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Characterizations of Besov and Triebel–Lizorkin spaces via averages on balls ☆

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ABSTRACT

Let $\ell \in \mathbb{N}$ and $p \in (1,\infty]$. In this article, the authors prove that the sequence $\{f - B_{\ell,2^{-k}} f\}_{k\in\mathbb{Z}}$ consisting of the differences between f and the ball average $B_{\ell,2^{-k}} f$ characterizes the Besov space $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ with $q \in (0,\infty]$ and the Triebel–Lizorkin space $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ with $q \in (1,\infty]$ when the smoothness order $\alpha \in (0, 2\ell)$. More precisely, it is proved that $f - B_{\ell,2^{-k}} f$ plays the same role as the approximation to the identity $\varphi_{2^{-k}} * f$ appearing in the definitions of $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ and $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$. The corresponding results for inhomogeneous Besov and Triebel–Lizorkin spaces are also obtained. These results, for the first time, give a way to introduce Besov and Triebel–Lizorkin spaces with any smoothness order in $(0, 2\ell)$ on spaces of homogeneous type, where $\ell \in \mathbb{N}$.

1. Introduction

It is well known that the theory of function spaces with smoothness is a central topic of the analysis on spaces of homogeneous type in the sense of Coifman and Weiss [3,4]. Recall that the first order Sobolev space on spaces of homogeneous type was originally introduced by Hajłasz in [15] and later Shanmugalingam [21] introduced another kind of a first order Sobolev space which has strong locality and hence is more suitable for problems related to partial differential equations on spaces of homogeneous type. Recently, Alabern et al.

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[1] gave a way to introduce Sobolev spaces of any order bigger than 1 on spaces of homogeneous type in spirit closer to the square function and Dai et al. [6] gave several other ways, different from [1], to introduce Sobolev spaces of order 2ℓ on spaces of homogeneous type in spirit closer to the pointwise characterization as in [15], where $\ell \in \mathbb{N} := \{1, 2, \ldots\}$. Later, motivated by [1], Yang et al. [32] gave a way to introduce Besov and Triebel–Lizorkin spaces with smoothness order in $(0, 2)$ on spaces of homogeneous type. It is still an open question how to introduce Besov and Triebel–Lizorkin spaces with smoothness order not less than 2 on spaces of homogeneous type.

In this article, we establish a characterization of Besov and Triebel–Lizorkin spaces which can have any positive smoothness order on $\mathbb{R}^n$ via the difference between functions themselves and their ball averages. Since the average operator used in this article is also well defined on spaces of homogeneous type, this characterization can be used to introduce Besov and Triebel–Lizorkin spaces with any positive smoothness order on any space of homogeneous type and hence our results give an answer to the above open question.

Let us now give a detailed description of the main ideas used in this article. It is well known that a locally integrable function f belongs to the Sobolev space $W^{\alpha,p}(\mathbb{R}^n)$, with $\alpha \in (0, 1)$ and $p \in (1, \infty)$, if and only if $f \in L^p(\mathbb{R}^n)$ and

$$s_\alpha(f) := \left\{ \int_0^\infty \left[\fint_{B(\cdot,\, t)} |f(\cdot) - f(y)|\, dy \right]^2 \frac{dt}{t^{1+2\alpha}} \right\}^{1/2} \in L^p(\mathbb{R}^n)$$

(see, for example, [30,23,25,31]). Here and hereafter, $B(x, t)$ denotes an open ball with center at $x \in \mathbb{R}^n$ and radius $t \in (0, \infty)$, and $\fint_{B(x,t)} f(y)\, dy$ denotes the *integral average* of $f \in L^1_{\rm loc}(\mathbb{R}^n)$ on the ball $B(x, t) \subset \mathbb{R}^n$, namely,

$$\fint_{B(x,t)} f(y)\, dy := \frac{1}{|B(x,t)|} \int_{B(x,t)} f(y)\, dy =: B_t f(x). \tag{1.1}$$

However, when $\alpha \in [1, \infty)$, $s_\alpha(f)$ is not able to characterize $W^{\alpha,p}(\mathbb{R}^n)$, since, in this case, $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $\|s_\alpha(f)\|_{L^p(\mathbb{R}^n)} < \infty$ imply that f must be a constant function (see [13, Section 4] for more details).

Recently, Alabern et al. [1] established a remarkable characterization of Sobolev spaces of smooth order bigger than 1 and they proved that a function $f \in W^{\alpha,p}(\mathbb{R}^n)$, with $\alpha \in (0, 2)$ and $p \in (1, \infty)$, if and only if $f \in L^p(\mathbb{R}^n)$ and the square function $S_\alpha(f) \in L^p(\mathbb{R}^n)$, where

$$S_\alpha(f)(\cdot) := \left\{ \int_0^\infty \left| \fint_{B(\cdot,\, t)} [f(\cdot) - f(y)]\, dy \right|^2 \frac{dt}{t^{1+2\alpha}} \right\}^{1/2}, \quad f \in L^1_{\rm loc}(\mathbb{R}^n)$$

(see [1, Theorem 1 and p. 591]). Comparing S_α and s_α, we see that the only difference exists in that the absolute value $|f(\cdot) - f(y)|$ in $s_\alpha(f)$ is replaced by $f(\cdot) - f(y)$ in $S_\alpha(f)$. However, this slight change induces a quite different behavior between $s_\alpha(f)$ and S_α when characterizing Sobolev spaces. The former characterizes Sobolev spaces only with smoothness order less than 1, while the later characterizes Sobolev spaces with smoothness order less than 2. Such a difference follows from the following observation: for all $f \in C^2(\mathbb{R}^n)$ and $t \in (0, 1)$,

$$\fint_{B(x,\, t)} [f(x) - f(y)]\, dy = O(t^2), \quad x \in \mathbb{R}^n, \tag{1.2}$$

which follows from the Taylor expansion of f up to order 2:

$$f(y) = f(x) + \nabla f(x) \cdot (y - x) + O(|y - x|^2), \quad x, y \in \mathbb{R}^n;$$

in other words, the S_α-function provides smoothness up to order 2. We point out that this phenomenon was first observed by Wheeden in [29] (see also [30]), and later independently by Alabern, Mateu and Verdera [1].

By means of the fact (1.2), Alabern et al. [1, Theorems 2 and 3] also characterized Sobolev spaces of higher smoothness order and showed that $f \in W^{\alpha,p}(\mathbb{R}^n)$, with $\alpha \in [2N, 2N+2)$, $N \in \mathbb{N}$ and $p \in (1, \infty)$, if and only if $f \in L^p(\mathbb{R}^n)$ and there exist functions $g_1, \ldots, g_N \in L^p(\mathbb{R}^n)$ such that $S_\alpha(f, g_1, \ldots, g_N) \in L^p(\mathbb{R}^n)$, where

$$S_\alpha(f, g_1, \ldots, g_N)(\cdot) := \left\{ \int_0^\infty \left| \fint_{B(\cdot,t)} t^{-\alpha} R_N(y, \cdot)\, dy \right|^2 \frac{dt}{t} \right\}^{1/2}$$

with

$$R_N(y;\,\cdot) := f(y) - f(\cdot) - \sum_{j=1}^{N} g_j(\cdot)|y - \cdot|^{2j} \tag{1.3}$$

when $\alpha \in (2N, 2N+2)$, and

$$R_N(y;\,\cdot) := f(y) - f(\cdot) - \sum_{j=1}^{N-1} g_j(\cdot)|y - \cdot|^{2j} - B_t g_N(\cdot)|y - \cdot|^{2N} \tag{1.4}$$

when $\alpha = 2N$. Indeed, the function g_j was proved in [1, Theorems 2 and 3] to equal to $\frac{1}{L_j}\Delta^j f$ almost everywhere, where $L_j := \Delta^j |x|^{2j}$ for $j \in \{1, \ldots, N\}$. As the corresponding results for Triebel–Lizorkin spaces, Yang et al. [32, Theorems 1.1, 1.3 and 4.1] further proved that, for all $\alpha \in (2N, 2N+2)$, $N \in \mathbb{N}$ and $p \in (1, \infty]$, the Besov space $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$ with $q \in (0, \infty]$ and the Triebel–Lizorkin space $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ with $q \in (1, \infty]$ can be characterized via the function

$$S_{\alpha,\,q}(f)(x) := \left\{ \sum_{k\in\mathbb{Z}} 2^{k\alpha q} \left| \fint_{B(x,\,2^{-k})} \widetilde{R}_N(y;\,x)\, dy \right|^q \right\}^{1/q}, \quad x \in \mathbb{R}^n, \tag{1.5}$$

where, for all $x, y \in \mathbb{R}^n$ and $k \in \mathbb{Z}$,

$$\widetilde{R}_N(y;\,x) := f(y) - f(x) - \sum_{j=1}^{N} \frac{1}{L_j} \Delta^j f(x)|y - x|^{2j}. \tag{1.6}$$

It is an open question, posed in [32, Remark 4.1], whether there exists a corresponding characterization for $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$ and $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ when $\alpha = 2N$ with $N \in \mathbb{N}$. Moreover, only when $\alpha \in (0, 2)$, [32, Theorems 1.1 and 4.1] provide a way to introduce Besov and Triebel–Lizorkin spaces with smoothness order α on spaces of homogeneous type.

Via higher order differences, Triebel [27,28] and Haroske and Triebel [17,18] obtained another characterization of Sobolev spaces with order bigger than 1 on $\mathbb{R}^n$ without involving derivatives. Recall that, for

$\ell \in \mathbb{N}$, the *ℓ-th order (forward) difference operator* $\widetilde{\Delta}_h^\ell$ with $h \in \mathbb{R}^n$ is defined by setting, for all functions f and $x \in \mathbb{R}^n$,

$$\widetilde{\Delta}_h^1 f(x) := f(x+h) - f(x), \qquad \widetilde{\Delta}_h^\ell := \widetilde{\Delta}_h^1 \widetilde{\Delta}_h^{\ell-1}, \quad \ell \geq 2.$$

By means of $\widetilde{\Delta}_h^\ell f$, Triebel [27,28] and Haroske and Triebel [17,18] proved that the Sobolev space $W^{\ell,p}(\mathbb{R}^n)$ with $\ell \in \mathbb{N}$ and $p \in (1,\infty)$ can be characterized by a pointwise inequality in the spirit of Hajłasz [14] (see also Hu [15] and Yang [31]). Recall that the difference $\widetilde{\Delta}_h^\ell f$ can also be used to characterize Besov spaces and Triebel–Lizorkin spaces with smoothness order no more than ℓ. We refer the reader to Triebel's monograph [26, Section 3.4] for these difference characterizations of Besov and Triebel–Lizorkin spaces; see also [20, Section 3.1]. However, it is still unclear how to define higher than 1 order differences on spaces of homogeneous type.

On the other hand, recall that the averages of a function f can be used to approximate f itself in some function spaces; see, for example, [8,2]. Motivated by (1.2) and the pointwise characterization of Sobolev spaces with smoothness order no more than 1 (see Hajłasz [14], Hu [15] and Yang [31]), the authors established in [6] some pointwise characterizations of Sobolev spaces with smoothness order 2ℓ on $\mathbb{R}^n$ via ball averages of f, where $\ell \in \mathbb{N}$. To be precise, as the higher order variants of B_t in (1.1), for all $\ell \in \mathbb{N}$, $t \in (0,\infty)$ and $x \in \mathbb{R}^n$, we define the *2ℓ-th order average operator* $B_{\ell,t}$ by setting, for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$B_{\ell,t} f(x) := -\frac{2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} B_{jt} f(x), \tag{1.7}$$

here and hereafter, $\binom{2\ell}{\ell-j}$ denotes the binomial coefficients. Obviously, $B_{1,t} f = B_t f$. Moreover, it was observed in [6] that $f - B_{\ell,t} f$ is a 2ℓ-th order central difference of the function $t \mapsto B_t f(x)$ with step t at the origin, namely, for all $\ell \in \mathbb{N}$, $t \in (0,\infty)$, $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$f(x) - B_{\ell,t} f(x) = \frac{(-1)^\ell}{\binom{2\ell}{\ell}} \triangle_t^{2\ell} g(0) \tag{1.8}$$

with

$$g(t) := \begin{cases} B_t f(x), & t \in (0,\infty); \\ f(x), & t = 0; \\ B_{-t} f(x), & t \in (-\infty, 0). \end{cases} \tag{1.9}$$

Here and hereafter, for all functions h on $\mathbb{R}$ and $\theta, t \in \mathbb{R}$, let $T_\theta h(t) := h(t+\theta)$, and the *central difference operators* $\triangle_t^r$ are defined by setting

$$\triangle_\theta^1 h(t) := \triangle_\theta h(t) := h\left(t + \frac{\theta}{2}\right) - h\left(t - \frac{\theta}{2}\right) = \left(T_{\theta/2} - T_{-\theta/2}\right) h(t),$$
$$\triangle_\theta^r h(t) := \triangle_\theta(\triangle_\theta^{r-1} h)(t) = \sum_{j=0}^{r} \binom{r}{j} (-1)^j h\left(t + \frac{r\theta}{2} - j\theta\right), \quad r \in \{2, 3, \ldots\}.$$

The authors proved in [6] that $f \in W^{2\ell,p}(\mathbb{R}^n)$, with $\ell \in \mathbb{N}$ and $p \in (1,\infty)$, if and only if $f \in L^p(\mathbb{R}^n)$ and there exist a non-negative $g \in L^p(\mathbb{R}^n)$ and a positive constant C such that $|f(x) - B_{\ell,t} f(x)| \leq C t^{2\ell} g(x)$ for all $t \in (0,\infty)$ and almost every $x \in \mathbb{R}^n$. Various variants of this pointwise characterization were also presented in [6]. Recall that centered averages or their combinations were used to measure the smoothness and to characterize the K-functionals in [5,9,7].

Comparing the difference $f - B_{\ell,t}f$ with the usual difference $\widetilde{\Delta}_h^{2\ell} f$, we find that the former has an advantage that it involves only averages of f over balls, and hence can be easily generalized to any space of homogeneous type, whereas the difference operator $\widetilde{\Delta}_h^{2\ell} f$ cannot. We can also see their difference via (1.8). Indeed, it follows from (1.8) that $f - B_{\ell,t}f$ is a 2ℓ-th order central difference of a function g and the parameter related to such a difference is the radius $t \in (0, \infty)$ of the ball $B(x, t)$ with $x \in \mathbb{R}^n$, while the parameter related to $\widetilde{\Delta}_h^{2\ell} f$ is $h \in \mathbb{R}^n$, which also curbs the extension of $\widetilde{\Delta}_h^{2\ell} f$ to spaces of homogeneous type.

Although there exist differences between $f - B_{\ell,t}f$ and the usual difference $\widetilde{\Delta}_h^{2\ell} f$, the characterizations of $W^{2\ell,p}(\mathbb{R}^n)$ via $f - B_{\ell,t}f$ obtained in [6] imply that, in some sense, $f - B_{\ell,t}f$ also plays the role of 2ℓ-order derivatives. Therefore, it is natural to ask *whether we can use $f - B_{\ell,t}f$ to characterize Besov and Triebel–Lizorkin spaces with smoothness order less than 2ℓ or not.*

The main purpose of this article is to answer this question. To this end, we first recall some basic notions. Let $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ and $\mathcal{S}(\mathbb{R}^n)$ denote the collection of all *Schwartz functions* on $\mathbb{R}^n$, endowed with the usual topology, and $\mathcal{S}'(\mathbb{R}^n)$ its *topological dual*, namely, the collection of all bounded linear functionals on $\mathcal{S}(\mathbb{R}^n)$ endowed with the weak $*$-topology. Let $\mathcal{S}_\infty(\mathbb{R}^n)$ be the set of all Schwartz functions φ such that $\int_{\mathbb{R}^n} x^\gamma \varphi(x)\,dx = 0$ for all $\gamma \in \mathbb{Z}_+^n$, and $\mathcal{S}'_\infty(\mathbb{R}^n)$ its topological dual. For all $\alpha \in \mathbb{Z}_+^n$, $m \in \mathbb{Z}_+$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, let

$$\|\varphi\|_{\alpha,m} := \sup_{x\in\mathbb{R}^n,\, |\beta|\le|\alpha|} (1 + |x|)^m |\partial^\beta \varphi(x)|.$$

For all $\varphi \in \mathcal{S}'_\infty(\mathbb{R}^n)$, we use $\widehat{\varphi}$ to denote its *Fourier transform*. For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $t \in (0, \infty)$, we let $\varphi_t(\cdot) := t^{-n}\varphi(\cdot/t)$.

For all $a \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the *maximal integer* no more than a. For any $E \subset \mathbb{R}^n$, let χ_E be its *characteristic function*.

We now recall the notions of Besov and Triebel–Lizorkin spaces; see [25,26,11,34].

Definition 1.1. Let $\alpha \in (0, \infty)$, $p, q \in (0, \infty]$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy that

$$\operatorname{supp} \widehat{\varphi} \subset \{\xi \in \mathbb{R}^n :\ 1/2 \le |\xi| \le 2\} \text{ and } |\widehat{\varphi}(\xi)| \ge \text{constant} > 0 \text{ if } 3/5 \le |\xi| \le 5/3. \tag{1.10}$$

(i) The *homogeneous Besov space* $\dot{B}^\alpha_{p,\,q}(\mathbb{R}^n)$ is defined as the collection of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $\|f\|_{\dot{B}^\alpha_{p,\,q}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\dot{B}^\alpha_{p,\,q}(\mathbb{R}^n)} := \left[\sum_{k\in\mathbb{Z}} 2^{k\alpha q} \|\varphi_{2^{-k}} * f\|^q_{L^p(\mathbb{R}^n)}\right]^{1/q}$$

with the usual modifications made when $p = \infty$ or $q = \infty$.

(ii) The *homogeneous Triebel–Lizorkin space* $\dot{F}^\alpha_{p,\,q}(\mathbb{R}^n)$ is defined as the collection of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $\|f\|_{\dot{F}^\alpha_{p,\,q}(\mathbb{R}^n)} < \infty$, where, when $p \in (0, \infty)$,

$$\|f\|_{\dot{F}^\alpha_{p,\,q}(\mathbb{R}^n)} := \left\|\left[\sum_{k\in\mathbb{Z}} 2^{k\alpha q} |\varphi_{2^{-k}} * f|^q\right]^{1/q}\right\|_{L^p(\mathbb{R}^n)}$$

with the usual modification made when $q = \infty$, and

$$\|f\|_{\dot F^\alpha_{\infty,\,q}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n}\sup_{m\in\mathbb{Z}}\left\{ \fint_{B(x,\,2^{-m})} \sum_{k=m}^{\infty} 2^{k\alpha q}|\varphi_{2^{-k}} * f(y)|^q\,dy\right\}^{1/q}$$

with the usual modification made when $q=\infty$.

It is well known that the spaces $\dot B^\alpha_{p,\,q}(\mathbb{R}^n)$ and $\dot F^\alpha_{p,\,q}(\mathbb{R}^n)$ are independent of the choice of functions φ satisfying (1.10); see, for example, [12].

We also recall the corresponding inhomogeneous spaces.

Definition 1.2. Let $\alpha\in(0,\,\infty)$, $p,q\in(0,\,\infty]$, $\varphi\in\mathcal{S}(\mathbb{R}^n)$ satisfy (1.10) and $\Phi\in\mathcal{S}(\mathbb{R}^n)$ satisfy that

$$\operatorname{supp}\widehat{\Phi}\subset\{\xi\in\mathbb{R}^n:\ |\xi|\le 2\}\ \text{and}\ |\widehat{\Phi}(\xi)|\ge \text{constant}>0\ \text{if}\ |\xi|\le 5/3. \tag{1.11}$$

(i) The *inhomogeneous Besov space* $B^\alpha_{p,\,q}(\mathbb{R}^n)$ is defined as the collection of all $f\in\mathcal{S}'(\mathbb{R}^n)$ such that $\|f\|_{B^\alpha_{p,\,q}(\mathbb{R}^n)}<\infty$, where

$$\|f\|_{B^\alpha_{p,\,q}(\mathbb{R}^n)} := \left[\sum_{k\in\mathbb{Z}_+} 2^{k\alpha q}\|\varphi_{2^{-k}} * f\|^q_{L^p(\mathbb{R}^n)}\right]^{1/q}$$

with the usual modifications made when $p=\infty$ or $q=\infty$, where, when $k=0$, $\varphi_{2^{-k}}$ is replaced by Φ.

(ii) The *inhomogeneous Triebel–Lizorkin space* $F^\alpha_{p,\,q}(\mathbb{R}^n)$ is defined as the collection of all $f\in\mathcal{S}'(\mathbb{R}^n)$ such that $\|f\|_{F^\alpha_{p,\,q}(\mathbb{R}^n)}<\infty$, where, when $p\in(0,\infty)$,

$$\|f\|_{F^\alpha_{p,\,q}(\mathbb{R}^n)} := \left\|\left[\sum_{k\in\mathbb{Z}_+} 2^{k\alpha q}|\varphi_{2^{-k}} * f|^q\right]^{1/q}\right\|_{L^p(\mathbb{R}^n)}$$

with the usual modification made when $q=\infty$, and

$$\|f\|_{F^\alpha_{\infty,\,q}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n}\sup_{m\in\mathbb{Z}_+}\left\{ \fint_{B(x,\,2^{-m})} \sum_{k=m}^{\infty} 2^{k\alpha q}|\varphi_{2^{-k}} * f(y)|^q\,dy\right\}^{1/q}$$

with the usual modification made when $q=\infty$, where, when $k=0$, $\varphi_{2^{-k}}$ is replaced by Φ.

It is also well known that the spaces $B^\alpha_{p,\,q}(\mathbb{R}^n)$ and $F^\alpha_{p,\,q}(\mathbb{R}^n)$ are independent of the choice of functions φ and Φ satisfying (1.10) and (1.11), respectively; see, for example, [25].

As the main result of this article, we prove that the difference $f-B_{\ell,2^{-k}}f$ with $k\in\mathbb{Z}$ plays the same role of the approximation to the identity $\varphi_{2^{-k}}*f$ in the definitions of Besov and Triebel–Lizorkin spaces in the following sense.

Theorem 1.3. *Let $\ell\in\mathbb{N}$ and $\alpha\in(0,2\ell)$.*

(i) *Let $p\in(1,\infty]$ and $q\in(0,\infty]$. If $f\in\dot B^\alpha_{p,q}(\mathbb{R}^n)$, then there exists $g\in L^1_{\rm loc}(\mathbb{R}^n)\cap\mathcal{S}'_\infty(\mathbb{R}^n)$ such that $g=f$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and $|||g|||_{\dot B^\alpha_{p,q}(\mathbb{R}^n)}\le C\|f\|_{\dot B^\alpha_{p,q}(\mathbb{R}^n)}$ for some positive constant C independent of f, where*

$$|||g|||_{\dot B^\alpha_{p,q}(\mathbb{R}^n)} := \left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\|g-B_{\ell,2^{-k}}g\|^q_{L^p(\mathbb{R}^n)}\right\}^{1/q}.$$

Conversely, if $f \in L^1_{\rm loc}(\mathbb{R}^n) \cap \mathcal{S}'_\infty(\mathbb{R}^n)$ *and* $|||f|||_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)} < \infty$*, then* $f \in \dot{B}^\alpha_{p,q}(\mathbb{R}^n)$ *and* $\|f\|_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)} \le C|||f|||_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)}$ *for some positive constant* C *independent of* f*.*

(ii) *Let* $p \in (1,\infty]$ *and* $q \in (1,\infty]$*. If* $f \in \dot{F}^\alpha_{p,q}(\mathbb{R}^n)$*, then there exists* $g \in L^1_{\rm loc}(\mathbb{R}^n) \cap \mathcal{S}'_\infty(\mathbb{R}^n)$ *such that* $g = f$ *in* $\mathcal{S}'_\infty(\mathbb{R}^n)$ *and* $|||g|||_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)}$ *for some positive constant* C *independent of* f*, where, when* $p \in (1,\infty)$*,*

$$|||g|||_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} := \left\| \left\{ \sum_{k\in\mathbb{Z}} 2^{k\alpha q} |g - B_{\ell,2^{-k}} g|^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)}$$

and, when $p = \infty$*,*

$$|||g|||_{\dot{F}^\alpha_{\infty,q}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \sup_{m\in\mathbb{Z}} \left\{ \fint_{B(x,\,2^{-m})} \sum_{k=m}^{\infty} 2^{k\alpha q} |g(y) - B_{\ell,2^{-k}} g(y)|^q \, dy \right\}^{1/q}.$$

Conversely, if $f \in L^1_{\rm loc}(\mathbb{R}^n) \cap \mathcal{S}'_\infty(\mathbb{R}^n)$ *and* $|||f|||_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} < \infty$*, then* $f \in \dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ *and* $\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \le C|||f|||_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)}$ *for some positive constants* C *independent of* f*.*

Remark 1.4. (i) Notice that $f - B_{\ell,2^{-k}} f$ can be easily defined on any space of homogeneous type. Thus, the characterizations of $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$ and $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ obtained in Theorem 1.3 provide a possible way to introduce Besov and Triebel–Lizorkin spaces with arbitrary positive smoothness order on spaces of homogeneous type, while the characterizations of Besov and Triebel–Lizorkin spaces via the usual differences cannot.

(ii) Observing that $f - B_{1,t} f = f - B_t f$, we see that, when $\alpha \in (0,2)$, Theorem 1.3 just coincides with [32, Theorems 1.1 and 4.1]. When $\alpha \in (2,\infty)$, comparing with Theorem 1.3 and [32, Theorems 1.2 and 4.1], we find that the former provides a way to introduce Besov and Triebel–Lizorkin spaces with smoothness order no less than 2 on spaces of homogeneous type, while the later has a restriction that α cannot be any even positive integer and also cannot be generalized to any space of homogeneous type, due to the lack of derivatives on spaces of homogeneous type.

(iii) In [16], a concept of RD-spaces was introduced, namely, a space of homogeneous type whose measure also satisfies the inverse doubling condition is called an *RD-space* (see also [16,33] for several equivalent definitions of RD-spaces). Via approximations to the identity, a theory of Besov and Triebel–Lizorkin spaces with smoothness order in $(-1,1)$ on RD-spaces was also systematically developed in [16]. Then, a natural and interesting question is: on RD-spaces, whether the Besov and Triebel–Lizorkin spaces with smoothness order in $(0,1)$ defined in [16] coincide with those defined via $f - B_{1,2^{-k}} f$ in spirit of Theorem 1.3 or not. We will not seek an answer of this question in this article.

Theorem 1.3 is proved in Section 2. Comparing with those proofs for various pointwise characterizations of Sobolev spaces $W^{2\ell,p}(\mathbb{R}^n)$ via $f - B_{\ell,t} f$ in [6], the proof of Theorem 1.3 is much more complicated. Indeed, the main idea of the proof for Theorem 1.3 is to write $f - B_{\ell,2^{-k}} f$ as a convolution operator, then control $f - B_{\ell,2^{-k}} f$ by certain maximal functions via calculating pointwise estimates of the related operator kernel and finally apply the vector-valued maximal inequality of Fefferman and Stein in [10]. The Calderón reproducing formula on $\mathbb{R}^n$ (see, for example, [12]) also plays a key role in this proof.

In Section 3, we further show that the inhomogeneous variant of Theorem 1.3 also holds true (see Theorem 3.1 below). We also show that Theorems 1.3 and 3.1 still hold true on Euclidean spaces with non-Euclidean metrics.

Finally, we make some conventions on notation. The *symbol* C denotes a positive constant which depends only on the fixed parameters n, α, p, q and possibly on auxiliary functions, unless otherwise stated; its value may vary from line to line. We use the *symbol* $A \lesssim B$ to denote that there exists a positive constant C such that $A \le C B$. The symbol $A \sim B$ is used as an abbreviation of $A \lesssim B \lesssim A$. We also use the *symbol* $\lfloor s \rfloor$ for any $s \in \mathbb{R}$ to denote the maximal integer not more than s.

2. Proof of Theorem 1.3

To prove Theorem 1.3, we need some technical lemmas. Let, for all $t \in (0,\infty)$ and $x \in \mathbb{R}^n$, $I(x) := \frac{1}{|B(0,1)|}\chi_{B(0,1)}(x)$ and $I_t(x) := t^{-n}I(x/t)$. Then

$$(B_{\ell,t}f)(x) = \frac{-2}{\binom{2\ell}{\ell}}\sum_{j=1}^{\ell}(-1)^j\binom{2\ell}{\ell-j}(f * I_{jt})(x), \quad x \in \mathbb{R}^n, \quad t \in (0,\infty),$$

and hence

$$(B_{\ell,t}f)^\wedge(\xi) = m_\ell(t\xi)\widehat{f}(\xi), \qquad \xi \in \mathbb{R}^n, \tag{2.1}$$

where

$$m_\ell(x) := \frac{-2}{\binom{2\ell}{\ell}}\sum_{j=1}^{\ell}(-1)^j\binom{2\ell}{\ell-j}\widehat{I}(jx), \qquad x \in \mathbb{R}^n. \tag{2.2}$$

A straightforward calculation shows that

$$\widehat{I}(x) = \gamma_n\int_0^1 \cos(u|x|)(1-u^2)^{\frac{n-1}{2}}\,du, \quad x \in \mathbb{R}^n, \tag{2.3}$$

with $\gamma_n := [\int_0^1(1-u^2)^{\frac{n-1}{2}}\,du]^{-1}$ (see also Stein's book [24, p. 430, Section 6.19]).

Lemma 2.1. *For all $\ell \in \mathbb{N}$ and $x \in \mathbb{R}^n$,*

$$m_\ell(x) = 1 - A_\ell(|x|), \tag{2.4}$$

where

$$A_\ell(s) := \gamma_n\frac{4^\ell}{\binom{2\ell}{\ell}}\int_0^1(1-u^2)^{\frac{n-1}{2}}\left(\sin\frac{us}{2}\right)^{2\ell}du, \qquad s \in \mathbb{R}. \tag{2.5}$$

Furthermore, $s^{-2\ell}A_\ell(s)$ is a smooth function on $\mathbb{R}$ satisfying that there exist positive constants c_1 and c_2 such that

$$0 < c_1 \le \frac{A_\ell(s)}{s^{2\ell}} \le c_2, \qquad s \in (0,4] \tag{2.6}$$

and

$$\sup_{s\in\mathbb{R}}\left|\left(\frac{d}{ds}\right)^i\left(\frac{A_\ell(s)}{s^{2\ell}}\right)\right| < \infty, \quad i \in \mathbb{N}.$$

Proof. Combining (2.2) with (2.3), we obtain

$$m_\ell(x) = \frac{-2\gamma_n}{\binom{2\ell}{\ell}} \int_0^1 \left[\sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell-j} \cos(ju|x|)\right] (1-u^2)^{\frac{n-1}{2}}\, du, \quad x \in \mathbb{R}^n. \tag{2.7}$$

However, a straightforward calculation shows that, for all $s \in \mathbb{R}$,

$$4^\ell \left(\sin\frac{s}{2}\right)^{2\ell} = \binom{2\ell}{\ell} + 2\sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell-j} \cos js.$$

This, together with (2.7), implies (2.4).

Next we show (2.6). By the mean value theorem, we know that, for all $u \in (0,1)$ and $s \in \mathbb{R}$, there exists $\theta \in (0,1)$ such that

$$\left(\sin\frac{us}{2}\right)^{2\ell} = \left(\frac{1}{2}us\right)^{2\ell} \left(\cos\frac{us\theta}{2}\right)^{2\ell}.$$

From this and (2.5), we deduce that, for all $s \in (0,4]$,

$$\frac{A_\ell(s)}{s^{2\ell}} \le \gamma_n \frac{4^\ell}{2\binom{2\ell}{\ell}} \int_0^1 (1-u^2)^{\frac{n-1}{2}} u^{2\ell}\, du =: c_2 < \infty$$

and

$$\begin{aligned} \frac{A_\ell(s)}{s^{2\ell}} &\ge \gamma_n \frac{4^\ell}{2\binom{2\ell}{\ell}} \int_0^{\min\{1,\frac{2\pi}{3s}\}} (1-u^2)^{\frac{n-1}{2}} u^{2\ell} \left(\cos\frac{us\theta}{2}\right)^{2\ell} du \\ &\ge \gamma_n \frac{1}{2\binom{2\ell}{\ell}} \int_0^{\min\{1,\frac{2\pi}{3s}\}} (1-u^2)^{\frac{n-1}{2}} u^{2\ell}\, du \\ &\ge \gamma_n \frac{1}{2\binom{2\ell}{\ell}} \int_0^{\frac{\pi}{6}} (1-u^2)^{\frac{n-1}{2}} u^{2\ell}\, du =: c_1 > 0. \end{aligned}$$

These prove (2.6).

Finally, by the mean value theorem again, an argument similar to the above also implies that

$$\sup_{s\in\mathbb{R}} \left| \left(\frac{d}{ds}\right)^i \left(\frac{A_\ell(s)}{s^{2\ell}}\right) \right| < \infty$$

for all $i \in \mathbb{N}$. This finishes the proof of Lemma 2.1. $\square$

Recall that the *Hardy–Littlewood maximal operator* M is defined by setting, for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$Mf(x) := \sup_{B\subset\mathbb{R}^n} \fint_B |f(y)|\, dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ containing x. The following two lemmas can be verified straightforwardly.

Lemma 2.2. *Let $\{T_t\}_{t\in(0,\infty)}$ be a family of multiplier operators given by setting, for all $f \in L^2(\mathbb{R}^n)$,*

$$(T_t f)^\wedge(\xi) := m(t\xi)\widehat{f}(\xi), \qquad \xi \in \mathbb{R}^n, \quad t \in (0,\infty)$$

for some $m \in L^\infty(\mathbb{R}^n)$. If

$$\|\nabla^{n+1} m\|_{L^1(\mathbb{R}^n)} + \|m\|_{L^1(\mathbb{R}^n)} \le C_1 < \infty,$$

then there exists a positive constant C such that, for all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\sup_{t\in(0,\infty)} |T_t f(x)| \le CC_1\, Mf(x).$$

Proof. For all $t \in (0,\infty)$, $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, by the Fubini theorem, we see that

$$\begin{aligned}
|T_t f(x)| &= \left| \int_{\mathbb{R}^n} m(t\xi)\widehat{f}(\xi) e^{ix\cdot\xi}\, d\xi \right| \\
&= \left| \int_{\mathbb{R}^n} f(y) \int_{\mathbb{R}^n} m(t\xi) e^{i(x-y)\cdot\xi}\, d\xi\, dy \right| \\
&\le \left| \int_{|x-y|<t} f(y) \int_{\mathbb{R}^n} m(t\xi) e^{i(x-y)\cdot\xi}\, d\xi\, dy \right| + \left| \int_{|x-y|\ge t} \cdots \right| =: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

It is easy to see that $\mathrm{I} \lesssim \|m\|_{L^1(\mathbb{R}^n)} Mf(x)$.

For II, via the Fubini theorem and the integration by parts, we also have

$$\begin{aligned}
\mathrm{II} &\lesssim \int_{|x-y|\ge t} \frac{|f(y)|}{|x-y|^{n+1}} \int_{\mathbb{R}^n} t^{n+1} |\nabla^{n+1} m(t\xi)|\, d\xi\, dy \\
&\lesssim \|\nabla^{n+1} m\|_{L^1(\mathbb{R}^n)} \sum_{j=1}^\infty t \int_{2^j t \le |x-y| < 2^{j+1} t} \frac{|f(y)|}{|x-y|^{n+1}}\, dy \\
&\lesssim \|\nabla^{n+1} m\|_{L^1(\mathbb{R}^n)} \sum_{j=1}^\infty 2^{-j} Mf(x) \lesssim \|\nabla^{n+1} m\|_{L^1(\mathbb{R}^n)} Mf(x),
\end{aligned}$$

which completes the proof of Lemma 2.2. □

Remark 2.3. (i) The above proof of Lemma 2.2 actually yields the following more subtle estimate, which is also needed in the proof of Theorem 1.3: Assume that $f \in L^2(\mathbb{R}^n)$, $x \in B(z,s)$ for some $z \in \mathbb{R}^n$ and $s \in (0,\infty)$. Then there exists a positive constant C, independent of t, s, f and x, such that, for all $l \in \mathbb{N} \cap (n,\infty)$ and $t \in (0,\infty)$,

$$|T_t f(x)| \le C[\|m\|_{L^1(\mathbb{R}^n)} + \|\nabla^l m\|_{L^1(\mathbb{R}^n)}] \sum_{i=0}^\infty 2^{-i(l-n)} M(f\chi_{B(z,2^i t+s)})(x).$$

Indeed, notice that $x \in B(z,s)$, $y \in B(x,t)$ and $t \in (0,\infty)$ imply that $y \in B(z,t+s)$. Thus, by the Fubini theorem, we find that

$$\mathrm{I} \le \left| \int_{|x-y|<t} f(y)\chi_{B(z,t+s)}(y) \int_{\mathbb{R}^n} m(t\xi)e^{i(x-y)\cdot\xi}\,d\xi\,dy \right| \lesssim \|m\|_{L^1(\mathbb{R}^n)} M(f\chi_{B(z,t+s)})(x).$$

Similarly, by the Fubini theorem and the integration by parts, together with $\ell \in \mathbb{N} \cap (n,\infty)$, we know that

$$\begin{aligned}
\mathrm{II} &\lesssim \int_{|x-y|\ge t} \frac{|f(y)|}{|x-y|^l} \int_{\mathbb{R}^n} t^l |\nabla^l m(t\xi)|\,d\xi\,dy \\
&\lesssim \|\nabla^l m\|_{L^1(\mathbb{R}^n)} \int_{|x-y|\ge t} \frac{t^{l-n}|f(y)|}{|x-y|^l}\,dy \\
&\lesssim \|\nabla^l m\|_{L^1(\mathbb{R}^n)} \sum_{i=1}^{\infty} (2^i t)^{-l} t^{l-n} \int_{|x-y|\sim 2^i t} |f(y)|\chi_{B(z,2^i t+s)}(y)\,dy \\
&\lesssim \|\nabla^{n+1} m\|_{L^1(\mathbb{R}^n)} \sum_{i=1}^{\infty} 2^{-i(l-n)} M(f\chi_{B(z,2^i t+s)})(x),
\end{aligned}$$

where $|x-y| \sim 2^i t$ means $2^{i-1}t \le |x-y| < 2^i t$. This finishes the proof of the above claim.

(ii) We also point out that the conclusions of Lemma 2.2 and (i) of this remark remain true for all $f \in L^p(\mathbb{R}^n)$ with $p \in (1,\infty)$ and $m \in \mathcal{S}(\mathbb{R}^n)$.

From the Hölder inequality when $q \in [1,\infty]$ and the monotonicity of l^q when $q \in (0,1)$, we immediately deduce the following conclusions, the details being omitted.

Lemma 2.4. *Let $\{a_j\}_{j\in\mathbb{Z}} \subset \mathbb{C}$, $q \in (0,\infty]$ and $\beta \in (0,\infty)$. Then there exists a positive constant C, independent of $\{a_j\}_{j\in\mathbb{Z}}$, such that*

$$\left[\sum_{k\in\mathbb{Z}} 2^{k\beta q} \left(\sum_{j=k}^{\infty} |a_j|\right)^q\right]^{1/q} \le C \left(\sum_{k\in\mathbb{Z}} 2^{k\beta q}|a_k|^q\right)^{1/q}$$

and

$$\left[\sum_{k\in\mathbb{Z}} 2^{-k\beta q} \left(\sum_{j=-\infty}^{k} |a_j|\right)^q\right]^{1/q} \le C \left(\sum_{k\in\mathbb{Z}} 2^{-k\beta q}|a_k|^q\right)^{1/q}.$$

Now we prove Theorem 1.3.

Proof of Theorem 1.3. We only prove (ii), the proof of (i) being similar and easier.

To show (ii), let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (1.10) and $\sum_{j\in\mathbb{Z}} \widehat{\varphi}_j \equiv 1$ on $\mathbb{R}^n \setminus \{0\}$. Assume first that $\alpha \in (0, 2\ell)$, $p \in (1,\infty)$ and $q \in (1,\infty]$. Let $f \in \dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$. We know that $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n) \hookrightarrow L^1_{\mathrm{loc}}(\mathbb{R}^n)$ in the sense of distributions; see, for example, [19, Proposition 4.2], [33, Proposition 5.1] or [34, Proposition 8.2] for a proof. Indeed, it was proved therein that there exists a sequence $\{P_j\}_{j\in\mathbb{Z}}$ of polynomials of degree not more than $\lfloor \alpha - n/p \rfloor$ such that the summation $\sum_{j\in\mathbb{Z}}(\varphi_j * f + P_j)$ converges in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\mathcal{S}'_\infty(\mathbb{R}^n)$ to a function $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, which is known to be the Calderón reproducing formula (see, for example, [11,12]). The function g serves as a representative of f. Thus, in the below proof, we identify f with g. Then $g \in L^1_{\rm loc}(\mathbb{R}^n) \cap \mathcal{S}'_\infty(\mathbb{R}^n)$. Now we show $|||g|||_{\dot F^\alpha_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{\dot F^\alpha_{p,q}(\mathbb{R}^n)}$, namely,

$$\left\| \left\{ \sum_{k\in\mathbb{Z}} 2^{k\alpha q} |g - B_{\ell,2^{-k}} g|^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\dot F^\alpha_{p,q}(\mathbb{R}^n)}. \tag{2.8}$$

To this end, for all $k, j \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$, define $T_{k,j}$ as

$$(T_{k,j} f)^\wedge(\xi) := \widehat{\varphi}(2^{-j}\xi) A_\ell(2^{-k}|\xi|) \widehat{f}(\xi), \quad \xi \in \mathbb{R}^n. \tag{2.9}$$

Noticing that the degree of each P_j is not more than $\lfloor \alpha - n/p \rfloor < 2\ell$ and $P - B_{\ell,2^{-k}} P = 0$ for all polynomials P of degree less than 2ℓ, we then find that

$$g - B_{\ell,2^{-k}} g = \sum_{j\in\mathbb{Z}} T_{k,j} f. \tag{2.10}$$

We split the sum $\sum_{j\in\mathbb{Z}}$ in this last equation into two parts $\sum_{j\ge k}$ and $\sum_{j<k}$. The first part is relatively easy to deal with. Indeed, for $j \ge k$, by (2.9), we see that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} |T_{k,j} f(x)| &= |(I - B_{\ell,2^{-k}})(f * \varphi_{2^{-j}})(x)| \\ &\le |f * \varphi_{2^{-j}}(x)| + C_\ell \sum_{i=1}^{\ell} |B_{i2^{-k}}(f * \varphi_{2^{-j}})(x)| \\ &\lesssim M(f * \varphi_{2^{-j}})(x). \end{aligned} \tag{2.11}$$

From this and Lemma 2.4, it follows that

$$\begin{aligned} \sum_{k\in\mathbb{Z}} 2^{k\alpha q} \left| \sum_{j\ge k} T_{k,j} f \right|^q &\lesssim \sum_{k\in\mathbb{Z}} 2^{k\alpha q} \left[\sum_{j\ge k} M(f * \varphi_{2^{-j}}) \right]^q \\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{jq\alpha} [M(f * \varphi_{2^{-j}})]^q. \end{aligned} \tag{2.12}$$

Now we handle the sum $\sum_{j<k}$. Since φ satisfies (1.10), by [12, Lemma (6.9)], there exists $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (1.10) such that

$$\sum_{j\in\mathbb{Z}} \widehat{\varphi}(2^{-j}\xi) \widehat{\psi}(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$

Thus, for all $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$(T_{k,j} f)^\wedge(\xi) = \widehat{\varphi}(2^{-j}\xi) A_\ell(2^{-k}|\xi|) \widehat{f_j}(\xi) = m_{k,j}(\xi) \widehat{f_j}(\xi),$$

where $f_j := \sum_{i=-1}^{1} f * \psi_{2^{i-j}}$ and

$$m_{k,j}(\xi) := \widehat{\varphi}(2^{-j}\xi) \frac{A_\ell(2^{-k}|\xi|)}{(2^{-k}|\xi|)^{2\ell}} (2^{-k}|\xi|)^{2\ell}, \qquad \xi \in \mathbb{R}^n \setminus \{0\}.$$

Write $\widetilde{m}_{k,j}(\xi) := m_{k,j}(2^j\xi)$. From Lemma 2.1, it follows that, for all $j < k$ and $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$|\partial^\beta \widetilde{m}_{k,j}(\xi)| \lesssim 2^{2\ell(j-k)} \chi_{\overline{B(0,2)}\setminus B(0,1/2)}(\xi), \qquad \beta \in \mathbb{Z}^d_+, \tag{2.13}$$

and hence $\|\widetilde{m}_{k,j}\|_{L^1(\mathbb{R}^n)} + \|\nabla^{n+1}\widetilde{m}_{k,j}\|_{L^1(\mathbb{R}^n)} \lesssim 2^{2\ell(j-k)}$, which, together with Lemma 2.2, implies that

$$|T_{k,j}f(x)| \lesssim 2^{2\ell(j-k)} Mf_j(x), \qquad x \in \mathbb{R}^n.$$

Thus, by Lemma 2.4, for $\alpha \in (0, 2\ell)$, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} 2^{k\alpha q} \left|\sum_{j=-\infty}^{k} T_{k,j}f\right|^q &\lesssim \sum_{k\in\mathbb{Z}} 2^{k(\alpha-2\ell)q} \left(\sum_{j=-\infty}^{k} 2^{2\ell j} Mf_j\right)^q \\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{j\alpha q}[Mf_j]^q. \end{aligned}\tag{2.14}$$

Combining (2.12) and (2.14) with (2.10), and using the Fefferman–Stein vector-valued maximal inequality (see [10] or [24]), we see that

$$\begin{aligned}\left\|\left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha q}|g - B_{\ell,2^{-k}}g|^q\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)} &\lesssim \left\|\left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha q}[M(f * \varphi_{2^{-k}})]^q\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)}.\end{aligned}$$

This proves (2.8) and hence finishes the proof of the first part of Theorem 1.3(ii).

To see the inverse conclusion, we only need to prove

$$\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \lesssim \left\|\left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha q}|f - B_{\ell,2^{-k}}f|^q\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)} \tag{2.15}$$

whenever $f \in L^1_{\rm loc}(\mathbb{R}^n) \cap \mathcal{S}'_\infty(\mathbb{R}^n)$ and the right-hand side of (2.15) is finite. To this end, we first claim that

$$|f * \varphi_{2^{-j}}(x)| \lesssim M(f - B_{\ell,2^{-j}}f)(x), \qquad j \in \mathbb{Z}, \quad x \in \mathbb{R}^n. \tag{2.16}$$

Indeed, we see that, for all $j \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$(f * \varphi_{2^{-j}})^\wedge(\xi) = \frac{\widehat{\varphi}(2^{-j}\xi)}{A_\ell(2^{-j}|\xi|)}(f - B_{\ell,2^{-j}}f)^\wedge(\xi) =: \eta(2^{-j}\xi)(f - B_{\ell,2^{-j}}f)^\wedge(\xi),$$

where $\eta(\xi) := \frac{\widehat{\varphi}(\xi)}{A_\ell(|\xi|)}$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$, which is well defined due to (2.6). By Lemma 2.1, we know that $\eta \in C_c^\infty(\mathbb{R}^n)$ and $\operatorname{supp}\eta \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} \le |\xi| \le 2\}$. The claim (2.16) then follows from Lemma 2.2.

Now, using the claim (2.16) and the Fefferman–Stein vector-valued maximal inequality (see [10] or [24]), we find that

$$\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \lesssim \left\|\left\{\sum_{j\in\mathbb{Z}} 2^{j\alpha q}\left[M(f - B_{\ell,2^{-j}}f)\right]^q\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)}$$

$$\lesssim \left\| \left[\sum_{j\in\mathbb{Z}} 2^{j\alpha q} |f - B_{\ell,2^{-j}} f|^q \right]^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \sim |||f|||_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)}.$$

This proves the desired conclusion when $\alpha \in (0, 2\ell)$, $p \in (1, \infty)$ and $q \in (1, \infty]$.

It remains to consider the case that $\alpha \in (0, 2\ell)$, $p = \infty$ and $q \in (1, \infty]$. The proof is similar to that of the case $p \in (1, \infty)$ but more subtle. Assume first that $f \in \dot{F}^\alpha_{\infty,q}(\mathbb{R}^n)$. By an argument similar to the above, in this case, we need to show

$$\sup_{x\in\mathbb{R}^n} \sup_{m\in\mathbb{Z}} \left\{ \fint_{B(x,\,2^{-m})} \sum_{k=m}^{\infty} 2^{k\alpha q} |g(y) - B_{\ell,2^{-k}} g(y)|^q \, dy \right\}^{1/q} \lesssim \|f\|_{\dot{F}^\alpha_{\infty,q}(\mathbb{R}^n)}. \tag{2.17}$$

Notice that, if $y \in B(x, 2^{-m})$ and $z \in B(y, i2^{-k})$ with $k \ge m$ and $i \in \{1, \dots, \ell\}$, then $z \in B(x, (\ell+1)2^{-m})$. Then, similar to (2.11), we know that, for $j \ge k$ and $y \in B(x, 2^{-m})$,

$$\begin{aligned} |T_{k,j} f(y)| &= |(I - B_{\ell,2^{-k}})(f * \varphi_{2^{-j}})(y)| \\ &\le |f * \varphi_{2^{-j}}(y)| + C_\ell \sum_{i=1}^{\ell} |B_{i2^{-k}}(f * \varphi_{2^{-j}})(y)| \\ &\lesssim M(|f * \varphi_{2^{-j}}| \chi_{B(x,(\ell+1)2^{-m})})(y), \end{aligned}$$

which, together with (2.10) and Lemma 2.4, implies that

$$\begin{aligned} \sum_{k\ge m} 2^{k\alpha q} \left| \sum_{j\ge k} T_{k,j} f(y) \right|^q &\lesssim \sum_{k\ge m} 2^{k\alpha q} \left[\sum_{j\ge k} M(|f * \varphi_{2^{-j}}| \chi_{B(x,(\ell+1)2^{-m})})(y) \right]^q \\ &\lesssim \sum_{j\ge m} 2^{j\alpha q} [M(|f * \varphi_{2^{-j}}| \chi_{B(x,(\ell+1)2^{-m})})(y)]^q. \end{aligned} \tag{2.18}$$

When $m \le j < k$, by (2.13), and using Remark 2.3(i) instead of Lemma 2.2, we find that, for all $k \ge m$, integer $l \ge n + 1$ and $y \in B(x, 2^{-m})$,

$$|T_{k,j} f(y)| \lesssim 2^{2\ell(j-k)} \sum_{i=0}^{\infty} 2^{-i(l-n)} M(f_j \chi_{B(x,2^{i-j}+2^{-m})})(y)$$

and hence, by Lemma 2.4 and the Minkowski inequality, we see that

$$\begin{aligned} &\left\{ \sum_{k\ge m} 2^{k\alpha q} \left| \sum_{j=m}^{k} T_{k,j} f(y) \right|^q \right\}^{1/q} \\ &\lesssim \sum_{i=0}^{\infty} 2^{-i(l-n)} \left\{ \sum_{k\ge m} 2^{k(\alpha-2\ell)q} \left[\sum_{j=m}^{k} 2^{2\ell j} M(f_j \chi_{B(x,(2^i+1)2^{-m})})(y) \right]^q \right\}^{1/q} \\ &\lesssim \sum_{i=0}^{\infty} 2^{-i(l-n)} \left\{ \sum_{j=m}^{\infty} 2^{j\alpha q} [M(f_j \chi_{B(x,(2^i+1)2^{-m})})(y)]^q \right\}^{1/q}. \end{aligned} \tag{2.19}$$

When $j < m \le k$, we invoke (2.13) and the proof of Remark 2.3(i) to find that, for all $y \in \mathbb{R}^n$,

$$\begin{aligned} |T_{k,j}f(y)| &\le \left| \int_{|z-y|<2^{-j}} f_j(z) \int_{\mathbb{R}^n} \widetilde{m}_{k,j}(2^{-j}\xi) e^{i(x-y)\cdot\xi}\, d\xi\, dz \right| + \left| \int_{|z-y|\ge 2^{-j}} \cdots \right| \\ &\lesssim 2^{2\ell(j-k)} \fint_{|z-y|<2^{-j}} |f_j(z)|\, dz \\ &\quad + \int_{|z-y|\ge 2^{-j}} \frac{|f_j(z)|}{|z-y|^l} \int_{\mathbb{R}^n} 2^{-jl} |\nabla^l \widetilde{m}_{k,j}(2^{-j}\xi)|\, d\xi\, dz \\ &\lesssim 2^{2\ell(j-k)} \sum_{i=0}^{\infty} 2^{-i(l-n)} \fint_{|z-y|\sim 2^{i-j}} |f_j(z)|\, dz \\ &\lesssim 2^{2\ell(j-k)} \sum_{i=0}^{\infty} 2^{-i(l-n)} 2^{-j\alpha} \|f\|_{\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)} \\ &\lesssim 2^{2\ell(j-k)} 2^{-j\alpha} \|f\|_{\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)}, \end{aligned} \tag{2.20}$$

where $|z-y| \sim 2^{i-j}$ means that $2^{i-j-1} \le |z-y| < 2^{i-j}$ and we chose $l > n$.

Combining (2.10), (2.18), (2.19) and (2.20), and applying the Minkowski inequality and the boundedness of M on $L^q(\mathbb{R}^n)$ with $q \in (1, \infty]$, we know that, for all $m \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} &\left\{ \fint_{B(x,\,2^{-m})} \sum_{k=m}^{\infty} 2^{k\alpha q} |g(y) - B_{\ell,2^{-k}} g(y)|^q\, dy \right\}^{1/q} \\ &\lesssim \left\{ \fint_{B(x,\,2^{-m})} \sum_{j\ge m} 2^{j\alpha q} [M(|f * \varphi_{2^{-j}}| \chi_{B(x,(\ell+1)2^{-m})})(y)]^q\, dy \right\}^{1/q} \\ &\quad + \sum_{i=0}^{\infty} 2^{-i(l-n)} \left\{ \fint_{B(x,\,2^{-m})} \sum_{j\ge m} 2^{j\alpha q} [M(f_j \chi_{B(x,(2^i+1)2^{-m})})(y)]^q\, dy \right\}^{1/q} \\ &\quad + \left\{ \fint_{B(x,\,2^{-m})} \sum_{k\ge m} 2^{k\alpha q} \left[\sum_{j\le m-1} 2^{2\ell(j-k)} 2^{-j\alpha} \right]^q dy \right\}^{1/q} \|f\|_{\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)} \\ &\lesssim \left\{ \fint_{B(x,\,(\ell+1)2^{-m})} \sum_{j\ge m} 2^{j\alpha q} |f * \varphi_{2^{-j}}(y)|\, dy \right\}^{1/q} \\ &\quad + \sum_{i=0}^{\infty} 2^{-i(l-n)} 2^{in/q} \left\{ \fint_{B(x,(2^i+1)2^{-m})} \sum_{j\ge m} 2^{j\alpha q} |f_j(y)|^q\, dy \right\}^{1/q} + \|f\|_{\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)} \\ &\lesssim \|f\|_{\dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)}, \end{aligned}$$

where we took $l > n(1 + 1/q)$. This proves (2.17).

Finally, the inverse estimate of (2.17) is deduced from an argument similar to that used in the above proof for (2.17), with $\widetilde{m}_{k,j}$ and f_j therein replaced by $\eta := \frac{\widehat{\varphi}}{A_\ell(|\cdot|)}$ and $f - B_{\ell,2^{-j}}f$, respectively. This finishes the proof for the case $\alpha \in (0, 2\ell)$, $p = \infty$ and $q \in (1, \infty]$, and hence Theorem 1.3. □

3. Inhomogeneous spaces and further remarks

In this section, we first present the inhomogeneous version of Theorem 1.3. As a further generalization, we show that the conclusions of Theorems 1.3 and 3.1 remain valid on Euclidean spaces with non-Euclidean metrics.

It is known that, when $p \in (1, \infty)$ and $\alpha \in (0, \infty)$, then $B^\alpha_{p,\,q}(\mathbb{R}^n) \cup F^\alpha_{p,\,q}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$, while when $p = \infty$ and $\alpha \in (0, \infty)$, then $B^\alpha_{\infty,\,q}(\mathbb{R}^n) \cup F^\alpha_{\infty,\,q}(\mathbb{R}^n) \subset C(\mathbb{R}^n)$, where $C(\mathbb{R}^n)$ denotes the set of all complex-valued uniformly continuous functions on $\mathbb{R}^n$ equipped with the sup-norm; see, for example, [22, Theorem 3.3.1] and [20, Chapter 2.4, Corollary 2].

Theorem 3.1. *Let $\ell \in \mathbb{N}$ and $\alpha \in (0, 2\ell)$.*

(i) *Let $q \in (0, \infty]$. Then $f \in B^\alpha_{p,q}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ when $p \in (1, \infty)$ or $f \in C(\mathbb{R}^n)$ when $p = \infty$, and*

$$|||f|||_{B^\alpha_{p,q}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \left\{ \sum_{k=1}^\infty 2^{k\alpha q} \|f - B_{\ell,2^{-k}}f\|^q_{L^p(\mathbb{R}^n)} \right\}^{1/q} < \infty.$$

Moreover, $|||\cdot|||_{B^\alpha_{p,q}(\mathbb{R}^n)}$ is equivalent to $\|f\|_{B^\alpha_{p,q}(\mathbb{R}^n)}$.

(ii) *Let $p \in (1, \infty]$ and $q \in (1, \infty]$. Then $f \in F^\alpha_{p,q}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ when $p \in (1, \infty)$ or $f \in C(\mathbb{R}^n)$ when $p = \infty$, and $|||f|||_{F^\alpha_{p,q}(\mathbb{R}^n)} < \infty$, where, when $p \in (1, \infty)$,*

$$|||f|||_{F^\alpha_{p,q}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \left\| \left\{ \sum_{k=1}^\infty 2^{k\alpha q} |f - B_{\ell,2^{-k}}f|^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)}$$

and, when $p = \infty$,

$$|||f|||_{F^\alpha_{\infty,q}(\mathbb{R}^n)} := \|f\|_{L^\infty(\mathbb{R}^n)} + \sup_{x\in\mathbb{R}^n} \sup_{m\ge1} \left\{ \fint_{B(x,\,2^{-m})} \sum_{k=m}^\infty 2^{k\alpha q} |f(y) - B_{\ell,2^{-k}}f(y)|^q \, dy \right\}^{1/q}.$$

Moreover, $|||\cdot|||_{F^\alpha_{p,q}(\mathbb{R}^n)}$ is equivalent to $\|\cdot\|_{F^\alpha_{p,q}(\mathbb{R}^n)}$.

Proof. By similarity, we only consider (ii). The proof is similar to that of Theorem 1.3, and we mainly describe the difference. We need to use the following well-known result: when $\alpha \in (0, \infty)$ and $p, q \in (1, \infty]$, then, for all $f \in F^\alpha_{p,q}(\mathbb{R}^n)$,

$$\|f\|_{F^\alpha_{p,q}(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)} + \widetilde{\|f\|}_{F^\alpha_{p,q}(\mathbb{R}^n)}, \tag{3.1}$$

where $\widetilde{\|f\|}_{F^\alpha_{p,q}(\mathbb{R}^n)}$ is defined as $\|f\|_{F^\alpha_{p,q}(\mathbb{R}^n)}$ in Definition 1.2 with $k \in \mathbb{Z}_+$ and $m \in \mathbb{Z}_+$ therein replaced, respectively, by $k \in \mathbb{N}$ and $m \in \mathbb{N}$ (which can be easily seen from [22, Theorem 3.3.1] and [20, Chapter 2.4, Corollary 2]).

Assume first that $f \in F^\alpha_{p,q}(\mathbb{R}^n)$. By [22, Theorem 3.3.1] and [20, Chapter 2.4, Corollary 2], we know that $f \in L^p(\mathbb{R}^n)$ when $p \in (1, \infty)$ or $f \in C(\mathbb{R}^n)$ when $p = \infty$. On the other hand, repeating the proof of Theorem 1.3, we see that, when $p \in (1, \infty)$,

$$\left\| \left\{ \sum_{k=1}^{\infty} 2^{k\alpha q} |f - B_{\ell,2^{-k}} f|^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{F^\alpha_{p,q}(\mathbb{R}^n)}$$

and, when $p = \infty$,

$$\sup_{x\in\mathbb{R}^n} \sup_{m\ge 1} \left\{ \fint_{B(x,\,2^{-m})} \sum_{k=m}^{\infty} 2^{k\alpha q} |f(y) - B_{\ell,2^{-k}} f(y)|^q \, dy \right\}^{1/q} \lesssim \|f\|_{F^\alpha_{\infty,q}(\mathbb{R}^n)}$$

which show $|||f|||_{F^\alpha_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{F^\alpha_{p,q}(\mathbb{R}^n)}$.

Conversely, assume that $f \in L^p(\mathbb{R}^n)$ when $p \in (1,\infty)$ or $f \in C(\mathbb{R}^n)$ when $p = \infty$, and $|||f|||_{F^\alpha_{p,q}(\mathbb{R}^n)} < \infty$. Again the proof of Theorem 1.3 shows that, when $p \in (1,\infty)$,

$$\widetilde{\|f\|}_{F^\alpha_{p,q}(\mathbb{R}^n)} \lesssim \left\| \left\{ \sum_{k=1}^{\infty} 2^{k\alpha q} |f - B_{\ell,2^{-k}} f|^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)}$$

and, when $p = \infty$,

$$\widetilde{\|f\|}_{F^\alpha_{\infty,q}(\mathbb{R}^n)} \lesssim \sup_{x\in\mathbb{R}^n} \sup_{m\ge 1} \left\{ \fint_{B(x,\,2^{-m})} \sum_{k=m}^{\infty} 2^{k\alpha q} |f(y) - B_{\ell,2^{-k}} f(y)|^q \, dy \right\}^{1/q} \lesssim \|f\|_{F^\alpha_{\infty,q}(\mathbb{R}^n)}.$$

This, together with (3.1), further implies that $\|f\|_{F^\alpha_{\infty,q}(\mathbb{R}^n)} \lesssim |||f|||_{F^\alpha_{\infty,q}(\mathbb{R}^n)}$, and hence finishes the proof of Theorem 3.1. □

Finally, we point out that the *conclusions of Theorems 1.3 and 3.1 are independent of the choice of the metric in* $\mathbb{R}^n$. To be precise, let $\|\cdot\|$ be a norm in $\mathbb{R}^n$, which is not necessarily the usual Euclidean norm. Then $(\mathbb{R}^n, \|\cdot\|)$ is a finite dimensional normed vector space with the unit ball

$$K := \{x \in \mathbb{R}^n : \ \|x\| \le 1\}.$$

Clearly, K is a compact and symmetric convex set in $\mathbb{R}^n$ satisfying that $-K = K$ and $B(0,\delta_1) \subset K \subset B(0,\delta_2)$ for some $\delta_1, \delta_2 \in (0,\infty)$.

For all $\ell \in \mathbb{N}$, $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define

$$B_{\ell,t,K} f(x) := \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} B_{jt,K} f(x).$$

Then we have the following conclusion.

Theorem 3.2. *The conclusions of Theorems 1.3 and 3.1 remain valid with $B_{\ell,t}$ therein replaced by $B_{\ell,t,K}$.*

Since the proof of Theorem 3.2 is essentially similar to the proofs of Theorems 1.3 and 3.1, we only describe the main differences, the other details being omitted.

We first observe that

$$(B_{\ell,t,K} f)^\wedge(\xi) := m_{\ell,K}(t\xi)\widehat{f}(\xi), \qquad \xi \in \mathbb{R}^n,$$

where

$$m_{\ell,K}(x) := \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} \widehat{I_K}(jx), \quad x \in \mathbb{R}^n.$$

Similar to the proof of Lemma 2.1, by means of the symmetry property of K, a straightforward calculation shows that, for all $x \in \mathbb{R}^n$,

$$m_{\ell,K}(x) = \fint_K \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} \cos(jx \cdot y)\, dy =: 1 - A_{\ell,K}(x),$$

where

$$A_{\ell,K}(x) := \frac{4^\ell}{\binom{2\ell}{\ell}} \fint_K \left(\sin \frac{x \cdot u}{2}\right)^{2\ell} du.$$

Furthermore, we have the following estimates: for all $x \in \mathbb{R}^n$ with $|x| \le 4$,

$$0 < C_1 \le \frac{A_{\ell,K}(x)}{|x|^{2\ell}} \le C_2 \tag{3.2}$$

and

$$|\nabla^i A_{\ell,K}(x)| \le C \min\{|x|^{2\ell - i}, 1\}, \quad i \in \mathbb{N}, \tag{3.3}$$

where C_1, C_2 and C are positive constants independent of x. Similar to the proof of Lemma 2.2, by (3.2) and (3.3), we observe that

$$\sup_{t \in (0,\infty)} \fint_K |f(x + ty)|\, dy \lesssim Mf(x)$$

for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

Finally, notice that, by the equivalence of norms on finite-dimensional vector spaces, the spaces $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$, $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ and their inhomogeneous counterparts are essentially independent of the choice of the norm of the underlying space $\mathbb{R}^n$. By means of this observation and using (3.2), (3.3) in place of Lemma 2.1, we obtain Theorem 3.2 via some arguments similar to those used in the proofs of Theorems 1.3 and 3.1, the details being omitted.

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