On measure valued solutions to the compressible Euler equations

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Euler Equations

We consider the compressible isentropic Euler system

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho v) = 0 \\ \partial_t(\rho v) + \mathrm{div}_x (\rho v \otimes v) + \nabla_x [p(\rho)] = 0 \\ \rho(\cdot, 0) = \rho^0 \\ v(\cdot, 0) = v^0 . \end{cases} \tag{1}$$

Unknowns:

- $\rho(x,t)$ ... density
- $v(x,t)$ ... velocity

The pressure $p(\rho)$ is given.

Weak solutions

$$\int_0^T \int_\Omega \partial_t \psi \rho + \nabla_x \psi \cdot \rho v \mathrm{d}x\mathrm{d}t + \int_\Omega \psi(0,x)\rho_0(x)\mathrm{d}x = 0,$$

$$\int_0^T \int_\Omega \partial_t \varphi \cdot \rho v + \nabla_x \varphi : \rho v \otimes v + \mathrm{div}_x \varphi p(\rho) \mathrm{d}x\mathrm{d}t$$
$$+ \int_\Omega \varphi(0,x) \cdot \rho_0(x) v_0(x) \mathrm{d}x = 0$$

for all $\psi \in C_c^\infty([0,T) \times \Omega)$ and all $\varphi \in C_c^\infty([0,T) \times \Omega; \mathbb{R}^3)$

Properties

- Weak solutions nonunique $\Rightarrow$ Admissibility criteria $\Rightarrow$ Entropy conditions
- Existence of entropy weak solutions in general in multi-D not known
- What is correct notion of solution still unclear (ill-posedness results of entropy weak solutions by Chiodaroli, De Lellis and K.)
- Measure-valued solutions (MVS) introduced by DiPerna for general systems of conservation laws
- Existence of MVS for compressible Euler by Neustupa
- MVS criticized for being too weak, on the other hand may be useful in the weak-strong uniqueness results (Feireisl, Gwiazda, Świerczewska-Gwiazda, Wiedemann)
- Numerical schemes may not converge to entropy solutions, MVS are suggested instead (Fjordholm, Käppeli, Mishra, Tadmor)

Incompressible Euler system

Theorem 1 (Székelyhidi, Wiedemann, 2012)

Any measure-valued solution to the incompressible Euler system can be approximated by a sequence of weak solutions

This means that MVS are not substatially weaker than weak solutions, i.e. weak solutions are too weak.

What is the situation in the compressible case?

Young measures

For simplicity we work only with bounded measure-valued solutions - we ignore the effects of concentrations and avoid using generalized Young measures.
Young measure: map $\nu \in L^\infty_w(\Omega; \mathcal{M}^1(\mathbb{R}^d))$ ... assigns to almost every point $x \in \Omega$ a probability measure $\nu_x \in \mathcal{M}^1(\mathbb{R}^d)$ on the phase space $\mathbb{R}^d$.
Denote $\langle \nu_x, f \rangle := \int_{\mathbb{R}^d} f(z) \mathrm{d}\nu_x(z)$ ... the expectation of f with respect to the probability measure ν_x.

Measure valued solutions

In our context the domain takes the form $[0, T] \times \Omega$ and the phase space (we work in 3D) is $\mathbb{R}^+ \times \mathbb{R}^3$.
Denote the state variables by $\xi \in \mathbb{R}^+ \times \mathbb{R}^3$ and introduce

$$\xi = [\xi_0, \xi'] = [\xi_0, \xi_1, \xi_2, \xi_3] \in \mathbb{R}^+ \times \mathbb{R}^3$$

$$\langle \nu_{t,x}, \xi_0 \rangle = \overline{\rho}$$

$$\langle \nu_{t,x}, \sqrt{\xi_0}\xi' \rangle = \overline{\rho v}$$

$$\langle \nu_{t,x}, \xi' \otimes \xi' \rangle = \overline{\rho v \otimes v}$$

$$\langle \nu_{t,x}, p(\xi_0) \rangle = \overline{p(\rho)}.$$

In such a way ξ_0 is the state of the density ρ and ξ' is the state of $\sqrt{\rho} v$

Measure valued solutions II

Definition 2 (Measure-valued solution)

A *measure-valued solution to the compressible Euler equations* (1) is a Young measure $\nu_{t,x}$ on $\mathbb{R}^+ \times \mathbb{R}^3$ with parameters in $[0, T] \times \Omega$ which satisfies the Euler equations in an average sense, i.e.

$$\int_0^T \int_\Omega \partial_t \psi \overline{\rho} + \nabla_x \psi \cdot \overline{\rho v} \mathrm{d}x\mathrm{d}t + \int_\Omega \psi(0,x)\rho_0(x)\mathrm{d}x = 0$$

$$\int_0^T \int_\Omega \partial_t \varphi \cdot \overline{\rho v} + \nabla_x \varphi : \overline{\rho v \otimes v} + \mathrm{div}_x \varphi \overline{p(\rho)} \mathrm{d}x\mathrm{d}t$$
$$+ \int_\Omega \varphi(0,x) \cdot \rho_0(x) v_0(x) \mathrm{d}x = 0$$

for all $\psi \in C_c^\infty([0,T) \times \Omega)$ and all $\varphi \in C_c^\infty([0,T) \times \Omega; \mathbb{R}^3)$.

Some remarks

- Every weak solution defines naturally an atomic measure valued solution $\nu_{t,x} := \delta_{\rho(t,x),\sqrt{\rho}v(t,x)}$.
- We say that sequence $\{z_n\}$ *generates* the Young measure ν if for all bounded Carathéodory functions $f : \widetilde{\Omega} \times \mathbb{R}^d \to \mathbb{R}$

$$\lim_{n\to\infty} \int_{\widetilde{\Omega}} f(y, z_n(y))\varphi(y)\mathrm{d}y = \int_{\widetilde{\Omega}} \langle \nu_y, f(y,\cdot)\rangle \varphi(y)\mathrm{d}y$$

  for all $\varphi \in L^1(\widetilde{\Omega})$.
- Any sequence of functions bounded in $L^p(\Omega)$ (for any $p \geq 1$) generates, up to a subsequence, some Young measure [Fundamental theorem of Young measures].

Subsolutions

In order to formulate our first Theorem we need to define subsolutions. As usual we take the linearized system

$$\begin{aligned} \partial_t \rho + \operatorname{div}_x m &= 0 \\ \partial_t m + \operatorname{div}_x U + \nabla_x q &= 0, \end{aligned} \tag{2}$$

associated to the compressible Euler system. Here, as usual, $U \in S_0^3$ is a symmetric trace-free 3×3 matrix which replaces the traceless part of the matrix $\rho v \otimes v = \frac{m \otimes m}{\rho}$.
Weak solutions to (2) are functions (ρ, m, U, q) which satisfy (2) in the sense of distributions.

Measure valued subsolutions

We use the following notation:

$$
\begin{aligned}
[\zeta_0, \zeta', \mathbf{Z}, \tilde{\zeta}] &\in \mathbb{R}^+ \times \mathbb{R}^3 \times S_0^3 \times \mathbb{R}^+ \\
\langle \mu_{t,x}, \zeta_0 \rangle &= \overline{\rho} \\
\langle \mu_{t,x}, \zeta' \rangle &= \overline{m} \\
\langle \mu_{t,x}, \mathbf{Z} \rangle &= \overline{U} \\
\langle \mu_{t,x}, \tilde{\zeta} \rangle &= \overline{q}
\end{aligned}
$$

Measure values subsolutions II

Definition 3 (Measure valued subsolution)

A *measure-valued solution to the linear system* is a Young measure $\mu_{t,x}$ on $\mathbb{R}^+ \times \mathbb{R}^3 \times S_0^3 \times \mathbb{R}^+$ with parameters in $[0, T] \times \Omega$ which satisfies the linear system (2) in an average sense, i.e.

$$\int_0^T \int_\Omega \partial_t \psi \overline{\rho} + \nabla_x \psi \cdot \overline{m} \mathrm{d}x\mathrm{d}t + \int_\Omega \psi(0,x)\rho_0(x)\mathrm{d}x = 0$$

$$\int_0^T \int_\Omega \partial_t \varphi \cdot \overline{m} + \nabla_x \varphi : \overline{U} + \mathrm{div}\,_x \varphi \overline{q} \mathrm{d}x\mathrm{d}t + \int_\Omega \varphi(0,x) \cdot m_0(x)\mathrm{d}x = 0$$

for all $\psi \in C_c^\infty([0, T) \times \Omega)$ and all $\varphi \in C_c^\infty([0, T) \times \Omega; \mathbb{R}^3)$.

Lift

Similarly, as are subsolutions connected to solutions as functions, we need a procedure to connect measure valued solutions and subsolutions.

Definition 4 (Lift)

Let $\nu_{t,x}$ be a measure valued solution to the Euler equations. Denote $Q : \mathbb{R}^+ \times \mathbb{R}^3 \mapsto \mathbb{R}^+ \times \mathbb{R}^3 \times S_0^3 \times \mathbb{R}^+$

$$Q(\xi) := (\xi_0, \sqrt{\xi_0}\xi', \xi' \otimes \xi' - \frac{1}{3}\left|\xi'\right|^2 \mathbf{I}, p(\xi_0) + \frac{1}{3}\left|\xi'\right|^2).$$

We define the *lifted measure* $\tilde{\nu}_{t,x}$ as

$$\langle \tilde{\nu}_{t,x}, f \rangle := \langle \nu_{t,x}, f \circ Q \rangle$$

for $f \in C_0(\mathbb{R}^+ \times \mathbb{R}^3 \times S_0^3 \times \mathbb{R}^+)$ and a.e. (t, x).

Tartar framework

The linear system (2) fits into the so-called $\mathcal{A}$-free framework for linear partial differential constraints, introduced by Tartar. Consider a general linear system of l differential equations in $\mathbb{R}^N$ written as

$$\mathcal{A}z := \sum_{i=1}^{N} A^{(i)} \frac{\partial z}{\partial x_i} = 0, \tag{3}$$

where $A^{(i)}$ $(i = 1, ..., N)$ are $l \times d$ matrices and $z : \mathbb{R}^N \to \mathbb{R}^d$ is a vector-valued function.

We will present the formulation of the linear system (2) in the Tartar framework in a moment.

Constant rank property

Next, we define the $l \times d$ matrix

$$\mathbb{A}(w) := \sum_{i=1}^{N} w_i A^{(i)}$$

for $w \in \mathbb{R}^N$.

Definition 5 (Constant rank)

We say that $\mathcal{A}$ has the *constant rank property* if there exists $r \in \mathbb{N}$ such that

$$\operatorname{rank} \mathbb{A}(w) = r$$

for all $w \in \mathcal{S}^{N-1}$.

Definition 6 ($\mathcal{A}$-Quasiconvexity)

A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be $\mathcal{A}$-*quasiconvex* if

$$f(z) \leq \int_{(0,1)^N} f(z + w(x))\mathrm{d}x \tag{4}$$

for all $z \in \mathbb{R}^d$ and all $w \in C^\infty_{per}((0,1)^N; \mathbb{R}^d)$ such that $\mathcal{A}w = 0$ and $\int_{(0,1)^N} w(x)\mathrm{d}x = 0$.

Finally recall that a sequence $\{z_n\}$ is called *p-equiintegrable* if the sequence $\{|z_n|^p\}$ is equiintegrable in the usual sense.

Theorem of Fonseca and Müller

Theorem 7

Let $1 \leq p < \infty$ and let $\{\nu_x\}_{x\in\Omega}$ be a weakly measurable family of probability measures on $\mathbb{R}^d$. Let $\mathcal{A}$ have the constant rank property. There exists a p-equi-integrable sequence $\{z_n\}$ in $L^p(\Omega;\mathbb{R}^d)$ that generates the Young measure ν and satisfies $\mathcal{A}z_n = 0$ in Ω if and only if the following conditions hold:

(i) *there exists $z \in L^p(\Omega;\mathbb{R}^d)$ such that $\mathcal{A}z = 0$ and $z(x) = \langle \nu_x, \mathrm{id}\rangle$ a.e. $x \in \Omega$;*

(ii) $\int_\Omega \int_{\mathbb{R}^d} |w|^p \, \mathrm{d}\nu_x(w)\mathrm{d}x < \infty$;

(iii) *for a.e. $x \in \Omega$ and all $\mathcal{A}$-quasiconvex functions g that satisfy $|g(w)| \leq C(1 + |w|^p)$ for some $C > 0$ and all $w \in \mathbb{R}^d$ one has*

$$\langle \nu_x, g\rangle \geq g(\langle \nu_x, \mathrm{id}\rangle). \tag{5}$$

Our first main theorem is as follows

Theorem 8

Suppose the pressure function satisfies $c\rho^\gamma \leq p(\rho) \leq C\rho^\gamma$ *for some* $\gamma \geq 1$ *and* $\{(\rho_n, v_n)\}$ *is a sequence of weak solutions to the compressible Euler system* (1) *such that* $\{\rho_n\}$ *is* γ*-equiintegrable and* $\{\sqrt{\rho_n} v_n\}$ *is 2-equiintegrable. Suppose moreover* $\{(\rho_n, \sqrt{\rho_n} v_n)\}$ *generates a Young measure* ν *on* $\mathbb{R}^+ \times \mathbb{R}^3$*. Then* ν *is a measure-valued solution to the compressible Euler system* (1) *and the lifted measure* $\tilde{\nu}$ *on* $\mathbb{R}^+ \times \mathbb{R}^3 \times \mathcal{S}_0^3 \times \mathbb{R}^+$ *satisfies*

$$\langle \tilde{\nu}_{t,x}, g\rangle \geq g(\langle \tilde{\nu}_{t,x}, \mathrm{id}\rangle) \tag{6}$$

for all $\mathcal{A}_L$*-quasiconvex functions* g*.*

Linear system in Tartar framework

Denote $t = x_0$. We rewrite the linear system (2) in Tartar framework as follows. We define the state vector

$$z := (\rho, m_1, m_2, m_3, U_{11}, U_{12}, U_{13}, U_{22}, U_{23}, q) \in \mathbb{R}^{10}.$$

Accordingly, the 4×10 matrices $A_L^{(i)}$ for $i = 0, ..., 3$ have the following form

$$A_L^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (7)$$

$$A_L^{(1)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad (8)$$

$$A_L^{(2)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad (9)$$

$$A_L^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}. \quad (10)$$

Constant rank for Euler

With this notation, the system (2) takes the form $\mathcal{A}_L z := \sum_{i=0}^{3} A_L^{(i)} \frac{\partial z}{\partial x_i} = 0$. We can prove the following lemma.

Lemma 9

The operator $\mathcal{A}_L$ *defined as* $\mathcal{A}_L z := \sum_{i=0}^{3} A_L^{(i)} \frac{\partial z}{\partial x_i}$, *with the choice of matrices* $A_L^{(i)}$ *given by* (7)–(10), *has the constant rank property with* $r = 4$.

This follows from an easy linear algebra computation.

Proof of Theorem 8 I

The proof of Theorem 8 is a direct consequence of the general Theorem 7 of Fonseca and Müller. We proceed in the following steps

Step 1: The fact that the Young measure ν is a measure valued solution to the compressible Euler equations (1) is a direct consequence of the Fundamental Theorem of Young measures and the lack of concentrations implied by the equiintegrability assumptions.

Step 2: The sequence $\{(\rho_n, v_n)\}$ naturally gives rise to a sequence of weak solutions $\{z_n\} = \{\rho_n, m_n, U_n, q_n\}$ to the linear system (2), by defining

$$m_n = \rho_n v_n, \quad U_n := \rho_n v_n \otimes v_n - \frac{1}{3}\rho_n |v_n|^2 \mathbf{I}, \quad q_n := p(\rho_n) + \frac{1}{3}\rho_n |v_n|^2$$

where $\mathbf{I}$ is the 3×3 identity matrix.

Step 3: We show that the lifted measure $\tilde{\nu}$ is generated by the sequence $\{z_n\}$. This is an easy consequence of the definition of the lifted measure and the fact that ν is generated by $\{(\rho_n, \sqrt{\rho_n} v_n)\}$:

$$\begin{aligned}
&\int_0^T \int_\Omega \langle \tilde{\nu}_{t,x}, g(t,x,\cdot)\rangle \varphi(x,t) \mathrm{d}x\mathrm{d}t \\
&\qquad = \int_0^T \int_\Omega \langle \nu_{t,x}, (g \circ Q)(t,x,\cdot)\rangle \varphi(x,t) \mathrm{d}x\mathrm{d}t \\
&\qquad = \lim_{n\to\infty} \int_0^T \int_\Omega g(t,x,Q(\rho_n, \sqrt{\rho_n} v_n)(t,x)) \varphi(x,t) \mathrm{d}x\mathrm{d}t \\
&\qquad = \lim_{n\to\infty} \int_0^T \int_\Omega g(t,x,z_n(t,x)) \varphi(x,t) \mathrm{d}x\mathrm{d}t
\end{aligned}$$

for all test functions φ.

Proof of Theorem 8 III

Step 4: Applying Theorem 7 with the choice $\mu = \tilde{\nu}$, we obtain that $\tilde{\nu}$ is a measure valued solution to the linear system (2) and it satisfies the Jensen inequality (6).

The Theorem is proved.

Wave analysis

Our final aim is to give an example of a measure valued solution which cannot be generated by weak solutions. In fact we first prove a more general statement about $\mathcal{A}$-free rigidity which generalizes a well known result by Ball and James. Then we use this result to construct a desired example.
We start with some more definitions.

Wave cone

Definition 10

Consider a linear differential operator $\mathcal{A}$ as in (3). Its *wave cone* Λ is defined as the set of all $\bar{z} \in \mathbb{R}^d \setminus \{0\}$ for which there exists $\xi \in \mathbb{R}^N \setminus \{0\}$ such that

$$z(x) = h(x \cdot \xi)\bar{z}$$

satisfies $\mathcal{A}z = 0$ for any choice of profile function $h : \mathbb{R} \to \mathbb{R}$. Equivalently, $\bar{z} \in \Lambda$ if and only if $\bar{z} \neq 0$ and there exists $\xi \in \mathbb{R}^N \setminus \{0\}$ such that $\mathbb{A}(\xi)\bar{z} = 0$.

The wave cone Λ characterizes the directions of one dimensional oscillations compatible with (3).

Reformulation

Observe that

$$\sum_{i=1}^{N} A^{(i)} \frac{\partial z}{\partial x_i} = \left(\sum_{i=1}^{N} \sum_{k=1}^{d} A^{(i)}_{jk} \frac{\partial z_k}{\partial x_i} \right)_{j=1,\dots,l} = \left(\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \sum_{k=1}^{d} A^{(i)}_{jk} z_k \right)_{j=1,\dots,l}.$$

Therefore, if we define the $l \times N$-matrix $Z_{\mathcal{A}}$ by

$$(Z_{\mathcal{A}})_{ji} = \sum_{k=1}^{d} A^{(i)}_{jk} z_k, \qquad j = 1, \dots, l, \qquad i = 1, \dots, N, \tag{11}$$

then (3) can be rewritten as

$$\operatorname{div} Z_{\mathcal{A}} = 0. \tag{12}$$

Reformulation II

Moreover, the condition $\mathbb{A}(\xi)\bar{z} = 0$ from the definition of the wave cone translates to $\bar{Z}_{\mathcal{A}}\xi = 0$ (where $\bar{Z}_{\mathcal{A}}$ is obtained from $\bar{z}$ via (11)), so that the following are equivalent:

1. $\bar{z} \in \Lambda$;
2. $\bar{z} \neq 0$ and $\operatorname{rank} \bar{Z}_{\mathcal{A}} < N$.

It follows immediately that $\Lambda = \mathbb{R}^d \setminus \{0\}$ if $l < N$.

Rigidity theorem, part I

Theorem 11

Let $\Omega \subset \mathbb{R}^N$ *be a domain,* $\mathcal{A}$ *a linear operator of the form* (3)*, and* $1 < p < \infty$*. Let moreover* $z_n : \Omega \to \mathbb{R}^d$ *be a family of functions with*

$$\|z_n\|_{L^p(\Omega;\mathbb{R}^d)} \leq c,$$

$$\mathcal{A} z_n = 0 \text{ in } \mathcal{D}'(\Omega), \tag{13}$$

and suppose (z_n) *generates a compactly supported Young measure* $\nu_x \in \mathcal{M}^1(\mathbb{R}^d)$ *such that*

$$\mathrm{supp}[\nu_x] \subset \{\lambda \bar{z}_1 + (1-\lambda)\bar{z}_2,\ \lambda \in [0,1]\} \text{ for a.a. } x \in \Omega \tag{14}$$

and for some given constant states $\bar{z}_1, \bar{z}_2 \in \mathbb{R}^d$, $\bar{z}_1 \neq \bar{z}_2$*. Suppose that*

$$\bar{z}_2 - \bar{z}_1 \notin \Lambda.$$

Rigidity theorem, part II

Theorem 11 (cont.)

Then

$$z_n \to z_\infty \quad \textit{in } L^p(\Omega),$$

which implies that

$$\nu_x = \delta_{z_\infty(x)}, \; z_\infty(x) \in \{\lambda \bar{z}_1 + (1-\lambda)\bar{z}_2, \; \lambda \in [0,1]\} \textit{ for a.a. } x \in \Omega.$$

More specifically, z_∞ *is a constant function of the form*

$$z_\infty = \overline{\lambda}\bar{z}_1 + (1-\overline{\lambda})\bar{z}_2.$$

for some fixed $\overline{\lambda} \in [0,1]$.

Explicit example

Using the previous Theorem 11 we prove the following

Theorem 12

There exists a measure-valued solution of the compressible Euler system (1) *which is not generated by any sequence of* L^p*-bounded weak solutions to* (1) *(for any choice of* $p > 1$*).*

- Any reasonable sequence of approximate solutions of (1) will satisfy some uniform energy bound, so that the assumption of L^p-boundedness will always be met.
- As Theorem 11 did not require any equiintegrability, the statement of Theorem 12 is true even when the potential generating sequence is allowed to concentrate. I.e. there exists a generalized measure-valued solution which can not be generated by a sequence of weak solutions (take the measure from Theorem 12 as the oscillation part and choose the concentration part arbitrarily).

Proof of Theorem 12 I

To each state vector

$$z := (\rho, m_1, m_2, m_3, U_{11}, U_{12}, U_{13}, U_{22}, U_{23}, q) \in \mathbb{R}^{10}$$

we associate the 4×4 matrix $Z_{\mathcal{A}_L}$ given as

$$Z_{\mathcal{A}_L} = \begin{pmatrix} \rho & m_1 & m_2 & m_3 \\ m_1 & U_{11}+q & U_{12} & U_{13} \\ m_2 & U_{12} & U_{22}+q & U_{23} \\ m_3 & U_{13} & U_{23} & -U_{11}-U_{22}+q \end{pmatrix}. \qquad (15)$$

Hence, the wave cone for the operator $\mathcal{A}_L$ is equal to

$$\Lambda_L = \left\{ \bar{z} \in \mathbb{R}^{10} \text{ such that } \det(\bar{Z}_{\mathcal{A}_L}) = 0 \right\}$$

Moreover, by (12), the linear system (2) can be written

$$\mathrm{div}_{\,t,x} Z_{\mathcal{A}_L} = 0.$$

Proof of Theorem 12 II

We choose suitably $z^1 := (\rho^1, m^1, U^1, q^1)$ and $z^2 := (\rho^2, m^2, U^2, q^2)$ such that the homogeneous Young measure

$$\tilde{\nu} = \frac{1}{2}\delta_{z^1} + \frac{1}{2}\delta_{z^2}$$

cannot be a limit of bounded weak solutions to the linear system (2).
Set

- $\rho^1 = 1,\ m^1 = e_1,\ U^1 = \mathrm{diag}(2/3, -1/3, -1/3),\ q^1 = p(1) + \frac{1}{3}$
- $\rho^2 = \gamma,\ m^2 = e_1,\ U^2 = U^1/\gamma,\ q^2 = p(\gamma) + \frac{1}{3}\gamma$ for some $\gamma > 0$.

Notice that with this choice of z^1 and z^2, $\tilde{\nu}$ arises as the lifted Young measure of some measure-valued solution ν to (1)

Proof of Theorem 12 III

Observe that $\tilde{z} := z^2 - z^1$ is not in the wave cone for the operator $\mathcal{A}_L$. Indeed, the determinant of the matrix $\tilde{Z}$ is

$$\left(1 - \frac{1}{\gamma} + p(1) - p(\gamma)\right)(p(1) - p(\gamma))^2.$$

Finally, apply Theorem 11 with the choice $\bar{z}_1 := z^1$ and $\bar{z}_2 := z^2$. We easily see that $\tilde{\nu}$ cannot be generated by any sequence of subsolutions.

Moreover $\tilde{\nu}$ arises as the lifting of a measure valued solution ν to the original compressible Euler equations (1) of the form

$$\nu = \frac{1}{2}\delta_{\left(\rho^1, \frac{m^1}{\sqrt{\rho^1}}\right)} + \frac{1}{2}\delta_{\left(\rho^2, \frac{m^2}{\sqrt{\rho^2}}\right)}$$

which, as a consequence, cannot be generated by any sequence of weak solutions to (1), since this would contradict what we have shown at the level of subsolutions. This proves Theorem 12.

Proof of Theorem 11

Finally let us sketch the proof of Theorem 11.
We know that

$$z_n \rightharpoonup z_\infty \text{ weakly in } L^p(\Omega; \mathbb{R}^d),$$

so it is enough to show that $\{z_n\}_{n\geq 1}$ contains a subsequence converging in $L^p(\Omega)$.
In the first step we observe, that z_n generating Young measure satisfying (14) have to have a specific form, more precisely

$$z_n(x) = e_n(x) + \lambda_n(x)\bar{z}_1 + (1 - \lambda_n(x))\bar{z}_2, \tag{16}$$

where

$$e_n \to 0 \text{ in } L^p(\Omega; \mathbb{R}^d) \text{ as } n \to \infty,$$

and λ_n are bounded measurable functions

$$0 \leq \lambda_n \leq 1 \text{ a.e. in } \Omega.$$

Proof of Theorem 11 II

Using (16) we rewrite the condition $\mathcal{A}z_n = 0$ as

$$0 = \operatorname{div} E_n(x) + \operatorname{div}[\lambda_n(x)(\bar{Z}_1 - \bar{Z}_2)]$$

$$= \operatorname{div} E_n(x) - (\bar{Z}_2 - \bar{Z}_1)\nabla\lambda_n(x) \text{ in } \mathcal{D}'(\Omega).$$

Now we regularize by a suitable family of convolution kernels to obtain

$$(\bar{Z}_2 - \bar{Z}_1)\nabla(\lambda_n)_\varepsilon = \operatorname{div}(E_n)_\varepsilon \equiv \chi_{n,\varepsilon}, \tag{17}$$

where

$$\|\chi_{n,\varepsilon}\|_{W^{-1,p}(\Omega;\mathbb{R}^d)} \leq c_1(n) \to 0 \text{ as } n \to \infty \text{ independently of } \varepsilon. \tag{18}$$

Proof of Theorem 11 III

Since $(\bar{Z}_2 - \bar{Z}_1)$ has full rank, because $\bar{z}_2 - \bar{z}_1 \notin \Lambda$, we can multiply (17) by its inverse to obtain

$$\|\nabla(\lambda_n - \langle\lambda_n\rangle)_\varepsilon\|_{W^{-1,p}(\Omega;\mathbb{R}^d)} \leq c_2(n) \to 0;$$

whence

$$\|\nabla(\lambda_n - \langle\lambda_n\rangle)\|_{W^{-1,p}(\Omega;\mathbb{R}^d)} \leq c_2(n);$$

and finally (by Nečas' Lemma),

$$\|\lambda_n - \langle\lambda_n\rangle\|_{L^p(\Omega;\mathbb{R}^d)} \to 0 \text{ as } n \to \infty$$

where

$$\langle\lambda_n\rangle := \frac{1}{|\Omega|}\int_\Omega \lambda_n(x)\mathrm{d}x.$$

This implies not only the strong convergence of $\{z_n\}$ but also a convergence to a constant function. The theorem is proved.

Thank you

Thank you for your attention.