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Structure of the Set of Stationary Solutions to the Equations of Motion of a Class of Generalized Newtonian Fluids

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Abstract

We investigate the steady–state equations of motion of the generalized Newtonian fluid in a bounded domain $\Omega \subset \mathbb{R}^N$, when $N = 2$ or $N = 3$. Applying the tools of nonlinear analysis (Smale's theorem and properties of Fredholm operators, etc.), we show that if the dynamic stress tensor has the 2–structure then the solution set is finite and the solutions are C^1–functions of the external volume force $\mathbf{f}$ for generic $\mathbf{f}$. We also derive a series of properties of related operators in the case of a more general (p, δ)–structure, show that the solution set is compact if $p > 3N/(N + 2)$ and explain why the same method as in the case $p = 2$ cannot be applied in the case of general p.

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1 Introduction

1.1. Specification of the dynamic stress tensor. We investigate the structure of the set of steady solutions to the equations of motion of a class of generalized Newtonian fluids in a bounded Lipschitzian domain $\Omega \subset \mathbb{R}^N$, when $N = 2$ or $N = 3$. Concretely, we consider incompressible fluids, in which the dynamic stress tensor $\mathbb{S}$ depends on the rate of deformation tensor $\mathbb{D}\mathbf{v}$ (a symmetrical tensor defined in terms of the gradient of the velocity $\mathbf{v}$ and its transpose: $\mathbb{D}\mathbf{v} := \frac{1}{2}[\nabla\mathbf{v} + (\nabla\mathbf{v})^T]$) according to the law

$$\mathbb{S}(\mathbb{D}\mathbf{v}) := f\left(|\mathbb{D}\mathbf{v}|^2\right) \mathbb{D}\mathbf{v}. \tag{1.1}$$

The term $f\left(|\mathbb{D}\mathbf{v}|^2\right)$ represents the viscosity of the fluid. Throughout the paper, we assume that f is a positive function of the class C^1 on the interval $(0, \infty)$, satisfying

(i) $\lim_{t\to 0+} t\, f(t^2) \;=\; 0,$

(ii) there exist $\delta \geq 0$, $p > 1$ and positive constants c_1, c_2 such that

$$c_1\, (\delta + t)^{p-2} \;\leq\; \frac{\mathrm{d}}{\mathrm{d}t}\, \left[t\, f(t^2)\right] \;\leq\; c_2\, (\delta + t)^{p-2} \qquad \text{for all } t > 0. \tag{1.2}$$

Due to condition (i), the function $t\, f(t^2)$ can be continuously extended by zero from the interval $(0, \infty)$ to $[0, \infty)$. This enables us to extend naturally $\mathbb{S}(\mathbb{D}\mathbf{v})$ to the points $\mathbf{x} \in \Omega$, where $\mathbb{D}\mathbf{v}(\mathbf{x}) = \mathbb{O}$: we define $\mathbb{S}(\mathbb{O}) := \mathbb{O}$.

The fluid is said to be *shear–thinning* if function f is decreasing, and *shear–thickening* if f is increasing. The next lemma clarifies some important properties of the function f, which follow from conditions (i) and (ii).

Lemma 1. *There exist positive constants* $c_3,\ c_4,\ c_5$ *(depending only on* p*) such that*

$$c_3\, (\delta + t)^{p-2} \;\leq\; f(t^2) \;\leq\; c_4\, (\delta + t)^{p-2}, \tag{1.3}$$

$$t^2\, |f'(t^2)| \;\leq\; c_5\, (\delta + t)^{p-2}. \tag{1.4}$$

for all $t \in (0, \infty)$.

Proof. Condition (i) and the first inequality in (1.2) imply that

$$t\,f(t^2) \;=\; \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\,\left[\tau\,f(\tau^2)\right]\,\mathrm{d}\tau \;\geq\; \int_0^t c_1\,(\delta+\tau)^{p-2}\,\mathrm{d}\tau \;=\; c_1 t\int_0^1 (\delta+ts)^{p-2}\,\mathrm{d}s,$$

which yields

$$f(t^2) \;\geq\; c_1\int_0^1 (\delta s+ts)^{p-2}\,\mathrm{d}s \;=\; c_1\,(\delta+t)^{p-2}\int_0^1 s^{p-2}\,\mathrm{d}s \qquad \text{if } p\geq 2,$$

$$f(t^2) \;\geq\; c_1\int_0^1 \frac{\mathrm{d}s}{(\delta+t)^{2-p}} \;=\; c_1\,(\delta+t)^{p-2} \qquad \text{if } 1<p<1.$$

Similarly, due to (i) and the second inequality in (1.2), f satisfies

$$t\,f(t^2) \;\leq\; c_2\int_0^t (\delta+\tau)^{p-2}\;\mathrm{d}\tau \;=\; c_2\,t\int_0^1 (\delta+ts)^{p-2}\,\mathrm{d}s,$$

which yields

$$f(t^2) \;\leq\; c_2\int_0^1 (\delta+t)^{p-2}\,\mathrm{d}s \;=\; c_2\,(\delta+t)^{p-2} \qquad \text{if } p\geq 2,$$

$$f(t^2) \;\leq\; c_2\int_0^1 \frac{\mathrm{d}s}{(\delta s+ts)^{2-p}} \;=\; c_2\,(\delta+t)^{p-2}\int_0^1 s^{p-2}\,\mathrm{d}s \qquad \text{if } 1<p<1.$$

These inequalities imply (1.3). Since $2t^2\,f'(t^2)=[t\,f(t^2)]'-f(t^2)$ and $[t\,f(t^2)]'$ satisfies (1.2), we also have (1.4). $\square$

Conditions (i) and (ii) guarantee that the dynamic stress tensor $\mathbb{S}$ defined by (1.1) has the so called (p,δ)–structure. According to the definition (see e.g. [9]), tensor $\mathbb{S}\equiv(S_{ij})$ has the (p,δ)–structure (for $1<p<\infty$ and $\delta\geq 0$) if it is a C^0–mapping from $\mathbb{R}^{N\times N}_{\mathrm{sym}}:=\{\mathbb{A}\in\mathbb{R}^{N\times N};\ \mathbb{A}=\mathbb{A}^T\}$ to $\mathbb{R}^{N\times N}_{\mathrm{sym}}$ and a C^1–mapping from $\mathbb{R}^{N\times N}_{\mathrm{sym}}\smallsetminus\{\mathbb{O}\}$ to $\mathbb{R}^{N\times N}_{\mathrm{sym}}$, satisfying $\mathbb{S}(\mathbb{O})=\mathbb{O}$ and the inequalities

$$\sum_{i,j,k,l=1}^{N}\partial_{kl}S_{ij}(\mathbb{Q})\,P_{ij}\,P_{kl} \;\geq\; c_6\;(\delta+|\mathbb{Q}|)^{p-2}\;|\mathbb{P}|^2,$$

$$|\partial_{kl}S_{ij}(\mathbb{Q})| \;\leq\; c_7\;(\delta+|\mathbb{Q}|)^{p-2}$$

for all $\mathbb{P}\equiv(P_{ij})$ and $\mathbb{Q}\equiv(Q_{ij})$ in $\mathbb{R}^{N\times N}_{\mathrm{sym}}$ with $\mathbb{Q}\neq\mathbb{O}$ and all $i,j,k,l=1,\dots,N$. (c_6 and c_7 are positive constants independent of $\mathbb{P}$, $\mathbb{Q}$ and δ.) It follows from [9, Proposition 3] that if tensor $\mathbb{S}$ has the (p,δ)–structure then there exist positive constants c_8–c_{11} (depending only on p), such that

$$\begin{aligned} c_8\left(\delta+|\mathbb{B}|+|\mathbb{A}-\mathbb{B}|\right)^{p-2}|\mathbb{A}-\mathbb{B}|^2 \;&\leq\; [\mathbb{S}(\mathbb{A})-\mathbb{S}(\mathbb{B})]:(\mathbb{A}-\mathbb{B})\\ &\leq\; c_9\left(\delta+|\mathbb{B}|+|\mathbb{A}-\mathbb{B}|\right)^{p-2}|\mathbb{A}-\mathbb{B}|^2 \end{aligned} \tag{1.5}$$

$$\begin{aligned} c_{10}\left(\delta+|\mathbb{B}|+|\mathbb{A}-\mathbb{B}|\right)^{p-2}|\mathbb{A}-\mathbb{B}| \;&\leq\; |\mathbb{S}(\mathbb{A})-\mathbb{S}(\mathbb{B})|\\ &\leq\; c_{11}\left(\delta+|\mathbb{B}|+|\mathbb{A}-\mathbb{B}|\right)^{p-2}|\mathbb{A}-\mathbb{B}| \end{aligned} \tag{1.6}$$

for all $\mathbb{A}$, $\mathbb{B}\in\mathbb{R}^{3\times 3}_{\mathrm{sym}}$.

1.2. Examples. Typical examples of f, which provide tensor $\mathbb{S}$ with the (p,δ)–structure, are

$$f(t^2)=\mu\,(\delta+t)^{p-2}\quad\dots\quad\text{corresponding to}\quad \mathbb{S}(\mathbb{D}\mathbf{v})=\mu\,(\delta+|\mathbb{D}\mathbf{v}|)^{p-2}\;\mathbb{D}\mathbf{v}, \tag{1.7}$$

$$f(t^2)=\mu\left(\delta^2+t^2\right)^{\frac{p-2}{2}}\quad\dots\quad\text{corresponding to}\quad \mathbb{S}(\mathbb{D}\mathbf{v})=\mu\left(\delta^2+|\mathbb{D}\mathbf{v}|^2\right)^{\frac{p-2}{2}}\;\mathbb{D}\mathbf{v}, \tag{1.8}$$

where $\mu > 0$. (See e.g. [7] or [9].) If $p = 2$ then the inequalities in the definition of the (p, δ)–structure are independent of δ, so we may speak only about the 2–structure. In this case, condition (1.2) takes the simple form

$$c_1 \leq \frac{\mathrm{d}}{\mathrm{d}t} \left[t f(t^2) \right] \leq c_2 \qquad \text{for all } t > 0. \tag{1.9}$$

and inequalities (1.3) and (1.4) reduce to

$$c_3 \leq f(t^2) \leq c_4, \qquad t^2 \, |f'(t^2)| \leq c_5. \tag{1.10}$$

Examples of functions f, which provide tensor $\mathbb{S}$ with the 2–structure, are

$$f(t^2) = \mu_0 \pm (\mu_1 + t)^{-\gamma} \quad \dots \quad \text{corresponding to} \quad \mathbb{S}(\mathbb{D}\mathbf{v}) = \mu_0 \mathbb{D}\mathbf{v} \pm (\mu_1 + |\mathbb{D}\mathbf{v}|)^{-\gamma} \, \mathbb{D}\mathbf{v}, \tag{1.11}$$

$$f(t^2) = \mu_0 \pm \left(\mu_1^2 + t^2\right)^{-\frac{\gamma}{2}} \quad \dots \quad \text{corresponding to} \quad \mathbb{S}(\mathbb{D}\mathbf{v}) = \mu_0 \mathbb{D}\mathbf{v} \pm \left(\mu_1^2 + |\mathbb{D}\mathbf{v}|^2\right)^{-\frac{\gamma}{2}} \, \mathbb{D}\mathbf{v}, \tag{1.12}$$

where $\gamma \geq 0$ and $\mu_0,\, \mu_1 > 0$. (If the "$-$" sign is considered then we also assume that $\mu_0 > \mu_1^{-\gamma}$ in order to satisfy (1.3) and the first inequality in (1.2).) Obviously, if $\mathbb{S}(\mathbb{D}\mathbf{v})$ is as in (1.7) or (1.8) then the fluid is shear–thinning for $1 < p < 2$ and shear–thickening for $p > 2$. (It is Newtonian if $p = 2$.) If $\mathbb{S}(\mathbb{D}\mathbf{v})$ is as in (1.11) or (1.12) then the fluid is shear–thinning if $\gamma > 0$ and the sign "$+$" is considered and shear–thickening if $\gamma > 0$ and "$-$" is considered. (It is Newtonian if $\gamma = 0$.)

1.3. Notation, function spaces. Vector functions and spaces of vector functions are denoted by boldface letters.

- Let $1 < r < \infty$ and $k \in \{0\} \cup \mathbb{N}$. The norms of scalar– or vector– or tensor–valued functions, with components in the Lebesgue spaces $L^r(\Omega)$ (respectively the Sobolev spaces $W^{k,r}(\Omega)$) are denoted by $\| \, . \, \|_r$ (respectively $\| \, . \, \|_{k,r}$).
- $\mathbf{C}_{0,\sigma}^{\infty}(\Omega)$ denotes the space of infinitely differentiable divergence–free vector functions in Ω that have a compact support in Ω.
- $\mathbf{L}_{\sigma}^{p}(\Omega)$ is the closure of $\mathbf{C}_{0,\sigma}^{\infty}(\Omega)$ in $\mathbf{L}^p(\Omega)$.
- $\mathbf{W}_{0,\sigma}^{1,p}(\Omega) := \mathbf{L}_{\sigma}^{p}(\Omega) \cap \mathbf{W}_{0}^{1,p}(\Omega)$; since the norms $\| \, . \, \|_{1,p}$ and $\| \nabla \, . \, \|_p$ are equivalent in $\mathbf{W}_{0,\sigma}^{1,p}(\Omega)$, we mostly use the latter as a norm in $\mathbf{W}_{0,\sigma}^{1,p}(\Omega)$. Note that there exist $c_{12} > 0$ and $c_{13} > 0$ (depending on p and Ω) such that
$$c_{12} \, \| \nabla \mathbf{v} \|_p \leq \| \mathbb{D}\mathbf{v} \|_p \leq c_{13} \, \| \nabla \mathbf{v} \|_p \tag{1.13}$$
for all $\mathbf{v} \in \mathbf{W}_{0,\sigma}^{1,p}(\Omega)$. (This is the so called *Korn inequality*, see e.g. [11].)
- p' denotes the conjugate exponent to p.
- $\mathbf{W}_{0,\sigma}^{-1,p'}(\Omega)$ is the dual space to $\mathbf{W}_{0,\sigma}^{1,p}(\Omega)$. The duality between elements of $\mathbf{W}_{0,\sigma}^{-1,p'}(\Omega)$ and $\mathbf{W}_{0,\sigma}^{1,p}(\Omega)$ is denoted by $\langle \, . \, , \, . \, \rangle_{\sigma}$. The norm in $\mathbf{W}_{0,\sigma}^{-1,p'}(\Omega)$ is denoted by $\| \, . \, \|_{-1,p'}$.
- If X and Y are Banach spaces then $\mathcal{L}(X, Y)$ denotes the space of bounded linear operators from X to Y.
- If T is a closed densely defined linear operator from Banach space X to Banach space Y then $\mathrm{nul}\; T$ denotes the *nullity* of T, i.e. the dimension of the null space of T. The *deficiency* of T (i.e. the dimension of the quotient space $Y|_{R(T)}$, where $R(T)$ is the range of T) is denoted by $\mathrm{def}\; T$. The *index* of T, which equals the difference $\mathrm{nul}\; T - \mathrm{def}\; T$, is denoted by $\mathrm{ind}\, T$. The *approximate nullity* of T is denoted by $\mathrm{nul}'\, T$. (It is the maximum number $n \in \{0\} \cup \mathbb{N}$ with the property that to every $\epsilon > 0$ there exists an n–dimensional subspace X_n of X such that $x \in X_n$, $\|x\|_X = 1$ implies $\|Tx\|_Y \leq \epsilon$.) The *approximate deficiency* of T (i.e. the approximate nullity of the adjoint operator T^*) is denoted by $\mathrm{def}'\, T$. Note that $\mathrm{nul}\; T \leq \mathrm{nul}'\, T$ and $\mathrm{def}\; T \leq \mathrm{def}'\, T$. The equalities hold if $R(T)$ is closed, while $\mathrm{nul}'\, T = \mathrm{def}'\, T = \infty$ if $R(T)$ is not closed. (See [5, p. 233].)

1.4. The boundary–value problem, a weak solution. The boundary–value problem we intend to investigate consists of the equations

$$-\mathrm{div}\, \mathbb{S}(\mathbb{D}\mathbf{v}) + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla q \ = \ \mathbf{f}, \tag{1.14}$$

$$\mathrm{div}\, \mathbf{v} \ = \ 0 \tag{1.15}$$

in the domain Ω and the homogeneous Dirichlet boundary condition

$$\mathbf{v} \ = \ \mathbf{0} \tag{1.16}$$

on $\partial\Omega$. The unknowns are $\mathbf{v} \equiv (v_1, v_2, v_3)$ (the velocity) and q (the pressure). Equation (1.14) expresses the balance of momentum and equation (1.15) expresses the conservation of mass. The boundary condition (1.16) is also called the *no–slip condition.* Foundations of the qualitative theory of the system of equations (1.14), (1.15) and related models were given in the papers [6] (by J. Nečas, J. Málek and M. Růžička) and [1] (by H. Bellout, F. Bloom and J. Nečas). The existence of a weak solution to the problem (1.14)–(1.16) was proved by J. Frehse, J. Málek and M. Steinhauer [4] for general $N \geq 2$ under the condition that

$$\frac{2N}{N+2} \ < \ p \ < \ \infty. \tag{1.17}$$

The proof is based on the method of Lipschitz truncations. The procedure is also explained in detail in paper [9]. (Here, the author extends the existential result to the case $p = 2N/(N+2)$, $N \geq 3$.) By analogy with the Navier–Stokes equations, the pressure does not explicitly appear in the weak formulation. The right hand side $\mathbf{f}$ of equation (1.14) is supposed to be an element of $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$. The weak solution is a function $\mathbf{v} \in \mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ such that

$$\int_\Omega f\left(|\mathbb{D}\mathbf{v}|^2\right)\, \mathbb{D}\mathbf{v} : \mathbb{D}\boldsymbol{\varphi}\, \mathrm{d}\mathbf{x} - \int_\Omega (\mathbf{v} \otimes \mathbf{v}) : \mathbb{D}\boldsymbol{\varphi}\, \mathrm{d}\mathbf{x} \ = \ \langle \mathbf{f}, \boldsymbol{\varphi} \rangle_\sigma \tag{1.18}$$

for all $\boldsymbol{\varphi} \in \mathbf{W}^{1,s}_{0,\sigma}(\Omega)$, where

$$\frac{1}{s'} \ := \ \max\left\{\frac{1}{p'};\ \frac{2}{p} - \frac{2}{N}\right\}. \tag{1.19}$$

(The condition $(s')^{-1} \geq 2(p^{-1} - N^{-1})$ comes from the requirement that the function $(\mathbf{v} \otimes \mathbf{v}) : \mathbb{D}\boldsymbol{\varphi}$ is integrable in Ω.) Formula (1.19) implies that $p \leq s$, which means that $\mathbf{W}^{1,s}_{0,\sigma}(\Omega) \subset \mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ and $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega) \subset \mathbf{W}^{-1,s'}_{0,\sigma}(\Omega)$ (both with the continuous imbedding).

Obviously, if $(p')^{-1} \geq 2(p^{-1} - N^{-1})$ then $(s')^{-1} = (p')^{-1})$ and therefore $s = p$. The condition $(p')^{-1} \geq 2(p^{-1} - N^{-1})$ is equivalent to

$$p \ \geq \ \frac{3N}{N+2}. \tag{1.20}$$

If (1.20) holds then one can use (1.18) with $\boldsymbol{\varphi} = \mathbf{v}$. Thus, (1.18) yields

$$\int_\Omega f\left(|\mathbb{D}\mathbf{v}|^2\right)\, |\mathbb{D}\mathbf{v}|^2\, \mathrm{d}\mathbf{x} \ = \ \langle \mathbf{f}, \mathbf{v} \rangle_\sigma \ \leq \ \|\mathbf{f}\|_{-1,p'}\, \|\nabla \mathbf{v}\|_p.$$

(The integral of $(\mathbf{v} \otimes \mathbf{v}) : \mathbb{D}\mathbf{v}$ vanishes.) If $p \geq 2$ then, due to (1.3),

$$\begin{aligned}\|\mathbb{D}\mathbf{v}\|_p^p \ &\leq \ \int_\Omega (\delta + |\mathbb{D}\mathbf{v}|)^{p-2}\, |\mathbb{D}\mathbf{v}|^2\, \mathrm{d}\mathbf{x} \ \leq \ \frac{1}{c_3} \int_\Omega f\left(|\mathbb{D}\mathbf{v}|^2\right)\, |\mathbb{D}\mathbf{v}|^2\, \mathrm{d}\mathbf{x} \ = \ \frac{1}{c_3}\, \langle \mathbf{f}, \mathbf{v} \rangle_\sigma \\ &\leq \ \frac{1}{c_3}\, \|\mathbf{f}\|_{-1,p'}\, \|\nabla \mathbf{v}\|_p \ \leq \ \frac{1}{c_3\, c_{12}}\, \|\mathbf{f}\|_{-1,p'}\, \|\mathbb{D}\mathbf{v}\|_p \ \leq \ \frac{1}{2}\, \|\mathbb{D}\mathbf{v}\|_p^p + C(p)\, \|\mathbf{f}\|_{-1,p'}^{p'}.\end{aligned}$$

Here and further on, C denotes a generic constant. Hence

$$\|\mathbb{D}\mathbf{v}\|_p^p \;\leq\; c_{14}\, \|\mathbf{f}\|_{-1,p'}^{p'}, \tag{1.21}$$

where $c_{14} = c_{14}(p, c_3, c_{12})$. If $3N/(N+2) \leq p < 2$ then we denote $\Omega_1 := \{\mathbf{x} \in \Omega;\ |\mathbb{D}\mathbf{v}(\mathbf{x})| \leq \delta\}$ and $\Omega_2 := \Omega \smallsetminus \Omega_1$. We have

$$\begin{aligned}
\|\mathbb{D}\mathbf{v}\|_p^p \;&=\; \int_{\Omega_1} |\mathbb{D}\mathbf{v}|^p \,\mathrm{d}\mathbf{x} + \int_{\Omega_2} \frac{2^{2-p}\, |\mathbb{D}\mathbf{v}|^2}{(|\mathbb{D}\mathbf{v}| + |\mathbb{D}\mathbf{v}|)^{2-p}}\,\mathrm{d}\mathbf{x} \\
&\leq\; \delta^p\, |\Omega_1| + 2^{2-p} \int_{\Omega_2} \frac{|\mathbb{D}\mathbf{v}|^2}{(\delta + |\mathbb{D}\mathbf{v}|)^{2-p}}\,\mathrm{d}\mathbf{x} \\
&\leq\; \delta^p\, |\Omega| + \frac{2^{2-p}}{c_3} \int_{\Omega} f\left(|\mathbb{D}\mathbf{v}|^2\right)\, |\mathbb{D}\mathbf{v}|^2 \,\mathrm{d}\mathbf{x} \;=\; \delta^p\, |\Omega| + C(p)\, \langle \mathbf{f}, \mathbf{v}\rangle_\sigma \\
&\leq\; \delta^p\, |\Omega| + C(p)\, \|\mathbf{f}\|_{-1,p'}\, \|\nabla\mathbf{v}\|_p \;\leq\; \delta^p\, |\Omega| + C(p)\, \|\mathbf{f}\|_{-1,p'}\, \|\mathbb{D}\mathbf{v}\|_p \\
&\leq\; \delta^p\, |\Omega| + \frac{1}{2}\, \|\mathbb{D}\mathbf{v}\|_p^p + C(p)\, \|\mathbf{f}\|_{-1,p'}^{p'}.
\end{aligned}$$

Hence there exists $c_{15} = c_{15}(p, c_3, c_{12}) > 0$ such that

$$\|\mathbb{D}\mathbf{v}\|_p^p \;\leq\; 2\delta^p\, |\Omega| + c_{15}\, \|\mathbf{f}\|_{-1,p'}^{p'}. \tag{1.22}$$

1.5. An alternative form of equation (1.18). In order to represent equation (1.18) as an equation in the dual space to $\mathbf{W}_{0,\sigma}^{1,p}(\Omega)$, we assume from now on that condition (1.20) holds. Let the operators $\mathscr{A}$ and $\mathscr{B}$ from $\mathbf{W}_{0,\sigma}^{1,p}(\Omega)$ to $\mathbf{W}_{0,\sigma}^{-1,p'}(\Omega)$ be defined by the equations

$$\langle \mathscr{A}\mathbf{v}, \boldsymbol{\varphi}\rangle_\sigma \;:=\; \int_\Omega \mathbb{S}(\mathbb{D}\mathbf{v}) : \mathbb{D}\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x}, \tag{1.23}$$

$$\langle \mathscr{B}\mathbf{v}, \boldsymbol{\varphi}\rangle_\sigma \;:=\; \int_\Omega \mathbf{v}\cdot\nabla\mathbf{v}\cdot\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} \tag{1.24}$$

for all $\mathbf{v}$ and $\boldsymbol{\varphi}$ in $\mathbf{W}_{0,\sigma}^{1,p}(\Omega)$. Operator $\mathscr{A}$ is often called the *Stokes–type* or the *p–Stokes–type operator.* Equation (1.18) can now be written in the form

$$\mathscr{A}\mathbf{v} + \mathscr{B}\mathbf{v} \;=\; \mathbf{f}, \tag{1.25}$$

as an equation in the dual space $\mathbf{W}_{0,\sigma}^{-1,p'}(\Omega)$ to $\mathbf{W}_{0,\sigma}^{1,p}(\Omega)$.

1.6. Aims of this paper, previous related results. The structure of the set of strong solutions to the steady Navier–Stokes equations in a smooth bounded domain Ω in $\mathbb{R}^N$ was studied by C. Foias and R. Temam [3]. These authors have shown that there exists a dense open set $\mathcal{O}$ in $\mathbf{L}^2_\sigma(\Omega)$ such that for every $\mathbf{f} \in \mathcal{O}$ the set of solutions in $\mathbf{W}_{0,\sigma}^{1,2}(\Omega) \cap \mathbf{W}^{2,2}(\Omega)$ is finite, the number of solutions is constant for $\mathbf{f}$ in each component of $\mathcal{O}$ and every solution is a C^∞–function of $\mathbf{f}$ for $\mathbf{f} \in \mathcal{O}$. Similar results have also been obtained by J. Neustupa and D. A. Siginer in connection with the Bénard problem in a two–dimensional quadrangular cavity, heated/cooled on two opposite sides and insulated on the other two sides, see paper [8]. The methods, used in [3] and [8], are especially based on Smale's theorem (a Banach space generalization of the former Sard theorem, see [10]), which we cite here for completeness in the form following from [12, Theorem 4.K]:

Theorem 1 (Smale). *Let X and Y be Banach spaces and $\mathcal{F} : X \to Y$ be a proper Fredholm operator of class C^k for $k > \max\{\mathrm{ind}\,\mathcal{F},\, 0\}$. Then the set of regular values of $\mathcal{F}$ is dense and open in Y.*

Recall that a point $x \in X$ is said to be a *regular point* of $\mathcal{F}$ if the range of $\mathcal{F}'(x)$ is the whole space Y. The point x is called *singular* if it is not regular. A point $y \in Y$ is said to be a *singular value* of $\mathcal{F}$ if $\mathcal{F}^{-1}(y)$ contains a singular point. Otherwise y is called a *regular value.*

A closed linear densely defined operator is said to be *Fredholm* if its range is closed and both its nullity and deficiency are finite. A closed linear operator is said to be *semi–Fredholm* if its range is closed and its nullity or its deficiency is finite. Since both the approximate nullity and the approximate deficiency are automatically infinite if the range is not closed, and on the other hand, they coincide with the nullity and deficiency, respectively, if the range is closed, the semi–Fredholm operator can also be defined to be a closed operator, whose approximate nullity or approximate deficiency is finite. Naturally, if an operator is Fredholm then it is also semi–Fredholm. A nonlinear operator $\mathcal{F}$ of class C^1 is said to be *Fredholm* if the Fréchet differential $D\mathcal{F}(x)$ is a linear Fredholm operator at all points $x \in X$. In this case, $\operatorname{ind} D\mathcal{F}(x)$ is independent of x (see [5, Theorem IV.5.26]) and it is called the index of the operator $\mathcal{F}$.

Since we often use the notions of the Gâteaux and Fréchet differentials, recall that if the directional Gâteaux derivative $D\mathcal{F}(x)x^*$ (at point $x \in X$ in the direction x^*) exists in all directions x^* and it is a bounded linear operator from X to Y in dependence on x^* then $D\mathcal{F}(x)$ is called the *Gâteaux differential.* If, moreover, $\mathcal{F}(x+h) - \mathcal{F}(x) - D\mathcal{F}(x)h = o(h)$ for $\|h\|_X \to 0$ then $D\mathcal{F}(x)$ is said to be the Fréchet differential. It is well known that if the Gâteaux differential $D\mathcal{F}(x)$ depends continuously on x in the topology of $\mathcal{L}(X,Y)$ then it is the Fréchet differential. (See e.g. [2] or [12].)

The aim of this paper is to derive similar results as in papers [3] and [8], however for the boundary–value problem (1.14)–(1.16). In Section 2, we study in greater detail the operators $\mathscr{A}$ and $\mathscr{A}+\mathscr{B}$ and their differentials. We show that if $p \geq 2$ then $\mathscr{A}$ and $\mathscr{A}+\mathscr{B}$ are C^1–operators from $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$. Using the fact that $\mathscr{A}^{-1} \circ \mathscr{B}$ is compact, we deduce that the set of all solutions to equation (1.25) is compact (Theorem 2). However, it turns out that if $p > 2$ then neither $\mathscr{A}$ nor $\mathscr{A}+\mathscr{B}$ is a Fredholm operator because its Fréchet differentials $D\mathscr{A}(\mathbf{v})$ and $D\mathscr{A}(\mathbf{v}) + D\mathscr{B}(\mathbf{v})$ are not semi–Fredholm at any point $\mathbf{v} \in \mathbf{W}^{1,p}_{0,\sigma}(\Omega)$. (See Lemma 7 and Lemma 8.) Especially due to these “negative” results (see also the end of Section 2 for further explanation), we focus on the case $p = 2$ in Section 3 and we derive an analogous description of the solution set of equation (1.25) as in [3] and [8] (see Theorem 3). Recall that, in contrast to [3] and [8], our equation of motion (1.14) concerns the generalized Newtonian fluid and we deal with weak solutions.

2 More on the operators $\mathscr{A}$ and $\mathscr{B}$ and their differentials

Recall that p is supposed to satisfy condition (1.20), which implies that the numbers s' (defined in (1.19)) and s coincide with p' and p, respectively.

2.1. Basic properties of operators $\mathscr{A}$ and $\mathscr{B}$. The domain of $\mathscr{A}$ is the space $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ and the range of $\mathscr{A}$ is contained in $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$. Operator $\mathscr{A}$ is bounded and continuous, which follows easily from the second inequality in (1.6). The first inequality in (1.5) implies that $\mathscr{A}$ is strictly monotone and coercive. Hence $\mathscr{A}$ satisfies the assumptions of Browder’s theorem, see e.g. [2, p. 375]. Thus, the range of $\mathscr{A}$ is the whole dual space $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$. (Note that this also follows from [4] and [9].) Moreover, due to the strict monotonicity, operator $\mathscr{A}$ is one–to–one and therefore invertible. The boundedness of $\mathscr{A}^{-1}$ easily follows from (1.21) (the case $p \geq 2$) or from (1.22) (the case $3N/(N+2) \leq p < 2$). The next lemma summarizes these findings and adds an information on the continuity of the inverse operator $\mathscr{A}^{-1}$.

Lemma 2. *Let function f satisfy conditions (i) and (ii) and p satisfy condition (1.20). Then the p–Stokes–type operator $\mathscr{A}$ is a one–to–one bounded continuous operator, mapping the whole space $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ onto the whole space $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$. The inverse operator $\mathscr{A}^{-1}$ is bounded and continuous.*

Proof. We only need to prove that the operator $\mathscr{A}^{-1}$ is continuous. Let $\{\mathbf{f}_n\}$ be a sequence such that $\mathbf{f}_n \to \mathbf{f}$ in $\mathbf{W}^{-1,p}_{0,\sigma}$. Denote $\mathbf{v}_n := \mathscr{A}^{-1}\mathbf{f}_n$ and $\mathbf{v} := \mathscr{A}^{-1}\mathbf{f}$.

a) Assume at first that $p \geq 2$. We have

$$\langle \mathscr{A}\mathbf{v}_n - \mathscr{A}\mathbf{v}, \mathbf{v}_n - \mathbf{v} \rangle_\sigma \;=\; \langle \mathbf{f}_n - \mathbf{f}, \mathbf{v}_n - \mathbf{v} \rangle_\sigma \;\leq\; \|\mathbf{f}_n - \mathbf{f}\|_{-1,p'}\, \|\nabla(\mathbf{v}_n - \mathbf{v})\|_p$$

$$\leq \frac{1}{c_{12}} \|\mathbf{f}_n - \mathbf{f}\|_{-1,p'} \|\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}\|_p. \tag{2.1}$$

Due to (1.23) and (1.5), the left hand side is

$$= \int_\Omega [\mathbb{S}(\mathbb{D}\mathbf{v}_n) - \mathbb{S}(\mathbb{D}\mathbf{v})] : (\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v})\, \mathrm{d}\mathbf{x} \geq c_8 \int_\Omega \big(\delta + |\mathbb{D}\mathbf{v}| + |\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}|\big)^{p-2} |\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}|^2\, \mathrm{d}\mathbf{x}$$
$$\geq c_8 \int_\Omega |\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}|^{p-2} |\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}|^2\, \mathrm{d}\mathbf{x} = c_8 \|\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}\|_p^p.$$

Hence $c_8 \|\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}\|_p^{p-1} \leq c_{12}^{-1} \|\mathbf{f}_n - \mathbf{f}\|_{-1,p'}$. The right hand side tends to 0 for $n \to \infty$. Thus, $\mathbb{D}\mathbf{v}_n(\mathbf{x}) \to \mathbb{D}\mathbf{v}(\mathbf{x})$ in $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$. This implies that the operator $\mathscr{A}^{-1}$ is continuous.

b) The case $3N/(N+2) \leq p < 2$. Denote $\Omega_{1n} := \{\mathbf{x} \in \Omega;\ |\mathbb{D}\mathbf{v}_n(\mathbf{x}) - \mathbb{D}\mathbf{v}(\mathbf{x})| < \delta + |\mathbb{D}\mathbf{v}(\mathbf{x})|\}$ and $\Omega_{2n} := \Omega \smallsetminus \Omega_{1n}$. The right hand side of (2.1) can be estimated:

$$\frac{1}{c_{12}} \|\mathbf{f}_n - \mathbf{f}\|_{-1,p'} \|\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}\|_p$$
$$\leq \|\mathbf{f}_n - \mathbf{f}\|_{-1,p'} \left(\int_{\Omega_{1n}} |\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}|^p\, \mathrm{d}\mathbf{x}\right)^{\frac{1}{p}} + \|\mathbf{f}_n - \mathbf{f}\|_{-1,p'} \left(\int_{\Omega_{2n}} |\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}|^p\, \mathrm{d}\mathbf{x}\right)^{\frac{1}{p}}$$
$$\leq C(p) \|\mathbf{f}_n - \mathbf{f}\|_{-1,p'} \left(\int_{\Omega_{1n}} (\delta + |\mathbb{D}\mathbf{v}|)^p\, \mathrm{d}\mathbf{x}\right)^{\frac{1}{p}} + \epsilon \int_{\Omega_{2n}} |\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}|^p\, \mathrm{d}\mathbf{x} + C(p,\epsilon) \|\mathbf{f}_n - \mathbf{f}\|_{-1,p'}^{p'},$$

where $\epsilon > 0$ can be chosen arbitrarily small. The left hand side of (2.1) can be estimated from below by means of (1.5):

$$\int_\Omega [\mathbb{S}(\mathbb{D}\mathbf{v}_n) - \mathbb{S}(\mathbb{D}\mathbf{v})] : (\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v})\, \mathrm{d}\mathbf{x} \geq c_8 \int_\Omega \frac{|\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}|^2}{(\delta + |\mathbb{D}\mathbf{v}| + |\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}|)^{2-p}}\, \mathrm{d}\mathbf{x}$$
$$\geq c_8 \int_{\Omega_{1n}} \frac{|\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}|^2}{(\delta + |\mathbb{D}\mathbf{v}| + |\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}|)^{2-p}}\, \mathrm{d}\mathbf{x} + c_8 2^{p-2} \int_{\Omega_{2n}} |\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}|^p\, \mathrm{d}\mathbf{x}.$$

Thus, if we choose $\epsilon = \frac{1}{2} c_8 2^{p-2}$, we obtain

$$c_8 \int_{\Omega_{1n}} \frac{|\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}|^2}{(\delta + |\mathbb{D}\mathbf{v}| + |\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}|)^{2-p}}\, \mathrm{d}\mathbf{x} + c_8 2^{p-3} \int_{\Omega_{2n}} |\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}|^p\, \mathrm{d}\mathbf{x}$$
$$\leq C(p) \|\mathbf{f}_n - \mathbf{f}\|_{-1,p'} \left(\int_{\Omega_{1n}} (\delta + |\mathbb{D}\mathbf{v}|)^p\, \mathrm{d}\mathbf{x}\right)^{\frac{1}{p}} + C(p) \|\mathbf{f}_n - \mathbf{f}\|_{-1,p'}^{p'}.$$

Since the right hand side tends to zero for $n \to \infty$, the left hand side must tend to zero as well. Hence

$$\mathbb{D}\mathbf{v}_n(\mathbf{x}) \to \mathbb{D}\mathbf{v}(\mathbf{x}) \quad \text{a.e. in } \Omega \qquad \text{and} \qquad \int_{\Omega_{2n}} |\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}|^p\, \mathrm{d}\mathbf{x} \to 0 \tag{2.2}$$

for $n \to \infty$. Denote by χ_{1n} the characteristic function of set Ω_{1n}. Then

$$\int_\Omega |\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}|^p\, \mathrm{d}\mathbf{x} = \int_\Omega |\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}|^p \chi_{1n}\, \mathrm{d}\mathbf{x} + \int_{\Omega_{2n}} |\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}|^p\, \mathrm{d}\mathbf{x}.$$

The second integral on the right hand side tends to zero for $n \to \infty$ due to (2.2). The integrand in the first integral satisfies $0 \leq |\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}|^p \chi_{1n} \leq (\delta + |\mathbb{D}\mathbf{v}|)^p$ a.e. in Ω. Thus, due to Lebesgue's dominated theorem, the first integral on the right hand side tends to zero for $n \to \infty$, too. The proof is completed. $\square$

Applying operator $\mathscr{A}^{-1}$ to equation (1.25), we obtain its equivalent form

$$\mathbf{v} + \mathscr{A}^{-1}(\mathscr{B}\mathbf{v}) = \mathscr{A}^{-1}(\mathbf{f}), \tag{2.3}$$

which is an equation in $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$.

Lemma 3. *If p satisfies condition (1.20) then $\mathscr{B}$ is a continuous operator from $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$. If, moreover, function f satisfies conditions (i) and (ii) then the operator $\mathscr{A}^{-1} \circ \mathscr{B}$ is continuous in $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$. If, furthermore, p satisfies the stronger condition*

$$p \;>\; \frac{3N}{N+2} \tag{2.4}$$

instead of (1.20), then the operator $\mathscr{A}^{-1} \circ \mathscr{B}$ is compact in $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$.

Proof. Let us at first prove the continuity of $\mathscr{B}$. Thus, consider a sequence $\{\mathbf{v}_n\}$ in $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ such that $\mathbf{v}_n \to \mathbf{v}$ in $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$. If $p \geq N$ then $\mathbf{v}_n \to \mathbf{v}$ in $\mathbf{L}^s(\Omega)$ for each $s \in (1,\infty)$. If $3N/(N+2) \leq p < N$ then $\mathbf{v}_n \to \mathbf{v}$ in $\mathbf{L}^{Np/(N-p)}(\Omega)$ and $\mathbf{v}_n \otimes \mathbf{v}_n \to \mathbf{v} \otimes \mathbf{v}$ in $L^{Np/2(N-p)}(\Omega)^{N\times N}$. In any case, $\mathbf{v}_n \otimes \mathbf{v}_n \to \mathbf{v} \otimes \mathbf{v}$ in $L^{p'}(\Omega)^{N\times N}$, because condition (1.20) guarantees that $Np/2(N-p) \geq p'$. Since

$$|\langle \mathscr{B}\mathbf{v}_n - \mathscr{B}\mathbf{v}, \boldsymbol{\varphi}\rangle_\sigma| \;=\; \left| \int_\Omega [\mathbf{v}_n \otimes \mathbf{v}_n - \mathbf{v} \otimes \mathbf{v}] : \nabla\boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} \right| \;\leq\; \|\mathbf{v}_n \otimes \mathbf{v}_n - \mathbf{v} \otimes \mathbf{v}\|_{p'} \; \|\nabla\boldsymbol{\varphi}\|_p,$$

we observe that $\mathscr{B}\mathbf{v}_n \to \mathscr{B}\mathbf{v}$ in $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$. Hence $\mathscr{B}$ is a continuous operator from $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$. This and Lemma 2 imply that the operator $\mathscr{A}^{-1} \circ \mathscr{B}$ is continuous in $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$.

Further, let $\{\mathbf{v}_n\}$ be a bounded sequence in $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$. The space $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ is reflexive, hence there exists a subsequence (which we denote again by $\{\mathbf{v}_n\}$) such that $\mathbf{v}_n$ converges weakly to some $\mathbf{v}$ in $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$. Since $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ is compactly imbedded to $\mathbf{L}^s(\Omega)$ for each $1 < s < \infty$ (if $p \geq N$) or for each $1 < s < Np/(N-p)$ (if $p < N$), $\mathbf{v}_n \to \mathbf{v}$ strongly in $\mathbf{L}^s(\Omega)$ for all $1 < s < \infty$ (if $p \geq N$) or for all $1 < s < Np/(N-p)$ (if $p < N$). Hence $\mathbf{v}_n \otimes \mathbf{v}_n \to \mathbf{v} \otimes \mathbf{v}$ in $L^s(\Omega)^{N\times N}$ for all $1 < s < \infty$ (if $p \geq N$) or for all $1 < s < Np/2(N-p)$ (if $p < N$). Consequently, $\mathbf{v}_n \otimes \mathbf{v}_n \to \mathbf{v} \otimes \mathbf{v}$ in $L^{p'}(\Omega)^{N\times N}$, because condition (2.4) guarantees that $Np/2(N-p) > p'$. Thus, $\mathscr{B}\mathbf{v}_n \to \mathscr{B}\mathbf{v}$ in $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$ and $\mathscr{A}^{-1}\mathscr{B}\mathbf{v}_n \to \mathscr{A}^{-1}\mathscr{B}\mathbf{v}$ in $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$, which confirms the compactness of the operator $\mathscr{A}^{-1}\mathscr{B}$. $\square$

Since the set of all solutions to the equation (1.25) is bounded (due to the estimates (1.21) and (1.22)), and equation (1.25) is equivalent to (2.3), we can formulate the following theorem:

Theorem 2. *If p satisfies condition (2.4) and function f satisfies conditions (i) and (ii) then the set of all solutions of equation (1.25) (and therefore also the set of all weak solutions of the boundary–value problem (1.14)–(1.16)) is compact in $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$.*

2.2. The Gâteaux and Fréchet differential of the operators $\mathscr{A}$ and $\mathscr{B}$. The Gâteaux derivative $D\mathscr{A}(\mathbf{v})\mathbf{v}^*$ of $\mathscr{A}$ at point $\mathbf{v}$ in the direction $\mathbf{v}^*$ is a functional on $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$, given by the equation

$$\big\langle D\mathscr{A}(\mathbf{v})\mathbf{v}^*, \boldsymbol{\varphi}\big\rangle_\sigma \;=\; \int_\Omega f\left(|\mathbb{D}\mathbf{v}|^2\right) \mathbb{D}\mathbf{v}^* : \mathbb{D}\boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} + \int_\Omega 2\, f'\left(|\mathbb{D}\mathbf{v}|^2\right) \, (\mathbb{D}\mathbf{v} : \mathbb{D}\mathbf{v}^*) \, (\mathbb{D}\mathbf{v} : \mathbb{D}\boldsymbol{\varphi}) \, \mathrm{d}\mathbf{x}. \tag{2.5}$$

Let $p \geq 2$. Recall that so far function f has been defined on the interval $(0,\infty)$ with the possibility of a continuous extension of $t\, f(t^2)$ by zero at the point $t = 0$ due to condition (i). In order to give a natural sense to the integrands in (2.5) at the points $\mathbf{x} \in \Omega$ where $\mathbb{D}\mathbf{v}(\mathbf{x}) = \mathbb{O}$, we assume from now on that

(iii) $\lim\limits_{\tau\to 0+} f(\tau) = f_0$ (where $0 < f_0 < \infty$) and $\lim\limits_{\tau\to 0+} \tau\, f'(\tau) = 0$.

Obviously, (iii) $\Longrightarrow$ (i). Moreover, condition (iii) implies that the functions $f(\tau)$ and $\tau\, f'(\tau)$ (and also $f(t^2)$ and $t^2 f'(t^2)$) can be extended from the interval $(0,\infty)$ by continuity to $[0,\infty)$. Note that the concrete examples of function f, given in (1.11) and (1.12), satisfy condition (iii). Since $p \geq 2$, the examples (1.7), (1.8) satisfy (iii), too. (They satisfy (iii) also if $p < 2$, but in this case $\delta > 0$ is needed.)

Now, due to (1.3) and (1.4), we have

$$
\begin{aligned}
\left|\left\langle D\mathscr{A}(\mathbf{v})\mathbf{v}^*, \boldsymbol{\varphi}\right\rangle_\sigma\right| &\leq \int_\Omega \left|f\left(|\mathbb{D}\mathbf{v}|^2\right)\right| |\mathbb{D}\mathbf{v}^*|\, |\mathbb{D}\boldsymbol{\varphi}| \, \mathrm{d}\mathbf{x} + \int_\Omega \left|2 f'\left(|\mathbb{D}\mathbf{v}|^2\right)\right| |\mathbb{D}\mathbf{v}|^2\, |\mathbb{D}\mathbf{v}^*|\, |\mathbb{D}\boldsymbol{\varphi}| \, \mathrm{d}\mathbf{x} \\
&\leq (c_4 + 2c_5) \int_\Omega (\delta + |\mathbb{D}\mathbf{v}|)^{p-2}\, |\mathbb{D}\mathbf{v}^*|\, |\mathbb{D}\boldsymbol{\varphi}| \, \mathrm{d}\mathbf{x} \leq \\
&\leq C \left(\int_\Omega (\delta + |\mathbb{D}\mathbf{v}|)^p \, \mathrm{d}\mathbf{x} \right)^{\frac{p-2}{p}} \|\mathbb{D}\mathbf{v}^*\|_p \, \|\mathbb{D}\boldsymbol{\varphi}\|_p \\
&\leq C \left(\delta^{p-2} + \|\mathbb{D}\mathbf{v}\|_p^{p-2}\right) \|\mathbb{D}\mathbf{v}^*\|_p \, \|\mathbb{D}\boldsymbol{\varphi}\|_p.
\end{aligned}
$$

This shows that $D\mathscr{A}(\mathbf{v})$ is a bounded linear operator from $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$. Thus, $D\mathscr{A}(\mathbf{v})$ coincides with the Gâteaux differential of $\mathscr{A}$ at the point $\mathbf{v}$.

The condition $p \geq 2$ is important, because if $p < 2$ then, in addition to the problem possibly arising at points where $\mathbb{D}\mathbf{v}(\mathbf{x}) = \mathbb{O}$, the integrals on the right hand side of (2.5) generally do not converge because $\mathbb{D}\mathbf{v}$, $\mathbb{D}\mathbf{v}^*$ and $\mathbb{D}\boldsymbol{\varphi}$ are in $L^p(\Omega)^{N\times N}$ and the integrands need not be integrable. It means that the Gâteaux derivative $D\mathscr{A}(\mathbf{v})\mathbf{v}^*$, as an element of $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$, need not exist in all directions $\mathbf{v}^* \in \mathbf{W}^{1,p}_{0,\sigma}(\Omega)$, and this holds true even if $\mathbf{v}$ is e.g. in $\mathbf{C}^\infty_{0,\sigma}(\Omega)$. Consequently, mapping $\mathscr{A}$ is not Gâteaux–differentiable, and therefore also not Fréchet–differentiable. This is the main reason why we assume from now on that $p \geq 2$.

In order to show that the operator $D\mathscr{A}(\mathbf{v})$, as an element of $\mathcal{L}\big(\mathbf{W}^{1,p}_{0,\sigma}(\Omega), \mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)\big)$, depends continuously on $\mathbf{v} \in \mathbf{W}^{1,p}_{0,\sigma}(\Omega)$, consider a sequence $\{\mathbf{v}_n\}$, such that $\mathbf{v}_n \to \mathbf{v}$ in $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ (for $n \to \infty$). Assume that $p > 2$. (The case $p = 2$ can be treated similarly.) Writing $\mathbb{D}(\mathbf{v}_n) \equiv (d_{ij}(\mathbf{v}_n))$, $\mathbb{D}\mathbf{v} \equiv (d_{ij}(\mathbf{v}))$ and $\mathbb{D}\boldsymbol{\varphi} \equiv (d_{ij}(\boldsymbol{\varphi}))$, we obtain

$$
\begin{aligned}
&\left\langle D\mathscr{A}(\mathbf{v}_n)\mathbf{v}^*, \boldsymbol{\varphi}\right\rangle_\sigma - \left\langle D\mathscr{A}(\mathbf{v})\mathbf{v}^*, \boldsymbol{\varphi}\right\rangle_\sigma = \int_\Omega \left[f\left(|\mathbb{D}\mathbf{v}_n|^2\right) - f\left(|\mathbb{D}\mathbf{v}|^2\right)\right] \mathbb{D}\mathbf{v}^* : \mathbb{D}\boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} \\
&\quad + \int_\Omega \left[2 f'\left(|\mathbb{D}\mathbf{v}_n|^2\right) (\mathbb{D}\mathbf{v}_n : \mathbb{D}\mathbf{v}^*)\, (\mathbb{D}\mathbf{v}_n : \mathbb{D}\boldsymbol{\varphi}) - 2 f'\left(|\mathbb{D}\mathbf{v}|^2\right) (\mathbb{D}\mathbf{v} : \mathbb{D}\mathbf{v}^*)\, (\mathbb{D}\mathbf{v} : \mathbb{D}\boldsymbol{\varphi})\right] \mathrm{d}\mathbf{x} \\
&= \int_\Omega \left[f\left(|\mathbb{D}\mathbf{v}_n|^2\right) - f\left(|\mathbb{D}\mathbf{v}|^2\right)\right] \mathbb{D}\mathbf{v}^* : \mathbb{D}\boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} \\
&\quad + \int_\Omega \left[2 f'\left(|\mathbb{D}\mathbf{v}_n|^2\right) d_{ij}(\mathbf{v}_n)\, d_{ij}(\mathbf{v}^*)\, d_{kl}(\mathbf{v}_n)\, d_{kl}(\boldsymbol{\varphi}) - f'\left(|\mathbb{D}\mathbf{v}|^2\right) d_{ij}(\mathbf{v})\, d_{ij}(\mathbf{v}^*)\, d_{kl}(\mathbf{v})\, d_{kl}(\boldsymbol{\varphi})\right] \mathrm{d}\mathbf{x} \\
&= \int_\Omega \left[f\left(|\mathbb{D}\mathbf{v}_n|^2\right) - f\left(|\mathbb{D}\mathbf{v}|^2\right)\right] \mathbb{D}\mathbf{v}^* : \mathbb{D}\boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} \\
&\quad + \int_\Omega \left[2 f'\left(|\mathbb{D}\mathbf{v}_n|^2\right) \mathbb{D}\mathbf{v}_n \otimes \mathbb{D}\mathbf{v}_n - 2 f'\left(|\mathbb{D}\mathbf{v}|^2\right) \mathbb{D}\mathbf{v} \otimes \mathbb{D}\mathbf{v}\right] : (\mathbb{D}\mathbf{v}^* \otimes \mathbb{D}\boldsymbol{\varphi}) \, \mathrm{d}\mathbf{x} \\
&\leq \left(\mathcal{I}_{1n}^{\frac{p-2}{p}} + \mathcal{I}_{2n}^{\frac{p-2}{p}}\right) \|\mathbb{D}\mathbf{v}^*\|_p \, \|\mathbb{D}\boldsymbol{\varphi}\|_p,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{I}_{1n} &= \int_\Omega \left|f\left(|\mathbb{D}\mathbf{v}_n|^2\right) - f\left(|\mathbb{D}\mathbf{v}|^2\right)\right|^{\frac{p}{p-2}} \mathrm{d}\mathbf{x}, \\
\mathcal{I}_{2n} &= \int_\Omega \left|2 f'\left(|\mathbb{D}\mathbf{v}_n|^2\right) \mathbb{D}\mathbf{v}_n \otimes \mathbb{D}\mathbf{v}_n - 2 f'\left(|\mathbb{D}\mathbf{v}|^2\right) \mathbb{D}\mathbf{v} \otimes \mathbb{D}\mathbf{v}\right|^{\frac{p}{p-2}} \mathrm{d}\mathbf{x}.
\end{aligned}
$$

(Note that “$\otimes$” denotes the outer tensorial product and $\mathbb{D}\mathbf{v}_k \otimes \mathbb{D}\mathbf{v}_k$ and $\mathbb{D}\mathbf{v} \otimes \mathbb{D}\mathbf{v}$ are therefore the 4th order tensors. On the other hand, “:” denotes the inner product of the 2nd order and 4th order tensors.) Let $k > 0$. Denote

$$
\Omega_{1n} := \{\mathbf{x} \in \Omega;\ |\mathbb{D}\mathbf{v}_n(\mathbf{x})|^p \leq |\mathbb{D}\mathbf{v}(\mathbf{x})|^p + k\}, \qquad \Omega_{2n} := \Omega \smallsetminus \Omega_{1n}.
$$

It follows from the inequalities

$$\int_\Omega |\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}|^p \, \mathrm{d}\mathbf{x} \;\geq\; \int_{\Omega_{2n}} |\mathbb{D}\mathbf{v}_n - \mathbb{D}\mathbf{v}|^p \, \mathrm{d}\mathbf{x} \;\geq\; C(p) \int_{\Omega_{2n}} \left(|\mathbb{D}\mathbf{v}_n|^p - |\mathbb{D}\mathbf{v}|^p\right) \, \mathrm{d}\mathbf{x}$$
$$\geq\; C(p) \int_{\Omega_{2n}} k \; \mathrm{d}\mathbf{x} \;\geq\; C(p)\, k \operatorname{meas}(\Omega_{2n})$$

that $\operatorname{meas}(\Omega_{2n}) \to 0$ for $n \to \infty$. Hence $\int_{\Omega_{2n}} |\mathbb{D}\mathbf{v}|^p \, \mathrm{d}\mathbf{x} \to 0$ for $n \to \infty$. Consequently, $\int_{\Omega_{2n}} |\mathbb{D}\mathbf{v}_n|^p \, \mathrm{d}\mathbf{x} \to 0$ for $n \to \infty$, too. The term $\mathcal{I}_{1n}$ can be split to the sum of the integral on Ω_{1n} and the integral on Ω_{2n}. The latter can be estimated by means of (1.3):

$$\int_{\Omega_{2n}} \left| f\left(|\mathbb{D}\mathbf{v}_n|^2\right) - f\left(|\mathbb{D}\mathbf{v}|^2\right) \right|^{\frac{p}{p-2}} \mathrm{d}\mathbf{x} \;\leq\; C(p) \int_{\Omega_{2n}} \left(|f(|\mathbb{D}\mathbf{v}_n|^2)|^{\frac{p}{p-2}} + |f(|\mathbb{D}\mathbf{v}|^2)|^{\frac{p}{p-2}} \right) \, \mathrm{d}\mathbf{x}$$
$$\leq\; C(p) \int_{\Omega_{2n}} \left[c_4^{\frac{p}{p-2}} \; (\delta + |\mathbb{D}\mathbf{v}_n|)^p + c_4^{\frac{p}{p-2}} \; (\delta + |\mathbb{D}\mathbf{v}|)^p \right] \mathrm{d}\mathbf{x}$$
$$\leq\; C(p) \int_{\Omega_{2n}} \left[\delta^p + |\mathbb{D}\mathbf{v}_n|^p + |\mathbb{D}\mathbf{v}|^p \right] \mathrm{d}\mathbf{x} \;\longrightarrow\; 0 \quad \text{for } n \to \infty.$$

If we denote by χ_{1n} the characteristic function of set Ω_{1n} then

$$\int_{\Omega_{1n}} \left| f\left(|\mathbb{D}\mathbf{v}_n|^2\right) - f\left(|\mathbb{D}\mathbf{v}|^2\right) \right|^{\frac{p}{p-2}} \, \mathrm{d}\mathbf{x} \;=\; \int_\Omega \left| f\left(|\mathbb{D}\mathbf{v}_n|^2\right) - f\left(|\mathbb{D}\mathbf{v}|^2\right) \right|^{\frac{p}{p-2}} \chi_{1n} \, \mathrm{d}\mathbf{x}.$$

The integrand satisfies the estimates

$$\left| f(|\mathbb{D}\mathbf{v}_n|^2) - f(|\mathbb{D}\mathbf{v}|^2) \right|^{\frac{p}{p-2}} \chi_{1n} \;\leq\; C(p) \left(|f(|\mathbb{D}\mathbf{v}_n|^2)|^{\frac{p}{p-2}} + |f(|\mathbb{D}\mathbf{v}|^2)|^{\frac{p}{p-2}} \right) \chi_{1n}$$
$$\leq\; C(p) \left[c_4 \; (\delta + |\mathbb{D}\mathbf{v}_n|)^p + c_4 \; (\delta + |\mathbb{D}\mathbf{v}|)^p \right] \chi_{1n}$$
$$\leq\; C(p) \left[\delta^p + |\mathbb{D}\mathbf{v}_n|^p + |\mathbb{D}\mathbf{v}|^p \right] \chi_{1n} \;\; \leq\; C(p) \left[\delta^p + 2|\mathbb{D}\mathbf{v}|^p + k \right]$$

a.e. in Ω. The function on the right hand side is integrable in Ω. Moreover, as $\mathbf{v}_n \to \mathbf{v}$ point-wise a.e. in Ω and function f is continuous on $[0,\infty)$, $f(|\mathbb{D}\mathbf{v}_n|) - f(|\mathbb{D}\mathbf{v}|) \to 0$ (for $n \to \infty$) point-wise a.e. in Ω. Hence the integral of $\left| f\left(|\mathbb{D}\mathbf{v}_n|^2\right) - f\left(|\mathbb{D}\mathbf{v}|^2\right) \right|^{\frac{p}{p-2}} \chi_{1n}$ tends to zero as $n \to \infty$. Consequently, $\mathcal{I}_{1n} \to 0$ for $n \to \infty$ due to the Lebesgue dominated theorem. (Note that this is the point where we use condition (iii), particularly the continuity of f on $[0,\infty)$.) We can also prove, by analogy, that $\mathcal{I}_{2n} \to 0$ for $n \to \infty$. (Here, we only apply (1.4) instead of (1.3) and use the continuity of $t^2 \, f'(t^2)$ up to the point $t = 0$.) We have proven the lemma:

Lemma 4. *Let $p \geq 2$ and let function f satisfy conditions (ii) and (iii). Then operator $\mathscr{A}$ is of the class C^1 and $D\mathscr{A}(\mathbf{v})$ can be therefore identified with the Fréchet differential of $\mathscr{A}$ at the point $\mathbf{v} \in \mathbf{W}^{1,p}_{0,\sigma}(\Omega)$.*

The Gâteaux derivative of operator $\mathscr{B}$ at point $\mathbf{v} \in \mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ in the direction $\mathbf{v}^*$ is an element of $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$, satisfying

$$\langle D\mathscr{B}(\mathbf{v})\mathbf{v}^*, \boldsymbol{\varphi} \rangle_\sigma \;=\; \int_\Omega \left[\mathbf{v} \cdot \nabla \mathbf{v}^* \cdot \boldsymbol{\varphi} + \mathbf{v}^* \cdot \nabla \mathbf{v} \cdot \boldsymbol{\varphi} \right] \, \mathrm{d}\mathbf{x} \tag{2.6}$$

for all $\boldsymbol{\varphi} \in \mathbf{W}^{1,p}_{0,\sigma}(\Omega)$. By analogy with $\mathscr{A}$, we can prove that $D\mathscr{B}(\mathbf{v})$ is a bounded linear operator from $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$, and it can be therefore identified with the Gâteaux differential of $\mathscr{B}$ at the point $\mathbf{v}$. To prove the continuous dependence of $D\mathscr{B}$ on $\mathbf{v}$ in the topology of the space $\mathcal{L}\big(\mathbf{W}^{1,p}_{0,\sigma}(\Omega), \mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)\big)$, consider a sequence $\{\mathbf{v}_n\}$ such that $\mathbf{v}_n \to \mathbf{v}$ in $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ (for $n \to \infty$). Applying the continuous imbedding $\mathbf{W}^{1,p}_{0,\sigma}(\Omega) \hookrightarrow \mathbf{L}^{2p'}(\Omega)$, which follows from (1.20), we get

$$\left\langle D\mathscr{B}(\mathbf{v}_n)\mathbf{v}^*, \boldsymbol{\varphi} \right\rangle_\sigma - \left\langle D\mathscr{B}(\mathbf{v})\mathbf{v}^*, \boldsymbol{\varphi} \right\rangle_\sigma \;=\; \int_\Omega \left[(\mathbf{v}_n - \mathbf{v}) \cdot \nabla \mathbf{v}^* \cdot \boldsymbol{\varphi} + \mathbf{v}^* \cdot \nabla(\mathbf{v}_n - \mathbf{v}) \cdot \nabla\boldsymbol{\varphi} \right] \, \mathrm{d}\mathbf{x}$$

$$\leq \|\mathbf{v}_n - \mathbf{v}\|_{2p'} \|\nabla \mathbf{v}^*\|_p \|\boldsymbol{\varphi}\|_{2p'} + \|\mathbf{v}^*\|_{2p'} \|\nabla(\mathbf{v}_n - \mathbf{v})\|_p \|\boldsymbol{\varphi}\|_{2p'} \leq C \|\nabla(\mathbf{v}_n - \mathbf{v})\|_p \|\nabla \mathbf{v}^*\|_p \|\nabla \boldsymbol{\varphi}\|_p.$$

The continuity of $D\mathscr{B}$ is proven. This, together with Lemma 4, yields:

Lemma 5. *Let $p \geq 2$ and let function f satisfy conditions (ii) and (iii). Then the operator $\mathscr{A} + \mathscr{B}$ is of the class C^1 and its Gâteaux differential $D[\mathscr{A} + \mathscr{B}](\mathbf{v}) \equiv D\mathscr{A}(\mathbf{v}) + D\mathscr{B}(\mathbf{v})$ can be identified with the Fréchet differential at each point $\mathbf{v} \in \mathbf{W}^{1,p}_{0,\sigma}(\Omega)$.*

The next lemma provides more information on the Fréchet differential of $\mathscr{A}$ in the cases when $\{p > 2$ and $\delta + |\mathbb{D}\mathbf{v}| > 0$ a.e. in $\Omega\}$ or $p = 2$.

Lemma 6. *Let $\{p > 2$ and $\delta + |\mathbb{D}\mathbf{v}| > 0$ a.e. in $\Omega\}$ or $p = 2$. Let function f satisfy conditions (ii) and (iii). Then $\operatorname{nul} D\mathscr{A}(\mathbf{v}) = 0$, which means that the operator $D\mathscr{A}(\mathbf{v})$ is one–to–one.*

Proof. Denote $\Omega_+ := \{\mathbf{x} \in \Omega;\ f'(|\mathbb{D}\mathbf{v}(\mathbf{x})|^2) \geq 0$ and $\Omega_- := \Omega \smallsetminus \Omega_+$. Then, choosing $\boldsymbol{\varphi} = \mathbf{v}^*$ in (2.5) and applying (1.2), we obtain

$$\begin{aligned} \big\langle D\mathscr{A}(\mathbf{v})\mathbf{v}^*, \mathbf{v}^* \big\rangle_\sigma &= \left(\int_{\Omega_+} + \int_{\Omega_-} \right) \left[f\left(|\mathbb{D}\mathbf{v}|^2\right) |\mathbb{D}\mathbf{v}^*|^2 + 2\, f'\left(|\mathbb{D}\mathbf{v}|^2\right) (\mathbb{D}\mathbf{v} : \mathbb{D}\mathbf{v}^*)^2 \right] \mathrm{d}\mathbf{x} \\ &\geq \int_{\Omega_+} f\left(|\mathbb{D}\mathbf{v}|^2\right) |\mathbb{D}\mathbf{v}^*|^2 \,\mathrm{d}\mathbf{x} + \int_{\Omega_-} \left[f\left(|\mathbb{D}\mathbf{v}|^2\right) + 2\, f'\left(|\mathbb{D}\mathbf{v}|^2\right) |\mathbb{D}\mathbf{v}|^2 \right] |\mathbb{D}\mathbf{v}^*|^2 \,\mathrm{d}\mathbf{x} \\ &\geq \int_{\Omega_+} c_3\, (\delta + |\mathbb{D}\mathbf{v}|)^{p-2} |\mathbb{D}\mathbf{v}^*|^2 \,\mathrm{d}\mathbf{x} + \int_{\Omega_-} c_1\, (\delta + |\mathbb{D}\mathbf{v}|)^{p-2} |\mathbb{D}\mathbf{v}^*|^2 \,\mathrm{d}\mathbf{x}. \end{aligned} \tag{2.7}$$

If $p > 2$ and $\delta + |\mathbb{D}\mathbf{v}| > 0$ a.e. in Ω then the right hand side is equal to zero only if $|\mathbb{D}\mathbf{v}^*| = 0$ a.e. in Ω, which means that $\mathbf{v}^* = \mathbf{0}$. If $p = 2$ then the right hand of (2.7) reduces to $c_3 \int_{\Omega_+} |\mathbb{D}\mathbf{v}^*|^2 \,\mathrm{d}\mathbf{x} + c_1 \int_{\Omega_-} |\mathbb{D}\mathbf{v}^*|^2 \,\mathrm{d}\mathbf{x}$ which is also equal to zero only if $\mathbf{v}^* = \mathbf{0}$, independently of δ. Hence $\operatorname{nul} D\mathscr{A}(\mathbf{v}) = 0$ and the operator $D\mathscr{A}(\mathbf{v})$ is one–to–one. $\square$

As the case $p = 2$ will be studied separately in Section 3, we assume that $p > 2$ from now on till the end of Section 2.

Lemma 7. *Assume that $p > 2$ and function f satisfies conditions (ii) and (iii). Then the operator $D\mathscr{A}(\mathbf{v})$ is not semi–Fredholm from $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$ for any $\mathbf{v} \in \mathbf{W}^{1,p}_{0,\sigma}(\Omega)$.*

Proof. 1) Let us at first consider the case $\delta > 0$. Assume for a while that $\mathbf{v} \in \mathbf{C}^\infty_{0,\sigma}(\Omega)$. Then $D\mathscr{A}(\mathbf{v})$, which is a bounded linear operator from $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$, can also be considered to be a bounded linear operator from $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. Inequality (2.7) and Lax–Milgram's theorem imply that $D\mathscr{A}(\mathbf{v})$ maps $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ onto the whole space $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. Since $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ is dense in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$, the range of $D\mathscr{A}(\mathbf{v})$ (as an operator defined in $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ – we denote the range by $R(D\mathscr{A}(\mathbf{v}))$) is dense in $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. As $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ is dense in $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$, $R(D\mathscr{A}(\mathbf{v}))$ is also dense in $\mathbf{W}^{-1,p'}_{0,\sigma}$. Moreover, since the range of $D\mathscr{A}(\mathbf{v}))$ differs from $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$, it is not closed in $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$. Consequently, the operator $D\mathscr{A}(\mathbf{v})$ is not semi–Fredholm and $\operatorname{nul}' D\mathscr{A}(\mathbf{v}) = \operatorname{def}' D\mathscr{A}(\mathbf{v}) = \infty$.

Let us now show that the operator $D\mathscr{A}(\mathbf{v})$ is not semi–Fredholm at all points $\mathbf{v} \in \mathbf{W}^{1,p}_{0,\sigma}(\Omega)$. By contradiction: assume that $D\mathscr{A}(\mathbf{v})$ is semi–Fredholm at some point $\mathbf{v}_0 \in \mathbf{W}^{1,p}_{0,\sigma}(\Omega)$. Since $\mathscr{A}$ is of the class C^1, we can apply Theorem IV.5.22 in [5] and deduce that $D\mathscr{A}(\mathbf{v})$ is semi–Fredholm for all $\mathbf{v}$ in a sufficiently small neighborhood of $\mathbf{v}_0$. This is, however, impossible, because any neighborhood of $\mathbf{v}_0$ contains functions $\mathbf{v}$ from $\mathbf{C}^\infty_{0,\sigma}(\Omega)$, and we already know that $D\mathscr{A}(\mathbf{v})$ is not semi–Fredholm for these $\mathbf{v}$.

2) Now, assume that $\delta = 0$. Let $\xi > 0$. Let us write, for a while, f_0 instead of f, and put $f_\xi(t^2) := \xi + f_0(t^2)$. Obviously, $f_\xi(t^2)$ satisfies condition (iii). Since f_0 satisfies (1.2) with $\delta = 0$, function f_ξ satisfies (1.2) with $\delta = \xi^{1/(p-2)}$ and modified constants c_1 and c_2. Denote by $\mathscr{A}_0$ the operator $\mathscr{A}$ defined by (1.23) with $\mathbb{S}$ given

by (1.1), where $f \equiv f_0$. Let $\mathscr{A}_\xi$ be the operator defined by (1.23) with $\mathbb{S}$ corresponding to function f_ξ. The difference $D\mathscr{A}_\xi(\mathbf{v}) - D\mathscr{A}_0(\mathbf{v})$ satisfies

$$\big\langle [D\mathscr{A}_\xi(\mathbf{v}) - D\mathscr{A}_0(\mathbf{v})]\mathbf{v}^*, \boldsymbol{\varphi} \big\rangle_\sigma \;=\; \int_\Omega \big[f_\xi\left(|\mathbb{D}\mathbf{v}|^2\right) - f_0\left(|\mathbb{D}\mathbf{v}|^2\right) \big]\, \mathbb{D}\mathbf{v}^* : \mathbb{D}\boldsymbol{\varphi}\; \mathrm{d}\mathbf{x} \;=\; \xi \int_\Omega \mathbb{D}\mathbf{v}^* : \mathbb{D}\boldsymbol{\varphi}\; \mathrm{d}\mathbf{x}.$$

From this, we observe that

$$\big\| [D\mathscr{A}_\xi(\mathbf{v}) - D\mathscr{A}_0(\mathbf{v})]\mathbf{v}^* \big\|_{-1,p'} \;\leq\; \xi\, \|\nabla\mathbf{v}^*\|_p.$$

This inequality shows that if the operator $D\mathscr{A}_0(\mathbf{v})$ was semi–Fredholm, then $D\mathscr{A}_\xi(\mathbf{v})$ would also be semi–Fredholm for $\xi > 0$ sufficiently small. (See [5, Theorem IV.5.22].) This is, however, not true due to part 1) of this proof. $\square$

Provided that $p > 2$ and f satisfies conditions (ii), (iii), we have proven that the range of $D\mathscr{A}(\mathbf{v})$ is not closed in $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$ if $\delta > 0$. In fact, one can also show that $R(D\mathscr{A}(\mathbf{v}))$ is not closed in $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$ if $\delta = 0$ and $\mathbf{v} \not\equiv \mathbf{0}$. The proof is, however, subtler and more technical.

The next lemma unveils information analogous to Lemma 7, but it concerns the differential of $\mathscr{A} + \mathscr{B}$.

Lemma 8. *Let $p > 2$ and function f satisfy conditions (ii) and (iii). Then $D[\mathscr{A} + \mathscr{B}](\mathbf{v})$ is not a semi–Fredholm operator from $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$ for any $\mathbf{v} \in \mathbf{W}^{1,p}_{0,\sigma}(\Omega)$.*

Proof. We claim that $D\mathscr{B}(\mathbf{v})$ is a compact operator from $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$.

If $N = 2$ or $\{N = 3$ and $p > 3\}$ and $\mathbf{v} \in \mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ then $D\mathscr{B}(\mathbf{v})$ is a continuous operator from $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ to $\mathbf{L}^p_\sigma(\Omega)$ because

$$\begin{aligned} \big|\langle D\mathscr{B}(\mathbf{v})\mathbf{v}^*, \boldsymbol{\varphi}\rangle_\sigma\big| \;&=\; \left| \int_\Omega [\mathbf{v}\cdot\nabla\mathbf{v}^* + \mathbf{v}^*\cdot\nabla\mathbf{v}] \cdot \boldsymbol{\varphi}\; \mathrm{d}\mathbf{x} \right| \\ &\leq\; \big(\|\mathbf{v}\|_\infty\, \|\nabla\mathbf{v}^*\|_p + \|\mathbf{v}^*\|_\infty\, \|\nabla\mathbf{v}\|_p \big)\, \|\boldsymbol{\varphi}\|_{p'} \;\leq\; C\, \|\nabla\mathbf{v}\|_p\, \|\nabla\mathbf{v}^*\|_p\, \|\boldsymbol{\varphi}\|_{p'} \end{aligned}$$

holds for all $\mathbf{v}^*$, $\boldsymbol{\varphi} \in \mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ and $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ is dense in $\mathbf{L}^{p'}_\sigma(\Omega)$. (We use the continuous imbedding $\mathbf{W}^{1,p}_{0,\sigma} \hookrightarrow \mathbf{L}^\infty(\Omega)$, which follows from the inequality $p > N$.) The space $\mathbf{L}^p_\sigma(\Omega)$ is compactly imbedded to $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$, hence $D\mathscr{B}(\mathbf{v})$ is a compact mapping from $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$.

If $N = 3$ and $2 < p < 3$ then we use the estimates

$$\big|\langle D\mathscr{B}(\mathbf{v})\mathbf{v}^*, \boldsymbol{\varphi}\rangle_\sigma\big| \;\leq\; \big(\|\mathbf{v}\|_{\frac{3p}{3-p}}\, \|\nabla\mathbf{v}^*\|_p + \|\mathbf{v}^*\|_{\frac{3p}{3-p}}\, \|\nabla\mathbf{v}\|_p \big)\, \|\boldsymbol{\varphi}\|_s \;\leq\; C\, \|\nabla\mathbf{v}\|_p\, \|\nabla\mathbf{v}^*\|_p\, \|\boldsymbol{\varphi}\|_s,$$

where $p^{-1} + (3-p)/3p + s^{-1} = 1$, which yields $s^{-1} = \frac{4}{3} - 2p^{-1}$. (We apply the continuous imbedding $\mathbf{W}^{1,p}_{0,\sigma}(\Omega) \hookrightarrow \mathbf{L}^{3p/(3-p)}_\sigma(\Omega)$.) Hence the conjugate exponent s' satisfies $(s')^{-1} = 2p^{-1} - \frac{1}{3}$. The estimates of $\langle D\mathscr{B}(\mathbf{v})\mathbf{v}^*, \boldsymbol{\varphi}\rangle_\sigma$ show that $D\mathscr{B}(\mathbf{v})$ is a bounded linear operator from $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ to $\mathbf{L}^{s'}_\sigma(\Omega)$. Exponent s is less than $3p/(3-p)$ (which is equivalent to $s^{-1} \equiv \frac{4}{3} - 2p^{-1} > p^{-1} - \frac{1}{3}$), hence $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ is compactly imbedded to $\mathbf{L}^s_\sigma(\Omega)$. Consequently, $\mathbf{L}^{s'}_\sigma(\Omega)$ is compactly imbedded to $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$ and $D\mathscr{B}(\mathbf{v})$ is a compact operator from $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$.

The case $N = p = 3$ can be treated similarly.

As $D\mathscr{B}(\mathbf{v})$ is compact from $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$, it is also relatively compact with respect to $D[\mathscr{A} + \mathscr{B}](\mathbf{v})$. Thus, if $D[\mathscr{A} + \mathscr{B}](\mathbf{v})$ was semi–Fredholm from $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$ then $D[\mathscr{A} + \mathscr{B}](\mathbf{v}) - D\mathscr{B}(\mathbf{v}) = D\mathscr{A}(\mathbf{v})$ would also be semi–Fredholm, which is not true due to Lemma 7. $\square$

We observe from Lemma 8 that if $p > 2$ then the Smale–Sard theorem (see Theorem 1) or other related tools, based on the theory of Fredholm operators, cannot be applied to equation (1.25) in order to characterize the set of right hand sides $\mathbf{f}$ that are regular values of $\mathscr{A}+\mathscr{B}$. Moreover, as the differential $D[\mathscr{A}+\mathscr{B}]$ is not surjective from $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$, we also cannot apply the implicit function theorem or other related tools in order to derive an information on the structure of the solution set of equation (1.25) and the behavior of solutions if the right hand side $\mathbf{f}$ varies. These are the reasons why we focus just on the case $p=2$ in the next section.

3 The structure of the set of solutions to the boundary value problem (1.14)–(1.16) in the case $p=2$.

In this section, we assume that $p=2$, which means that the stress tensor $\mathbb{S}$, defined by (1.1), has the 2–structure. We suppose that function f satisfies conditions (ii) and (iii). Recall that as $p=2$, inequalities (1.2), (1.3) and (1.4) reduce to (1.9) and (1.10). We already know from Lemma 4 and Lemma 5 that $\mathscr{A}$ and $\mathscr{A}+\mathscr{B}$ are C^1–mappings from $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$.

Lemma 9. *The range of the Fréchet differential $D\mathscr{A}(\mathbf{v})$ (at each point $\mathbf{v}\in\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$) is the whole space $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. Thus, $D\mathscr{A}(\mathbf{v})$ is a Fredholm operator of index 0 from $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. Consequently, $\mathscr{A}$ is a Fredholm operator of index 0 from $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$, too.*

Proof. It follows from (2.7) that the bilinear form $\langle D\mathscr{A}(\mathbf{v})\mathbf{v}^*,\mathbf{v}^*\rangle_\sigma$ is elliptic in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. Thus, applying Lax–Milgram's theorem, we deduce that $R(D\mathscr{A}(\mathbf{v}))=\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ and $\operatorname{def}D\mathscr{A}(\mathbf{v})$ is therefore equal to zero. Since $\operatorname{nul}D\mathscr{A}(\mathbf{v})=0$ as well (see Lemma 6), we have $\operatorname{ind}D\mathscr{A}(\mathbf{v})=0$. This implies that $D\mathscr{A}(\mathbf{v})$ is a Fredholm operator of index 0. Hence $\mathscr{A}$ is a Fredholm operator of index 0 as well. $\square$

Lemma 10. *$\mathscr{A}+\mathscr{B}$ is a proper Fredholm operator from $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ of index 0.*

Proof. Let us at first prove that $\mathscr{A}+\mathscr{B}$ is proper, i.e. that the pre–image $[\mathscr{A}+\mathscr{B}]^{-1}(K)$ of any compact set K in $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ is compact in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. Thus, let $\{\mathbf{v}_n\}$ be a sequence in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ such that $\mathbf{f}_n:=[\mathscr{A}+\mathscr{B}](\mathbf{v}_n)$ are in a compact set K in $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. There exists a subsequence of $\{\mathbf{f}_n\}$ (which we denote again $\{\mathbf{f}_n\}$), that converges to some function $\mathbf{f}$ in $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. Due to the continuity of $\mathscr{A}^{-1}$, the sequence $\{\mathscr{A}^{-1}\mathbf{f}_n\}$ converges in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. If follows from estimate (1.21) that the sequence $\mathbf{v}_n$ is bounded in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. Hence there exists a subsequence (which we denote again by $\{\mathbf{v}_n\}$), weakly convergent to some $\mathbf{v}$ in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. Since the operator $\mathscr{A}^{-1}\circ\mathscr{B}$ is compact (see Lemma 3), $\mathscr{A}^{-1}\mathscr{B}\mathbf{v}_n\to\mathscr{A}^{-1}\mathscr{B}\mathbf{v}$ strongly in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. As $\mathbf{v}_n=-\mathscr{A}^{-1}\mathscr{B}\mathbf{v}_n+\mathscr{A}^{-1}\mathbf{f}_n$, we obtain that $\mathbf{v}_n\to\mathbf{v}$ strongly in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. This shows that $[\mathscr{A}+\mathscr{B}]^{-1}(K)$ is compact in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$.

In order to prove that the operator $\mathscr{A}+\mathscr{B}$ is Fredholm of index zero, we need to show that $D\mathscr{A}(\mathbf{v})+D\mathscr{B}(\mathbf{v})$ is a linear Fredholm operator of index zero at some point $\mathbf{v}\in\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. We have shown in the proof of Lemma 8 that if $p>2$ then $D\mathscr{B}(\mathbf{v})$ is a compact operator from $\mathbf{W}^{1,p}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,p'}_{0,\sigma}(\Omega)$. The proof can be modified so that it also works in the case $p=2$: let $\{\mathbf{v}_n\}$ be a bounded sequence in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. We want to show that there exists a subsequence such that $\mathscr{B}\mathbf{v}_n$ is convergent in $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. Using formula (2.6), one can simply verify that $D\mathscr{B}(\mathbf{v})$ is a bounded operator from $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ to $\mathbf{L}^2_\sigma(\Omega)$ (if $N=2$) or to $\mathbf{L}^{3/2}_\sigma(\Omega)$ (if $N=3$). Since the imbedding of $\mathbf{L}^2_\sigma(\Omega)$ to $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ (if $N=2$), respectively of $\mathbf{L}^{3/2}_\sigma(\Omega)$ to $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ (if $N=3$), is compact, $D\mathscr{B}(\mathbf{v})$ is a compact operator from $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$. Consequently, the operator $D\mathscr{B}(\mathbf{v})$ is also relatively compact with respect to $D\mathscr{A}(\mathbf{v})$. Hence, since $D\mathscr{A}(\mathbf{v})$ is a Fredholm operator of index 0, $D\mathscr{A}(\mathbf{v})+D\mathscr{B}(\mathbf{v})$ is a semi–Fredholm operator of the same index 0. (See [5, p. 238, Theorem 5.26].) In order to verify that $D\mathscr{A}(\mathbf{v})+D\mathscr{B}(\mathbf{v})$ is not only semi–Fredholm, but it is a Fredholm operator, it is sufficient to show that $\operatorname{nul}\,[D\mathscr{A}(\mathbf{v})+D\mathscr{B}(\mathbf{v})]<\infty$. (Then $\operatorname{def}\,[D\mathscr{A}(\mathbf{v})+D\mathscr{B}(\mathbf{v})]$ is also finite because the index is zero.) By contradiction: assume that $\operatorname{nul}\,[D\mathscr{A}(\mathbf{v})+D\mathscr{B}(\mathbf{v})]=\infty$. Then there exists a sequence

$\{\mathbf{v}_n\}$ on the unit ball in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$, such that its closure is not compact (i.e. the sequence is not pre-compact) and $[D\mathscr{A}(\mathbf{v}) + D\mathscr{B}(\mathbf{v})]\mathbf{v}_n = \mathbf{0}$. Hence $\mathbf{v}_n + [D\mathscr{A}(\mathbf{v})]^{-1} D\mathscr{B}(\mathbf{v})\mathbf{v}_n = \mathbf{0}$. Since the operator $D\mathscr{B}(\mathbf{v})$ is compact from $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ and $[D\mathscr{A}(\mathbf{v})]^{-1}$ is bounded from $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ to $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$, there exists a subsequence (which we also denote by $\{\mathbf{v}_n\}$) such that $\{[D\mathscr{A}(\mathbf{v})]^{-1} D\mathscr{B}(\mathbf{v})\mathbf{v}_n\}$ is a convergent sequence in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. Consequently, $\{\mathbf{v}_n\}$ is also convergent in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$. This is a contradiction with the assumption that $\{\mathbf{v}_n\}$ was not pre-compact. $\square$

Now, we are in a position to apply Theorem 1 (with $k = 1$) to equation (1.25). It provides the existence of a dense open set $\mathcal{O}$ in $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ such that all $\mathbf{f}$ in $\mathcal{O}$ are regular values of $\mathscr{A} + \mathscr{B}$. Denote by $\mathcal{S}(\mathbf{f})$ the set of all solutions of equation (1.25), i.e. $[\mathscr{A} + \mathscr{B}]\mathbf{v} = \mathbf{f}$. Since $\operatorname{ind}[\mathscr{A} + \mathscr{B}] = 0$, the set $\mathcal{S}(\mathbf{f})$ is discrete for each $\mathbf{f} \in \mathcal{O}$. (This follows from [10, Corollary 1.5] or [3, Theorem A].) Since it is also compact (see Theorem 2), it must be finite.

Assume that $\mathbf{f}_0$ and $\mathbf{f}_1$ are in the same component $\mathcal{O}'$ of $\mathcal{O}$. Then $\mathbf{f}_0$ and $\mathbf{f}_1$ can be connected by a continuous curve $t \in [0,1] \mapsto \mathbf{f}_t \in \mathcal{O}'$. Let $\mathbf{v}^0_1, \dots, \mathbf{v}^0_k$ be the elements of $\mathcal{S}(\mathbf{f}_0)$. Due to the implicit function theorem (see e.g. [12, Theorem 4.B]), there exist k continuous curves $t \in [0,1] \mapsto \mathbf{v}^i_t \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ $(i = 1, \dots, k)$ such that $\mathbf{v}^i_t \in \mathcal{S}(\mathbf{f}_t)$ for all $t \in [0,1]$. Any two different curves cannot intersect at any point $\mathbf{v}$, otherwise it would lead to a contradiction with the implicit function theorem around $\mathbf{v}$. Thus, the set $\mathcal{S}(\mathbf{f}_1)$ contains at least k different points $\mathbf{v}^1_1, \dots, \mathbf{v}^1_k$. Due to the symmetry, the number of elements in $\mathcal{S}(\mathbf{f}_1)$ is the same as the number of elements in $\mathcal{S}(\mathbf{f}_0)$. This shows that the number of elements of $\mathcal{S}(\mathbf{f})$ is constant for all $\mathbf{f}$ in the same component of $\mathcal{O}$. The fact that each element of $\mathcal{S}(\mathbf{f})$ is a C^1–function of $\mathbf{f}$ follows from the implicit function theorem or the so called pre-image theorem, see [12, Theorem 4.J]. We have proven the theorem:

Theorem 3. *There exists an open dense subset $\mathcal{O}$ of $\mathbf{W}^{-1,2}_{0,\sigma}(\Omega)$ such that*

1) *for every $\mathbf{f} \in \mathcal{O}$, the set $\mathcal{S}(\mathbf{f})$ is constituted by a finite number of solutions of equation (1.25),*
2) *the number of elements of $\mathcal{S}(\mathbf{f})$, for $\mathbf{f}$ in every connected component of $\mathcal{O}$, is constant,*
3) *each element of $\mathcal{S}(\mathbf{f})$ is a C^1–function of $\mathbf{f}$ for $\mathbf{f}$ in every connected component of $\mathcal{O}$.*

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