Lower bounds on eigenvalues of linear elliptic operators

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Supported by Neuron Fund for Support of Science, project no. 24/2016

SNA 2017, Ostrava, January 30–February 3, 2017

Outline

Model problem

$$\begin{aligned} -\Delta u_i &= \lambda_i u_i && \text{in } \Omega \\ u_i &= 0 && \text{on } \partial\Omega \end{aligned}$$

Lower bounds on eigenvalues:

$$? \leq \lambda_i \leq \Lambda_{h,i}$$

- Classical Weinstein's and Kato's lower bounds
- Weinstein's and Kato's bounds in weak setting
- Numerical examples

Lower bounds on eigenvalues

Old problem:
Temple 1928, Weinstein 1937, Kato 1949, Lehmann 1949, 1950, ...

Many results: M.G. Armentano, G. Barrenechea, H. Behnke, C. Carstensen, R.G. Duran, D. Galistl, J. Gedicke, F. Goerisch, L. Grubišić, Jun Hu, J.R. Kuttler, Y.A. Kuznetsov, Fubiao Lin, Qun Lin, Xuefeng Liu, M. Plum, S.I. Repin, V.G. Sigillito, Hehu Xie, Yidu Yang, Zhimin Zhang, ... *many others*

Weinstein's bounds

Eigenvalue problem: Find $u_i \in D(A) \setminus \{0\}$ and $\lambda_i \in \mathbb{R}$:

$$Au_i = \lambda_i u_i$$

Setting:

- V ... Hilbert space
- $A : D(A) \to V$ linear, symmetric operator
- $\{u_i\}$ form ON basis in V
- $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$

Weinstein's bounds

Eigenvalue problem: Find $u_i \in D(A) \setminus \{0\}$ and $\lambda_i \in \mathbb{R}$:

$$Au_i = \lambda_i u_i$$

Theorem 1 (Weinstein 1937):

- Let $u_* \in D(A) \setminus \{0\}$ and $\lambda_* \in \mathbb{R}$ be arbitrary.
- Let $\varepsilon = \|Au_* - \lambda_* u_*\| / \|u_*\|$.
- Let $\dfrac{\lambda_{n-1} + \lambda_n}{2} \leq \lambda_* \leq \dfrac{\lambda_n + \lambda_{n+1}}{2}$ for some n.

Then $\lambda_* - \varepsilon \leq \lambda_n$.

Kato's bound

Theorem 2 (Kato 1949):

- Let $1 \leq n \leq s$.
- Let $u_{*,i} \in D(A)$ and $\lambda_{*,i} \in \mathbb{R}$, $i = n, \dots, s$, satisfy

  $$\langle Au_{*,i}, v_* \rangle = \lambda_{*,i} \langle u_{*,i}, v_* \rangle \quad \forall v_* \in V_*, \quad \|u_{*,i}\| = 1,$$

  where $V_* = \operatorname{span}\{u_{*,i}, \; i = n, \dots, s\}$.
- Let $\varepsilon_i = \|Au_{*,i} - \lambda_{*,i} u_{*,i}\|$.
- Let $\lambda_{s-1} \leq \lambda_{*,s} < \nu \leq \lambda_{s+1}$.

Then $\displaystyle \lambda_{*,n} - \sum_{i=n}^{s} \frac{\varepsilon_i^2}{\nu - \lambda_{*,i}} \leq \lambda_n$.

Weak form

Eigenvalue problem: Find $u_i \in V \setminus \{0\}$ and $\lambda_i \in \mathbb{R}$:

$$a(u_i, v) = \lambda_i b(u_i, v) \quad \forall v \in V.$$

Setting:

- V is a Hilbert space
- $a(\cdot,\cdot)$ is a symmetric, continuous, V-elliptic bilinear form
- $b(\cdot,\cdot)$ is a symmetric, continuous, positive semidefinite bilinear form
- $\{u_i\}$ form ON basis in V, i.e. $b(u_i, u_j) = \delta_{ij}$
- $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$

Example:

- $a(u, v) = (\nabla u, \nabla v)$
- $b(u, v) = (u, v)$

Weinstein's bound in the weak form

Theorem 3:

- Let $u_* \in V \setminus \{0\}$ and $\lambda_* \in \mathbb{R}$ be arbitrary.
- Let $w \in V$ be given by

$$a(w, v) = a(u_*, v) - \lambda_* b(u_*, v) \quad \forall v \in V.$$

- Let $\|w\|_a \leq \eta$.
- Let $\sqrt{\lambda_{n-1}\lambda_n} \leq \lambda_* \leq \sqrt{\lambda_n \lambda_{n+1}}$

Then

$$\ell_n \leq \lambda_n, \quad \text{where } \ell_n = \frac{1}{4|u_*|_b^2} \left(-\eta + \sqrt{\eta^2 + 4\lambda_* |u_*|_b^2} \right)^2.$$

Kato's bound in the weak form

Theorem 4:

- Let $0 < n \leq s$.
- Let $u_{*,i} \in V$ and $\lambda_{*,i} \in \mathbb{R}$, $i = n, \dots, s$, satisfy

$$a(u_{*,i}, v_*) = \lambda_{*,i} b(u_{*,i}, v_*) \quad \forall v_* \in V_*, \quad |u_{*,i}|_b = 1,$$

  where $V_* = \operatorname{span}\{u_{*,i},\ i = n, \dots, s\}$.
- Let $w_i \in V$, $i = n, \dots, s$, be given by

$$a(w_i, v) = a(u_{*,i}, v) - \lambda_{*,i} b(u_{*,i}, v) \quad \forall v \in V.$$

- Let $\|w_i\|_a \leq \eta_i$ for all $i = n, \dots, s$.
- Let $\lambda_{s-1} \leq \lambda_{*,s} < \nu \leq \lambda_{s+1}$.

Then

$$L_n \leq \lambda_n, \quad \text{where } L_n = \lambda_{*,n} \left(1 + \nu \lambda_{*,n} \sum_{i=n}^{s} \frac{\eta_i^2}{\lambda_{*,i}^2 (\nu - \lambda_{*,i})} \right)^{-1}.$$

Complementary upper bound on the residual

Theorem 5:

- Let $V = H_0^1(\Omega)$, $a(u,v) = (\nabla u, \nabla v)$, and $b(u,v) = (u,v)$.
- Let $u_* \in V$ and $\lambda_* \in \mathbb{R}$ be arbitrary.
- Let $w \in V$ satisfy

$$a(w,v) = a(u_*,v) - \lambda_* b(u_*,v) \quad \forall v \in V.$$

- Let $\mathbf{q} \in \mathbf{H}(\mathrm{div},\Omega)$ be such that $-\operatorname{div}\mathbf{q} = \lambda_* u_*$.

Then

$$\|\nabla w\|_{L^2(\Omega)} \le \eta = \|\nabla u_* - \mathbf{q}\|_{L^2(\Omega)}.$$

[Synge 1957], [Haslinger, Hlaváček 1976], [Křížek, Hlaváček 1984], [Neittaanmäki, Repin 2004], [Braess 2007], ...

Flux reconstruction

- FEM eigenpairs: $\Lambda_{h,n} \in \mathbb{R}$, $u_{h,n} \in V_h$, $\|u_{h,n}\|_{L^2(\Omega)} = 1$, $n = 1, \dots, s$
- Flux reconstruction: $\mathbf{q}_{h,n} = \sum_{\mathbf{z} \in \mathcal{N}_h} \mathbf{q}_{\mathbf{z},n}$ [Braess, Schöberl 2006]
- Local mixed FEM: $\mathbf{q}_{\mathbf{z},n} \in \mathbf{W}_{\mathbf{z}}$, $d_{\mathbf{z},n} \in P_1^*(\mathcal{T}_{\mathbf{z}})$

$$
\begin{aligned}
(\mathbf{q}_{\mathbf{z},n}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} - (d_{\mathbf{z},n}, \operatorname{div} \mathbf{w}_h)_{\omega_{\mathbf{z}}} &= (\psi_{\mathbf{z}} \nabla u_{h,n}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} && \forall \mathbf{w}_h \in \mathbf{W}_{\mathbf{z}} \\
-(\operatorname{div} \mathbf{q}_{\mathbf{z},n}, \varphi_h)_{\omega_{\mathbf{z}}} &= (r_{\mathbf{z},n}, \varphi_h)_{\omega_{\mathbf{z}}} && \forall \varphi_h \in P_1^*(\mathcal{T}_{\mathbf{z}})
\end{aligned}
$$

where

- $\omega_{\mathbf{z}}$ is the patch of elements around vertex $\mathbf{z} \in \mathcal{N}_h$
- $\mathcal{T}_{\mathbf{z}}$ is the set of elements in $\omega_{\mathbf{z}}$
- $\mathbf{W}_{\mathbf{z}} = \{\mathbf{w}_h \in \mathbf{H}(\operatorname{div}, \omega_{\mathbf{z}}) : \mathbf{w}_h|_K \in \mathbf{RT}_1(K)\ \forall K \in \mathcal{T}_{\mathbf{z}}$ and $\mathbf{w}_h \cdot \mathbf{n}_{\omega_{\mathbf{z}}} = 0$ on $\Gamma^{\mathrm{ext}}_{\omega_{\mathbf{z}}}\}$
- $P_1^*(\mathcal{T}_{\mathbf{z}}) = \begin{cases} \{v_h \in P_1(\mathcal{T}_{\mathbf{z}}) : \int_{\omega_{\mathbf{z}}} v_h \, \mathrm{d}x = 0\} & \text{for } \mathbf{z} \in \mathcal{N}_h \setminus \partial\Omega \\ P_1(\mathcal{T}_{\mathbf{z}}) & \text{for } \mathbf{z} \in \mathcal{N}_h \cap \partial\Omega \end{cases}$
- $r_{\mathbf{z},n} = \Lambda_{h,n} \psi_{\mathbf{z}} u_{h,n} - \nabla \psi_{\mathbf{z}} \cdot \nabla u_{h,n}$

Flux reconstruction

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- Flux reconstruction: $\mathbf{q}_{h,n} = \sum_{\mathbf{z} \in \mathcal{N}_h} \mathbf{q}_{\mathbf{z},n}$ [Braess, Schöberl 2006]
- Local mixed FEM: $\mathbf{q}_{\mathbf{z},n} \in \mathbf{W}_{\mathbf{z}}$, $d_{\mathbf{z},n} \in P_1^*(\mathcal{T}_{\mathbf{z}})$

$$\begin{aligned} (\mathbf{q}_{\mathbf{z},n}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} - (d_{\mathbf{z},n}, \operatorname{div} \mathbf{w}_h)_{\omega_{\mathbf{z}}} &= (\psi_{\mathbf{z}} \nabla u_{h,n}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} \quad && \forall \mathbf{w}_h \in \mathbf{W}_{\mathbf{z}} \\ -(\operatorname{div} \mathbf{q}_{\mathbf{z},n}, \varphi_h)_{\omega_{\mathbf{z}}} &= (r_{\mathbf{z},n}, \varphi_h)_{\omega_{\mathbf{z}}} \quad && \forall \varphi_h \in P_1^*(\mathcal{T}_{\mathbf{z}}) \end{aligned}$$

- Error estimator: $\eta_n = \|\nabla u_{h,n} - \mathbf{q}_{h,n}\|_{L^2(\Omega)}$
- Weinstein's bound: $\ell_n = \left(-\eta_n + \sqrt{\eta_n^2 + 4\Lambda_{h,n}} \right)^2 / 4$

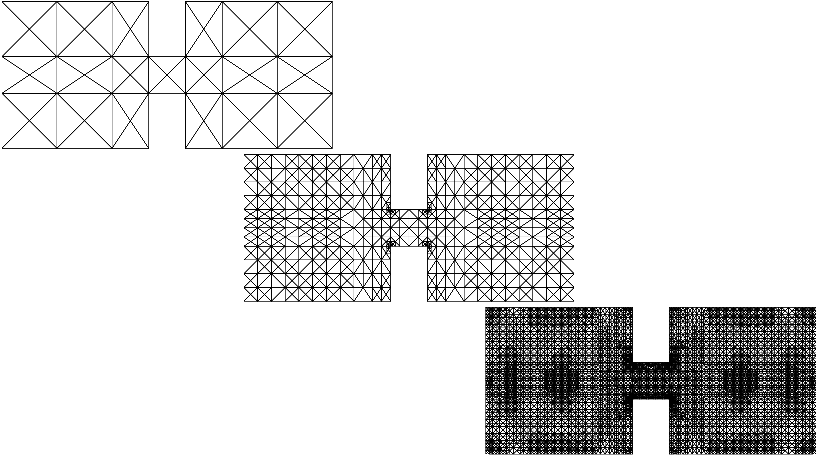
  provided $\Lambda_{h,n} \le \sqrt{\lambda_n \lambda_{n+1}}$.
- Kato's bound: $L_n = \Lambda_{h,n} \left(1 + \nu \Lambda_{h,n} \sum_{i=n}^{s} \frac{\eta_i^2}{\Lambda_{h,i}^2 (\nu - \Lambda_{h,i})} \right)^{-1}$

  provided $\Lambda_{h,s} < \nu \le \lambda_{s+1}$.

Example: Dumbbell – convergence

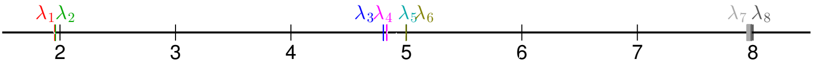
$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega = \text{dumbbell}$$
$$u_i = 0 \quad \text{on } \partial\Omega$$

Adaptively refined meshes:

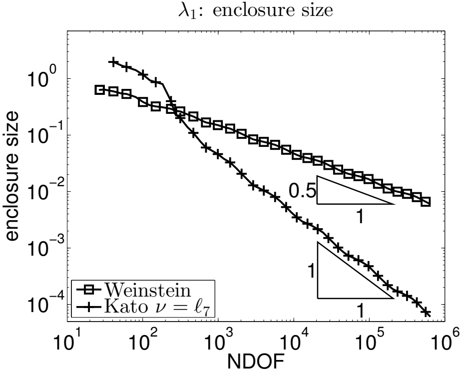


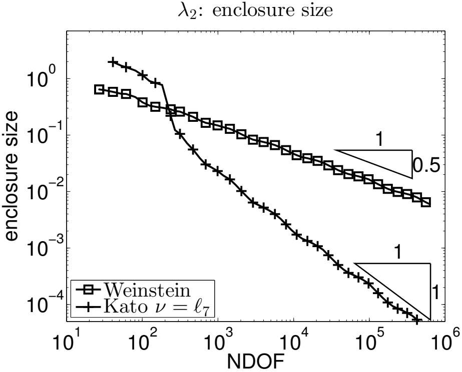
Example: Dumbbell – convergence

Spectrum:



Eigenvalue enclosure sizes:





Example: Dumbbell – convergence

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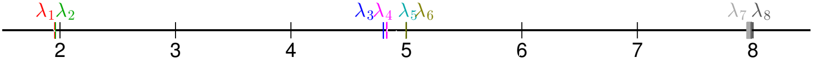

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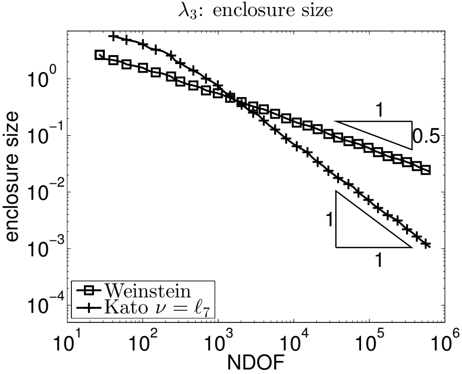

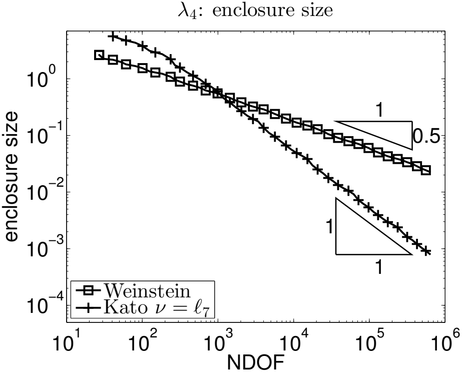

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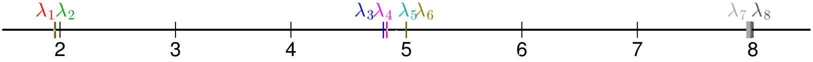

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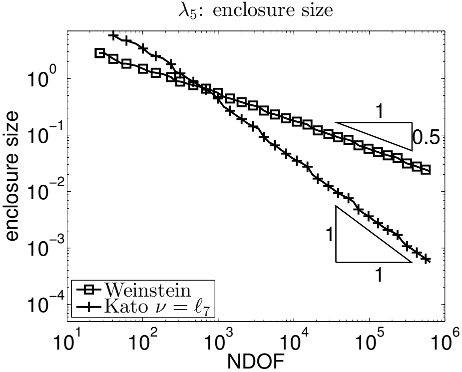

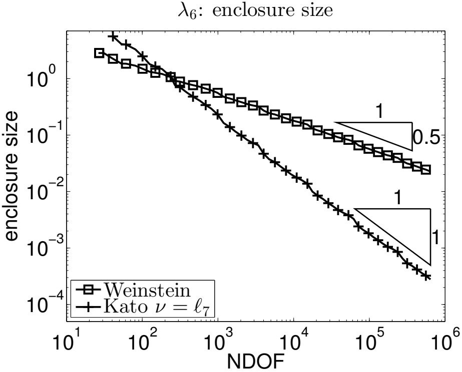

Conclusions

- Good for general symmetric elliptic second-order operators.
- Mixed boundary conditions (e.g. Steklov problem).
- Standard conforming finite element technology.
- Natural for adaptive mesh refinement.
- A priori information on spectrum needed.
- Combination of both Weinstein's and Kato's bound is useful.
- Homotopy method enables a guaranteed choice of ν.

Thank you for your attention

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