let $\mathcal{K}$ be a compact subset of $C_k(Y)$. We have to show that $\mathcal{K}$ is evenly continuous. Denote by $p^* : C_k(Y) \to C_k(X), p^*(h) := h(p(x))$, the adjoin continuous map.

Fix $y_0 \in Y$, $h_0 \in \mathcal{K}$ and an open neighborhood O_{z_0} of the point $z_0 := h_0(y_0)$. Set $f := p^*(h_0) \in C_k(X)$ and take arbitrarily a preimage x_0 of y_0, so $p(x_0) = y_0$. Since $p^*(\mathcal{K})$ is a compact subspace of $C_k(X)$ it is evenly continuous. Hence we can find neighborhoods $U_f \subset p^*(\mathcal{K})$ of f and $O_{x_0} \subset X$ of x_0 such that $U_f(O_{x_0}) \subset O_{z_0}$. Set $U_{h_0} := \mathcal{K} \cap \big(p^*\big)^{-1}(U_f)$ and $O_{y_0} := p\big(O_{x_0}\big)$ (which is a neighborhood of y_0 as p is open). For every $h \in U_{h_0}$ and each $y \in O_{y_0}$, take $x \in O_{x_0}$ with $p(x) = y$, so we obtain

$$h(y) = h(p(x)) = p^*(h)(x) \in O_{z_0}.$$

Thus $\mathcal{K}$ is evenly continuous, and therefore Y is Ascoli. □

Below we prove Theorem 1.6 and Corollary 1.7.

Proof of Theorem 1.6. Assume that E is infinite-dimensional. By Proposition 3.2 the space E is not normed. Let $(p_n)_n$ be a sequence of continuous seminorms providing the topology of E. For each $n \in \mathbb{N}$, let $E_n := E/p_n^{-1}(0)$ be the quotient endowed with the norm topology $p_n^* : [x] \mapsto p_n(x)$, where $[x]$ is the equivalence class of x in E. Since E is a quojection, the quotient E_n with the original quotient topology is a Banach space by [4, Proposition 3].

By Proposition 3.3 the space E_n endowed with the weak topology is Ascoli, so we apply Proposition 3.2 to deduce that each E_n is finite-dimensional. On the other hand, E embeds into the product $\prod_n E_n$. So E, being complete, is isomorphic to a closed subspace of the product $\mathbb{K}^{\mathbb{N}}$. Thus E is also isomorphic to $\mathbb{K}^{\mathbb{N}}$ by [27, Corollary 2.6.5]. □

Proof of Corollary 1.7. By Theorem 1.6 the space $C_k(X)$ is isomorphic to $\mathbb{R}^{\mathbb{N}}$, and since $\mathbb{R}^{\mathbb{N}}$ does not admit a weaker locally convex topology (see [27, Corollary 2.6.5]), $C_k(X) = C_p(X) = \mathbb{R}^{\mathbb{N}}$. Thus X is countable and discrete. The converse assertion is trivial. □

We do not know whether there exists a Fréchet space E such that E_w is an Ascoli non-metrizable space.

Remark 3.4. The first example of a non-distinguished Fréchet space (so also not quojection) was given by Grothendieck and Köthe, and it was the Köthe echelon space $\lambda_1(A)$ of order 1 for the Köthe matrix $A = (a_n)_n$ defined on $\mathbb{N} \times \mathbb{N}$ by $a_n(i,j) := j$ if $i < n$ and $a_n(i,j) = 1$ otherwise, see [5] also for more references. We do not know however if this space with the weak topology is an Ascoli space.

4. Proof of Theorem 1.9

To prove Theorem 1.9 we need the following key proposition, which proves, among others, that the unit ball B_{ℓ_1} in the weak topology is not Ascoli. In particular, since the k-space property is inherited by the closed subspaces, this shows also that any Banach space E whose weak unit ball B_w is a k-space contains no isomorphic copy of ℓ_1, i.e. the proposition proves (c)⇒(d) in Schlüchtermann–Wheeler's theorem 1.8. A sequence $\{x_i\}_{i\in\mathbb{N}} \subset E$ is called *trivial* if there is $n \in \mathbb{N}$ such that $x_i = x_n$ for all $i > n$.

Proposition 4.1. *Let $E = \ell_1$ and B_w its closed unit ball in the weak topology. Then there is a countable subset A of S_{ℓ_1} and a family $\mathcal{U} = \{U_a : a \in A\}$ of weakly open subsets of the unit ball B such that*

(1) *$a \in U_a$ for every $a \in A$;*
(2) *$\mathrm{dist}(U_a, U_b) \geq 1/5$ for every distinct $a, b \in A$;*
(3) *the zero 0 is the unique cluster point of A;*
(4) *$\big|\{a \in A : C \cap U_a \neq \emptyset\}\big| < \infty$ for every weakly compact subset C of B;*
(5) *$\overline{A}^w = A \cup \{0\}$ and every weakly compact subset of $\overline{A}^w$ is finite;*
(6) *A contains a sequence which is equivalent to the unit basis of ℓ_1;*
(7) *the set A does not have a non-trivial weakly fundamental subsequence;*
(8) *the countable space $\overline{A}^w$ and B_w are not Ascoli.*

Proof. Let $\{(e_i, e_i^*) : i \in \mathbb{N}\}$ be the standard biorthogonal basis in $\ell_1 \times \ell_1' = \ell_1 \times \ell_\infty$. Following [14], set $\Omega := \{(m, n) \in \mathbb{N}\times\mathbb{N} : m < n\}$ and

$$A := \left\{ a_{m,n} := \frac{1}{2}(e_m - e_n) : (m,n) \in \Omega \right\} \subset S_{\ell_1}.$$

For every $(m, n) \in \Omega$, define the following weak neighborhood of $a_{m,n}$

$$\begin{aligned} U_{m,n} &:= \left\{ x \in B : \ |\langle e_m^*, a_{m,n} - x\rangle| < \frac{1}{10} \text{ and } |\langle e_n^*, a_{m,n} - x\rangle| < \frac{1}{10} \right\} \\ &= \left\{ x = (x_i) \in B : \ \left|\frac{1}{2} - x_m\right| < \frac{1}{10} \text{ and } \left|\frac{1}{2} + x_n\right| < \frac{1}{10} \right\}. \end{aligned}$$

Then (1) holds trivially. Let us check (2). For every $k \notin \{m, n\}$ and each $x = (x_i) \in U_{m,n}$, one has

$$|x_k| \leq \|x\| - |x_m| - |x_n| < 1 - \left(\frac{1}{2} - \frac{1}{10}\right) - \left(\frac{1}{2} - \frac{1}{10}\right) = \frac{1}{5}.$$

So, if $(m, n) \neq (k, l)$ and $x = (x_i) \in U_{m,n}$, we obtain either

$$\left|\frac{1}{2} - x_k\right| > \frac{1}{2} - \frac{1}{5} = \frac{3}{10} \text{ if } k \notin \{m,n\}, \quad \text{ or } \quad \left|\frac{1}{2} + x_l\right| > \frac{3}{10} \text{ if } l \notin \{m,n\}.$$

Hence $\mathrm{dist}\big(U_{m,n}, U_{k,l}\big) \geq 3/10 - 1/10 = 1/5$ for all $(m,n) \neq (k,l)$. This proves (2). In particular, every point of A is weakly isolated.

To prove (3) we note first that $0 \in \overline{A}^w$ by Lemma 3.2 of [14]. We provide a proof of this result to keep the paper self-contained. Let U be a neighborhood of 0 of the canonical form

$$U=\left\{x\in\ell_1 : |\langle\chi_k,x\rangle|<\epsilon,\ \text{where } \chi_k=\big(\chi_k(i)\big)_{i\in\mathbb{N}}\in S_{\ell_\infty} \text{ for } 1\le k\le s\right\}.$$

Let I be an infinite subset of $\mathbb{N}$ such that, for every $1\le k\le s$, either $\chi_k(i)>0$ for all $i\in I$, or $\chi_k(i)=0$ for all $i\in I$, or $\chi_k(i)<0$ for all $i\in I$. Take a natural number $N>1/\epsilon$. Since I is infinite, by induction, one can find $(m,n)\in\Omega$ satisfying the following condition: for every $1\le k\le s$ there is $0<t_k\le N$ such that

$$\frac{t_k-1}{N}\le \min\big\{|\chi_k(m)|,|\chi_k(n)|\big\}\le \max\big\{|\chi_k(m)|,|\chi_k(n)|\big\}\le\frac{t_k}{N}. \tag{4.1}$$

Then, by the construction of I, we obtain

$$|\langle\chi_k,a_{m,n}\rangle|<1/N<\epsilon \text{ for every } 1\le k\le s.$$

Thus $a_{m,n}\in U$, and hence $0\in\overline{A}^w$.

Now fix arbitrarily a nonzero $z=(z_i)\in\ell_1$ and consider the following three cases.

(a) There is $z_i\not\in\{-1/2,0,1/2\}$, so $z\not\in A$. Set

$$\epsilon:=\frac{1}{2}\min\left\{|z_i|,\left|z_i-\frac{1}{2}\right|,\left|z_i+\frac{1}{2}\right|\right\} \text{ and } U:=\{x\in\ell_1:|\langle e_i^*,z-x\rangle|<\epsilon\}.$$

Clearly, $U\cap A=\emptyset$ and $z\not\in\overline{A}^w$.

(b) Assume that $z\not\in A$ and $z_i\in\{-1/2,0,1/2\}$ for every $i\in\mathbb{N}$. So there are distinct indices i and j such that $z_i=z_j\in\{-1/2,1/2\}$. Set

$$U:=\{x\in\ell_1:|\langle e_i^*+e_j^*,z-x\rangle|<1/10\}.$$

By the definition of A, we obtain $U\cap A=\emptyset$, and hence $z\not\in\overline{A}^w$.

(c) Assume that $z\in A$. Then z is not a cluster point of A because it is weakly isolated.

Now (a)–(c) prove (3). Let us prove (4). Fix a weakly compact subset C of ℓ_1. Assuming that $C\cap U_a\neq\emptyset$ for an infinite subset $J\subset A$ we choose $x_j\in C\cap U_j$ for every $j\in J$. Since ℓ_1 has the Schur property, C is also compact in the norm topology of ℓ_1. So we can assume that x_j converges to some $x_\infty\in C$ in the norm topology. But this contradicts (2) that proves (4).

(5) immediately follows from (3) and (4).

(6): Clearly, the sequence $\{a_{1,i}\}_{i>1}\subset A$ is equivalent to the unit basis of ℓ_1.

(7): Assuming the converse let $\{a_{m_i,n_i}\}_{i\in\mathbb{N}}$ be a faithfully indexed weakly fundamental subsequence of A. Then only the next two cases are possible.

Case 1. There is $k\in\mathbb{N}$ and $i_1<i_2<\dots$ such that $k=m_{i_1}=m_{i_2}=\dots$. Passing to a subsequence we can assume that $m_1=m_2=\cdots=k$ and $k<n_1<n_2<\dots$. Set

$$\chi:=(\chi_j)_{j\in\mathbb{N}}\in\ell_\infty,\ \text{where } \chi_j=\begin{cases}-1, \text{ if } j\in\{n_2,n_4,\dots\},\\ \ \ 0, \text{ if } j\not\in\{n_2,n_4,\dots\}.\end{cases}$$

Then $\chi\in S_{\ell_\infty}$ and

$$\langle\chi,a_{k,n_{2s}}-a_{k,n_{2s+1}}\rangle=\frac{1}{2},\quad \forall s\in\mathbb{N}.$$

Thus the sequence $\{a_{m_i,n_i}\}_{i\in\mathbb{N}}$ is not fundamental, a contradiction.

Case 2. $m_i \to \infty$ and $n_i \to \infty$. Passing to a subsequence if it is needed, we can assume that

$$m_1 < n_1 < m_2 < n_2 < \dots .$$

Defining $\chi \in S_{\ell_\infty}$ as in Case 1, we obtain

$$\langle \chi, a_{m_{2s},n_{2s}} - a_{m_{2s+1},n_{2s+1}} \rangle = \frac{1}{2}, \quad \forall s \in \mathbb{N}.$$

Thus the sequence $\{a_{m_i,n_i}\}_{i\in\mathbb{N}}$ is not weakly fundamental also in this case.

Therefore A does not have a weakly fundamental subsequence.

(8): The space $\overline{A}^w$ is not Ascoli by (5) and Corollary 2.2, and B_w is not Ascoli by (1)-(4) and Proposition 2.1. □

Recall that a (normalized) sequence (x_n) in a Banach space E is said to be equivalent to the standard basis of ℓ_1, or simply called an ℓ_1-sequence, if for some $\theta > 0$

$$\left\| \sum_{i=1}^n c_i x_i \right\| \geq \theta \cdot \sum_{i=1}^n |c_i|,$$

for any natural number n and any scalars $c_i \in \mathbb{R}$. We also call such a sequence a θ-ℓ_1-sequence if we want to specify the constant in the definition.

We need some measure-theoretic preparations. Let (T, Σ, μ) be a probability measure space. Measurable functions $g_n : T \to \mathbb{R}$ are said to be *stochastically independent with respect to* μ if

$$\mu\left(\bigcap_{n\leq k} g_n^{-1}(B_n)\right) = \prod_{n\leq k} \mu\left(g_n^{-1}(B_n)\right),$$

for every k and any Borel sets $B_n \subseteq \mathbb{R}$; see e.g. Fremlin [11, 272], for basic facts concerning independence. Recall (see [11, 272Q]) that, if integrable functions $f, g : T \to \mathbb{R}$ are independent with respect to μ, then $\int_T f \cdot g \, \mathrm{d}\mu = \left(\int_T f \, \mathrm{d}\mu\right) \cdot \left(\int_T g \, \mathrm{d}\mu\right)$.

Lemma 4.2. *Let (T, Σ, μ) and (S, Θ, ν) be probability measure spaces and let $\Phi : T \to S$ be a measurable mapping such that $\Phi[\mu] = \nu$, that is $\mu(\Phi^{-1}(E)) = \nu(E)$ for every $E \in \Theta$. If $(p_n)_n$ be a sequence of measurable functions $S \to \mathbb{R}$ which is stochastically independent with respect to ν, then the functions $g_n = p_n \circ \Phi$ are stochastically independent with respect to μ.*

Lemma 4.2 is standard and follows for instance from Theorem 272G in [11].

In the proof of crucial Proposition 4.5 we essentially use the following version of the Riemann-Lebesgue lemma, which is mentioned in Talagrand's [35], page 3.

Theorem 4.3. *Let (T, Σ, μ) be any probability space and let $(g_n)_n$ be a stochastically independent uniformly bounded sequence of measurable functions $T \to \mathbb{R}$ with $\int_T g_n \, \mathrm{d}\mu = 0$ for every n. Then*

$$\lim_{n\to\infty} \int_T f \cdot g_n \, \mathrm{d}\mu = 0,$$

for every bounded measurable function $f : T \to \mathbb{R}$.

Finally, let us recall the following fact, see e.g. [35], 1-2-5.

Lemma 4.4. *Let Φ be a continuous surjection of a compact space K onto a compact space L. If λ is a regular probability Borel measure on L then there exists a regular probability Borel measure μ on K such that $\Phi[\mu]=\lambda$, that is $\mu(\Phi^{-1}(B))=\lambda(B)$ for every Borel set $B\subseteq L$.*

Proposition 4.5. *If a Banach space E contains an isomorphic copy of ℓ_1, then B_w is not an Ascoli space.*

Proof. We show that B_w is not Ascoli in four steps.

Step 1. Since the Hilbert cube $H=[0,1]^{\mathbb{N}}$ is separable, one can find a continuous function Φ_0 from the discrete space $\mathbb{N}$ onto a dense subset of H. By Theorem 3.6.1 of [9], we can extend Φ_0 to a continuous map $\Phi:\beta\mathbb{N}\to H$. As $\Phi_0(\mathbb{N})$ is dense in H, we obtain that $\Phi(\beta\mathbb{N})=H$. Let $\pi_n: H\to[-1,1]$ be the projection onto the nth coordinate, and let $\lambda=\prod_n \mathrm{m}_n$ be the product measure of the normalized Lebesgue measures m_n on the interval $[-1,1]$. Then the sequence (π_n) is stochastically independent with respect to λ and

$$\int_H \pi_n \,\mathrm{d}\lambda=\int_H \pi_n\pi_m \,\mathrm{d}\lambda=0, \text{ and } \int_H \pi_n^2 \,\mathrm{d}\lambda=\frac{1}{2}\int_{-1}^{1} x^2 \,\mathrm{d}x=\frac{1}{3}, \tag{4.2}$$

for all $n,m\in\mathbb{N}$ and $n\neq m$. Moreover, the sequence $(\pi_n)_n$ is a 1-ℓ_1-sequence in $C(H)$. Indeed, for every $n\in\mathbb{N}$ and each scalars $c_1,\dots,c_n\in\mathbb{R}$, set

$$x:=\big(\mathrm{sign}(c_1),\dots,\mathrm{sign}(c_n),0,\dots\big)\in H.$$

Then $\sum_{i\le n} c_i\pi_i(x)=\sum_{i\le n}|c_i|$. Thus (π_n) is a 1-ℓ_1-sequence in $C(H)$.

Step 2. Let μ be a measure on $\beta\mathbb{N}$ such that $\Phi[\mu]=\lambda$, see Lemma 4.4. Set $g_n:=\pi_n\circ\Phi$ for every $n\in\mathbb{N}$. Then the sequence (g_n) is stochastically independent with respect to μ by Lemma 4.2. As Φ is surjective, (g_n) is also a 1-ℓ_1-sequence in $C(\beta\mathbb{N})$.

Step 3. Let Y be a subspace of E isomorphic to ℓ_1 and let $T_1: Y\to\ell_1$ be an isomorphism. For every $n\in\mathbb{N}$ choose $x_n\in Y$ such that $T_1(x_n)=e_n$, where (e_n) is the standard coordinate basis in ℓ_1. In turn, as (g_n) is a 1-ℓ_1-sequence in $C(\beta\mathbb{N})$, there is an isometric embedding $T_2:\ell_1\to C(\beta\mathbb{N})$, sending e_n to g_n.

As the space $C(\beta\mathbb{N})$ is 1-injective, the operator $T=T_2\circ T_1: Y\to C(\beta\mathbb{N})$ can be extended to an operator $\widetilde{T}: E\to C(\beta\mathbb{N})$ having the same norm; cf. Proposition 5.10 of [10].

Step 4. Set $d:=\sup\{\|x_n\|_E : n\in\mathbb{N}\}$ and $\gamma:=\sup\{\|\widetilde{T}(x)\| : x\in dB_E\}$. Let $h_{m,n}=(g_m-g_n)/2$ for $n,m\in\mathbb{N}, n>m$, and set

$$V_{m,n}=\left\{ f\in\gamma B_{C(\beta\mathbb{N})} : \left|\int_{\beta\mathbb{N}} f\cdot g_i \,\mathrm{d}\mu\right| > 1/4, \text{ for } i=m,n\right\}.$$

Denote by T^+ the map $\widetilde{T}$ from E_w into $C_w(\beta\mathbb{N})$. Clearly, T^+ is also continuous. Finally we set

$$A:=\{a_{m,n}:=(x_m-x_n)/2 :\ 1\le m<n\},$$

and

$$\mathcal{U}:=\{U_{m,n}:=(T^+)^{-1}(V_{m,n})\cap dB_E :\ 1\le m<n\}.$$

Now the following claim finishes the proof.

Claim. *The ball dB_E is not Ascoli in the weak topology.*

To prove the claim it is enough to check (i)-(iii) of Proposition 2.1 for the set A and the family $\mathcal{U}$.

(i): To show that $a_{m,n} \in U_{m,n}$ it is enough to prove that $h_{m,n} \in V_{m,n}$. But this follows from (4.2) since

$$2\int_{\beta\mathbb{N}} h_{m,n} \cdot g_n \, \mathrm{d}\mu = \int_{\beta\mathbb{N}} g_m \cdot g_n \, \mathrm{d}\mu - \int_{\beta\mathbb{N}} g_n^2 \, \mathrm{d}\mu = -\frac{2}{3} = -2\int_{\beta\mathbb{N}} h_{m,n} \cdot g_m \, \mathrm{d}\mu.$$

(iii): The zero function $\mathbf{0}$ is the weak cluster point of A by Proposition 4.1.

Let us check (ii), i.e. if $C \subseteq dB_E$ is weakly compact, then C can meet only finite number of $U_{m,n}$'s. Suppose otherwise: let $x_i \in C \cap U_{m_i,n_i}$, where the pairs (m_i, n_i) are distinct. As $m_i < n_i$ we may assume also that $n_i \neq n_{i'}$ for $i \neq i'$. Since C is weakly compact it is Fréchet–Urysohn by the Eberlein–Šmulyan theorem [10, 3.109]. So we can further assume that x_i converge weakly to some $x \in C$. Then also the functions $f_i := T^+(x_i) \in V_{m_i,n_i}$ converge weakly to $f := T^+(x) \in T^+(C) \subset \gamma B_{C(\beta\mathbb{N})}$, and they are uniformly bounded on $\beta\mathbb{N}$ and $f_i \to f$ pointwise.

Take arbitrarily $0 < \delta < 1/16(1+\gamma+2\gamma^2)$. By Theorem 4.3, there is $N_1 \in \mathbb{N}$ such that $\left|\int_{\beta\mathbb{N}} f \cdot g_{n_i} \, \mathrm{d}\mu\right| < \delta$ for all $i > N_1$. By the classical Egorov theorem, f_i converge almost uniformly to f, i.e. there is $B \subseteq \beta\mathbb{N}$ such that $\mu(\beta\mathbb{N} \setminus B) < \delta$ and f_i converge uniformly to f on B. Take $N_2 > N_1$ such that $|f_i - f| < \delta$ on B for all $i > N_2$. Taking into account that $|h| \leq \gamma$ for each $h \in \gamma B_{C(\beta\mathbb{N})}$, for every $i > N_2$ we obtain

$$\begin{aligned} \left|\int_{\beta\mathbb{N}} f_i \cdot g_{n_i} \, \mathrm{d}\mu\right| &\leq \left|\int_{\beta\mathbb{N}} f_i \cdot g_{n_i} \, \mathrm{d}\mu - \int_{\beta\mathbb{N}} f \cdot g_{n_i} \, \mathrm{d}\mu\right| + \left|\int_{\beta\mathbb{N}} f \cdot g_{n_i} \, \mathrm{d}\mu\right| \\ &\leq \int_{\beta\mathbb{N}} |f_i - f| \cdot |g_{n_i}| \, \mathrm{d}\mu + \delta \leq \int_B + \int_{\beta\mathbb{N}\setminus B} + \delta \\ &\leq \gamma \cdot \delta + 2\gamma^2 \cdot \delta + \delta = \delta(1+\gamma+2\gamma^2) < 1/16. \end{aligned}$$

On the other hand, $f_i \in V_{m_i,n_i}$ implies $\left|\int_{\beta\mathbb{N}} f_i \cdot g_{n_i} \, \mathrm{d}\mu\right| > 1/4$. This contradiction proves the claim. □

To prove Theorem 1.9 we need also the following simple lemma.

Lemma 4.6. *Let E be a Banach space and let B_w denote the unit ball of E equipped with the weak topology. For any function $f : B_w \to \mathbb{R}$ the following are equivalent*

(i) *f is sequentially continuous on B_w;*
(ii) *f is continuous on every compact subset of B_w.*

Proof. Let f be sequentially continuous on B_w and let C be a compact subset of B_w. For any closed set $H \subseteq \mathbb{R}$, the set $F = f^{-1}(H) \cap C$ is sequentially closed in C. Hence F is closed in C, since C, as a weakly compact set, has the Frechet–Urysohn property by the classical Eberlein–Šmulian theorem.

We have checked that (i) imples (ii); the reverse implication is obvious. □

Proof of Theorem 1.9. (i)⇒(iv) follows from Proposition 4.5. Theorem 1.8 implies (iv)⇒(iii). (iii)⇒(ii) follows from Lemma 4.6. Finally, the implication (ii)⇒(i) holds by [23]. □

5. On weakly sequentially continuous functions on the unit ball

Let E be a Banach space containing an isomorphic copy of ℓ_1 and let B_w denote the unit ball in E equipped with the weak topology. It follows from Theorem 1.9 that B_w is not a $k_{\mathbb{R}}$-space which, in view of Lemma 4.6, is equivalent to saying that there is a function $\Phi : B_w \to \mathbb{R}$ which is sequentially continuous but not continuous. We show below that such a function can be defined, in a sense, effectively by means of measure-theoretic properties of ℓ_1-sequences of continuous functions.

Proposition 5.1. *Let K be a compact space and let (g_n) be a normalized θ-ℓ_1-sequence in the Banach space $C(K)$. Then there exists a regular probability measure μ on K such that*

$$\int_K |g_n - g_k| \, \mathrm{d}\mu \geq \theta/2 \text{ whenever } n \neq k.$$

Proof. Suppose that (g_n) is θ-equivalent to the standard basis (e_n) in ℓ_1. Put

$$H = \overline{\mathrm{conv}}\left(\{|g_n - g_k| : n \neq k\}\right) \subseteq C(K).$$

Note that it is enough to check that $\|h\| \geq \theta/2$ for all $h \in H$ since in such a case, by the separation theorem, there is a norm-one $\mu \in C(K)^*$ such that $\int_K h \, \mathrm{d}\mu \geq \theta/2$ for every $h \in H$. As $h \geq 0$ for $h \in H$, we can then replace the signed measure μ by its variation $|\mu|$.

In turn, the fact that $\|h\| \geq \theta/2$ for $h \in H$ is implied by the following.

Claim. Suppose that $n_i \neq k_i$ for $i \leq p$. then for any convex coefficients $\alpha_1, \ldots, \alpha_p$

$$\left\| \sum_{i=1}^{p} \alpha_i |g_{n_i} - g_{k_i}| \right\| \geq \theta/2.$$

We shall verify the claim in two steps.

Step 1. There is $E \subseteq \{1, \ldots, p\}$ such that

$$\left\| \sum_{i \in E} \alpha_i (e_{n_i} - e_{k_i}) \right\| \geq 1/2.$$

Indeed, if L denotes the Cantor set $\{-1, 1\}^{\mathbb{N}}$, then the projections $\pi_n : L \to \{-1, 1\}$ form a sequence in $C(L)$ which is a 1-ℓ_1-sequence, so we have an isometric embedding $T : \ell_1 \to C(L)$, where $Te_n = \pi_n$ for every $n \in \mathbb{N}$.

Write λ for the standard product measure on L. We calculate directly that $\int_K |\pi_n - \pi_k| \, \mathrm{d}\lambda = 1$ for $n \neq k$ and therefore

$$\left\| \sum_{i=1}^{p} \alpha_i |\pi_{n_i} - \pi_{k_i}| \right\| \geq \int_L \sum_{i=1}^{p} \alpha_i |\pi_{n_i} - \pi_{k_i}| \, \mathrm{d}\lambda = 1.$$

Hence there is $t \in L$ such that $\sum_{i=1}^{p} \alpha_i |\pi_{n_i}(t) - \pi_{k_i}(t)| \geq 1$. Examining the signs of summands we conclude that for some set $E \subseteq \{1, \dots, p\}$ we have

$$\left| \sum_{i \in E} \alpha_i (\pi_{n_i}(t) - \pi_{k_i}(t)) \right| \geq 1/2.$$

This implies that

$$\left\| \sum_{i \in E} \alpha_i (e_{n_i} - e_{k_i}) \right\| = \left\| \sum_{i \in E} \alpha_i (Te_{n_i} - Te_{k_i}) \right\| \geq \left| \sum_{i \in E} \alpha_i (\pi_{n_i}(t) - \pi_{k_i}(t)) \right| \geq 1/2.$$

Step 2. Taking a set E from Step 1 we conclude that

$$\left\| \sum_{i=1}^{p} \alpha_i |g_{n_i} - g_{k_i}| \right\| \geq \left\| \sum_{i \in E} \alpha_i (g_{n_i} - g_{k_i}) \right\| \geq \theta \cdot \left\| \sum_{i \in E} \alpha_i (e_{n_i} - e_{k_i}) \right\| \geq \theta/2.$$

This verifies the claim and the proof is complete. □

Example 5.2. *Suppose that E is a Banach space containing an isomorphic copy of ℓ_1. Then there is a function $\Phi : B_w \to \mathbb{R}$ which is sequentially continuous but not continuous.*

Proof. Let K denote the dual unit ball B_{E^*} equipped with the *weak** topology. Write Ix for the function on K given by $Ix(x^*) = x^*(x)$ for $x^* \in K$. Then $I : E \to C(K)$ is an isometric embedding.

Since E contains a copy of ℓ_1, there is a normalized sequence (x_n) in E which is a θ-ℓ_1-sequence for some $\theta > 0$. Then the functions $g_n = Ix_n$ form a θ-ℓ_1-sequence in $C(K)$. By Proposition 5.1 there is a probability measure μ on K such that $\int_K |g_n - g_k| \, \mathrm{d}\mu \geq \theta/2$ whenever $n \neq k$.

Define a function Φ on E by $\Phi(x) = \int_K |Ix| \, \mathrm{d}\mu$. If $y_j \to y$ weakly in E then $Iy_j \to Iy$ weakly in $C(K)$, i.e. $(Iy_j)_j$ is a uniformly bounded sequence converging pointwise to Iy. Consequently, $\Phi(y_j) \to \Phi(y)$ by the Lebesgue dominated convergence theorem. Thus Φ is sequentially continuous.

We now check that Φ is not weakly continuous at 0 on B_w. Consider a basic weak neighbourhood of $0 \in B_w$ of the form

$$V = \{x \in B_w : |x_j^*(x)| < \varepsilon \text{ for } j = 1, \dots, r\}.$$

Then there is an infinite set $N \subseteq \mathbb{N}$ such that $\big(x_j^*(x_n)\big)_{n \in N}$ is a converging sequence for every $j \leq r$. Hence there are $n \neq k$ such that $|x_j^*(x_n - x_k)| < \varepsilon$ for every $j \leq r$, which means that $(x_n - x_k)/2 \in V$. On the other hand, $\Phi\big((x_n - x_k)/2\big) \geq \theta/4$ which demonstrates that Φ is not continuous at 0. □

6. Proof of Theorem 1.10 and final questions

In order to prove Theorem 1.10 we need the following two results also of independent interest.

Proposition 6.1 ([15])**.** *Let E be a metrizable lcs. Then every bounded subset of E is Fréchet–Urysohn in the weak topology of E if and only if every bounded sequence in E has a Cauchy subsequence in the weak topology of E.*

Proposition 6.2 ([32])**.** *Let E be a complete lcs such that every bounded set in E is metrizable. Then E does not contain a copy of ℓ_1 if and only if every bounded sequence in E has a Cauchy subsequence in the weak topology of E.*

Proof of Theorem 1.10. (i)⇒(ii): By Proposition 6.1 and Proposition 6.2 every bounded set A in E is even Fréchet-Urysohn in the weak topology of E. The converse implication (ii)⇒(i) follows from Theorem 1.9. □

We complete the paper with a few open questions. By Proposition 4.1, there is a countable (hence Lindelöf) non-Ascoli space A. So A is homeomorphic to a closed subspace of some $\mathbb{R}^\kappa$. As $\mathbb{R}^\kappa$ is a $k_\mathbb{R}$-space, we see that a $k_\mathbb{R}$-space may contain a countable closed non-Ascoli subspace. So the $k_\mathbb{R}$-space property and the Ascoli property are not preserved in general by closed subspaces.

Question 6.3. *Let X be an Ascoli space such that every closed subspace of X is Ascoli. Is X a k-space?*

Arhangelskii [9, 3.12.15] proved that a topological space X is a hereditarily k-space if and only if X is Fréchet–Urysohn.

Question 6.4. *Let X be a hereditarily Ascoli space. Is X Fréchet–Urysohn?*

Let $E = C_p(\omega_1) = \mathbb{R}^{\omega_1}$. Then the lcs E is a $k_\mathbb{R}$-space by [24, Theorem 5.6] and is not a k-space by [18, Problem 7.J(b)]. So the $k_\mathbb{R}$-space property and the Ascoli property are not equivalent to the k-space property for C_p-spaces, see the Pytkeev and Gerlits–Nagy Theorem [1, II.3.7].

Question 6.5. *For which Tychonoff spaces X the space $C_p(X)$ is Ascoli (or a $k_\mathbb{R}$-space)?*

It is well-known (see [1, III.1.2]) that, for a compact space K, the space $C_p(K)$ is a k-space if and only if K is scattered. Below we generalize this result.

Proposition 6.6. *Let K be a compact space. Then $C_p(K)$ is a $k_\mathbb{R}$-space if and only if K is scattered.*

Proof. If K is scattered, then $C_p(K)$ is Fréchet–Urysohn, and we are done, see [1, Theorem III.1.2]. Now assume that K is not scattered. Then there is a continuous map p from K onto $[0,1]$ by [34, 8.5.4]. Let λ be the Lebesgue measure on $[0,1]$. Take a measure μ on K such that $p[\mu] = \lambda$ (see Lemma 4.4). Note that the measure μ vanishes on points. If we define

$$\Psi(g) = \int_X \frac{|g|}{|g|+1}\, \mathrm{d}\mu,$$

then Ψ is easily seen to be sequentially continuous on $C_p(K)$ by the Lebesgue theorem. This implies that Ψ is continuous on every compact subset $\mathcal{K}$ of $C_p(X)$ (recall that $\mathcal{K}$ is Fréchet–Urysohn, see [1, Theorem III.3.6]). On the other hand, it is easy to construct a family $\mathfrak{G}$ of functions $g : K \to [0,1]$ such that $\int_K g \, d\mu \geq 1/2$ and the zero function lies in the pointwise closure of $\mathfrak{G}$, see [1, Theorem II.3.5]). This means that Ψ is not continuous on $C_p(K)$. □

Remark 6.7. Let κ be a cardinal number endowed with the discrete topology. Then $C_p(\kappa) = \mathbb{R}^\kappa$ is a $k_\mathbb{R}$-space by [24]. Recall also that in a model of set theory without weakly inaccessible cardinals, any sequentially continuous function on $\mathbb{R}^\kappa$ is in fact continuous, see [30] for further references.

Theorem 1.4 and Proposition 6.6 motivate the following problem.

Question 6.8. *Does there exist X such that $C_k(X)$ or $C_p(X)$ is Ascoli but is not a $k_\mathbb{R}$-space?*

For a Tychonoff space X denote by $L(X)$ (respectively, $F(X)$ and $A(X)$) the free locally convex space (the free or the free abelian topological group) over X.

Question 6.9. *Let $L(X)$ ($F(X)$ or $A(X)$) be an Ascoli space. Is X Ascoli?*

Question 6.10. *For which metrizable spaces X, the groups $F(X)$ and $A(X)$ are Ascoli?*

In [12] the first named author proved that the free lcs $L(X)$ over a Tychonoff space X is a k-space if and only if X is a discrete countable space.

Question 6.11. *Let $L(X)$ be an Ascoli space. Is X a discrete countable space?*

We do not know the answer even if "Ascoli" is replaced by a stronger assumption "$L(X)$ is a $k_\mathbb{R}$-space" (see [12, Question 3.6]).

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