Computing lower bounds on eigenvalues of elliptic operators

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Academy of Mathematics and Systems Science, CAS, Beijing, October 17, 2016

Laplace eigenvalue problem

Classical formulation

$$\begin{aligned} -\Delta u_i &= \lambda_i u_i \quad &&\text{in } \Omega \\ u_i &= 0 &&\text{on } \partial\Omega \end{aligned}$$

Countable sequence of eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$$

Weak formulation

$$\lambda_i \in \mathbb{R},\ 0 \neq u_i \in H_0^1(\Omega) : \quad (\nabla u_i, \nabla v) = \lambda_i (u_i, v) \quad \forall v \in H_0^1(\Omega)$$

Finite element method

$$V_h = \{v_h \in H_0^1(\Omega) : v_h|_K \in P_1(K),\ \forall K \in \mathcal{T}_h\}$$

$$\Lambda_{h,i} \in \mathbb{R},\ 0 \neq u_{h,i} \in V_h : \quad (\nabla u_{h,i}, \nabla v_h) = \Lambda_{h,i} (u_{h,i}, v_h) \quad \forall v_h \in V_h$$

Upper bound: $\lambda_i \le \Lambda_{h,i}, \quad i = 1, 2, \ldots, \dim V_h$

Laplace eigenvalue problem

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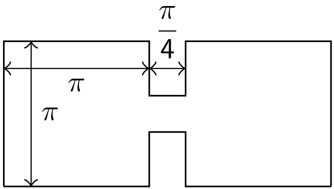
Lower bound: $\quad ? \le \lambda_i \le \Lambda_{h,i}, \quad i = 1, 2, \dots, \dim V_h$

Example – dumbbell

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega$$



[Trefethen, Betcke 2006]

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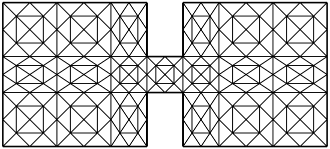
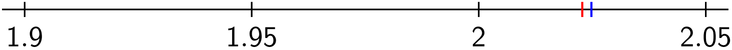
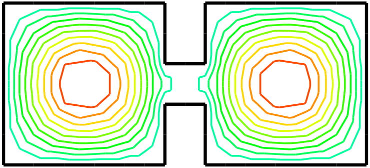
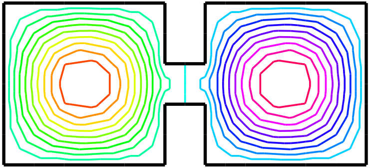




$\lambda_1 \approx 2.02280$ $\lambda_2 \approx 2.02481$





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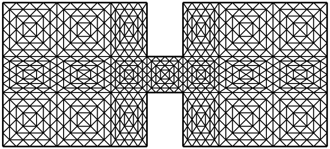

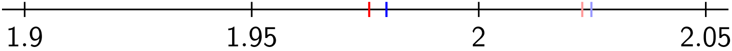

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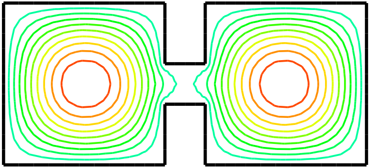

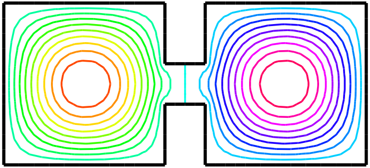

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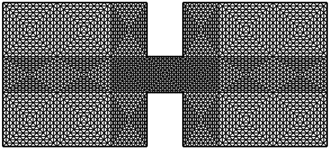

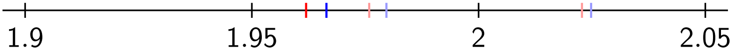

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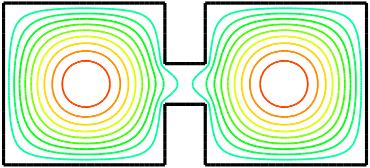

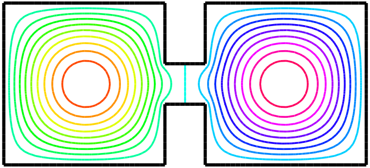

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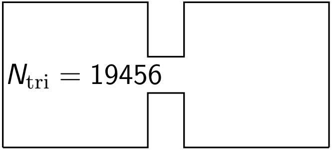

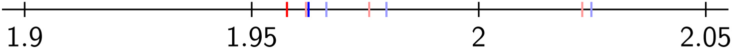

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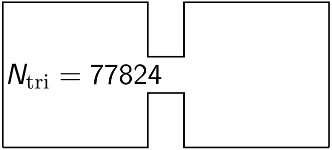

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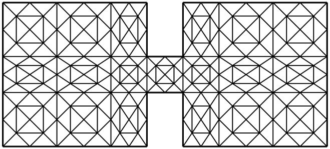
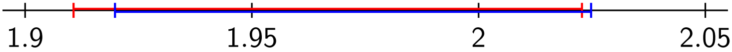
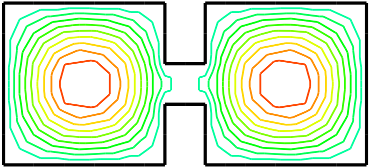
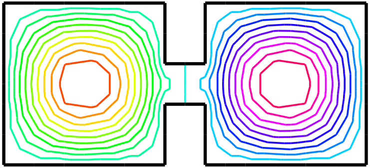




$$1.91067 \leq \lambda_1 \leq 2.02280 \qquad 1.91981 \leq \lambda_2 \leq 2.02481$$





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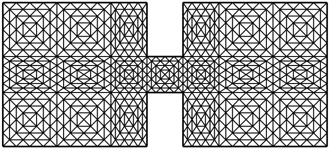

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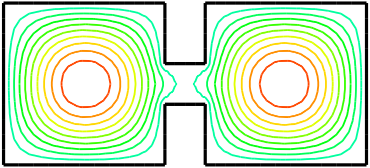

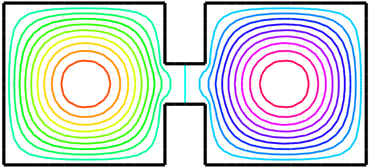

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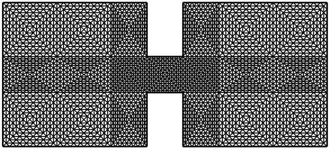

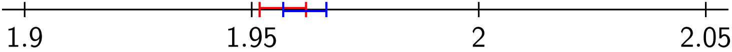

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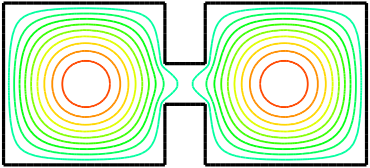

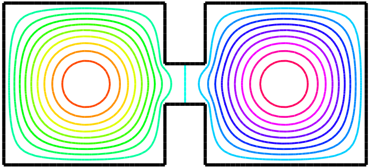

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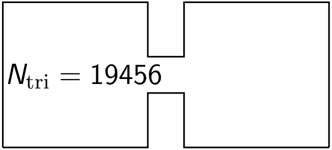

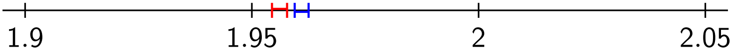

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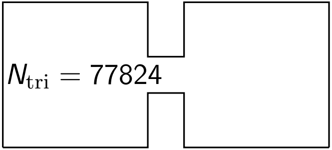

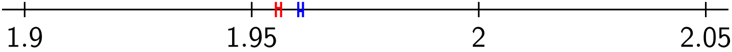

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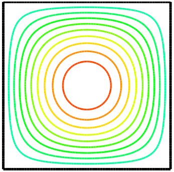
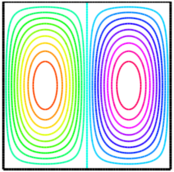
$1.95532 \le \lambda_1 \le 1.95646$	$1.96025 \le \lambda_2 \le 1.96129$

Example: Square

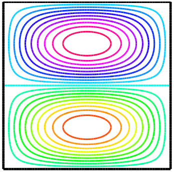
$\lambda_1 = 2,\ u_1(x,y) = \sin(x)\sin(y)$

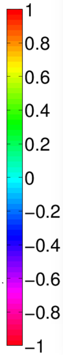


$\lambda_2 = 5,\ u_2(x,y) = \sin(2x)\sin(y)$



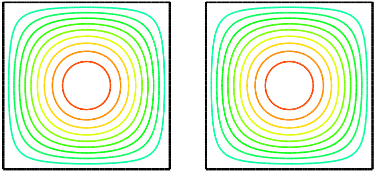
$\lambda_3 = 5,\ u_3(x,y) = \sin(x)\sin(2y)$



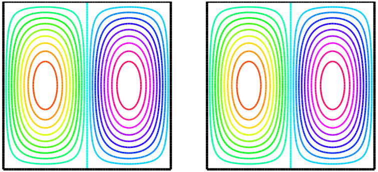


Example: Two squares

$\lambda_1 = 2$



$\lambda_2 = 5$



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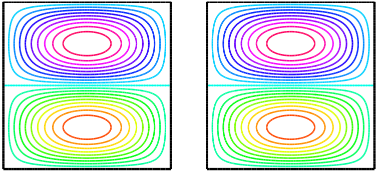

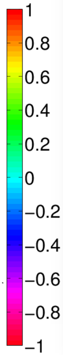

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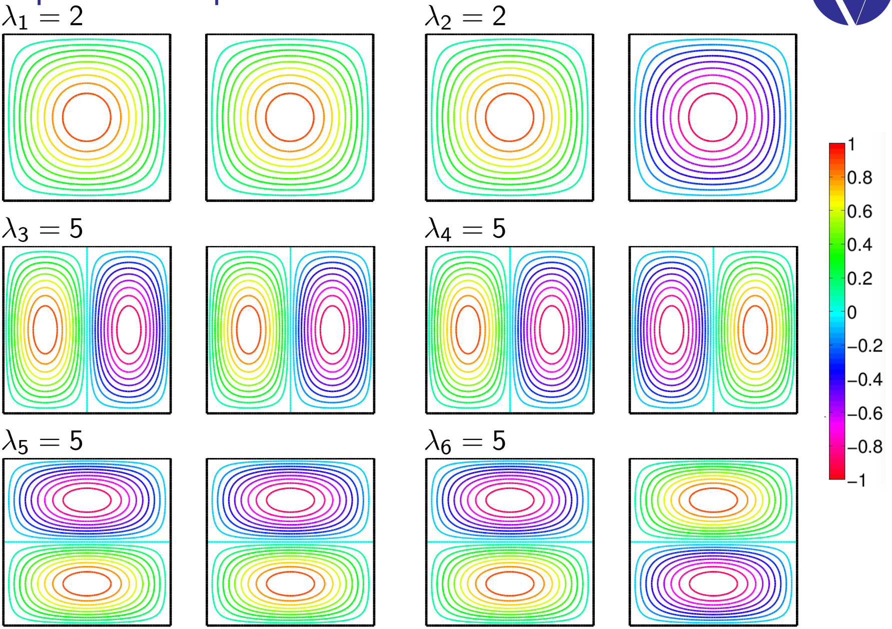
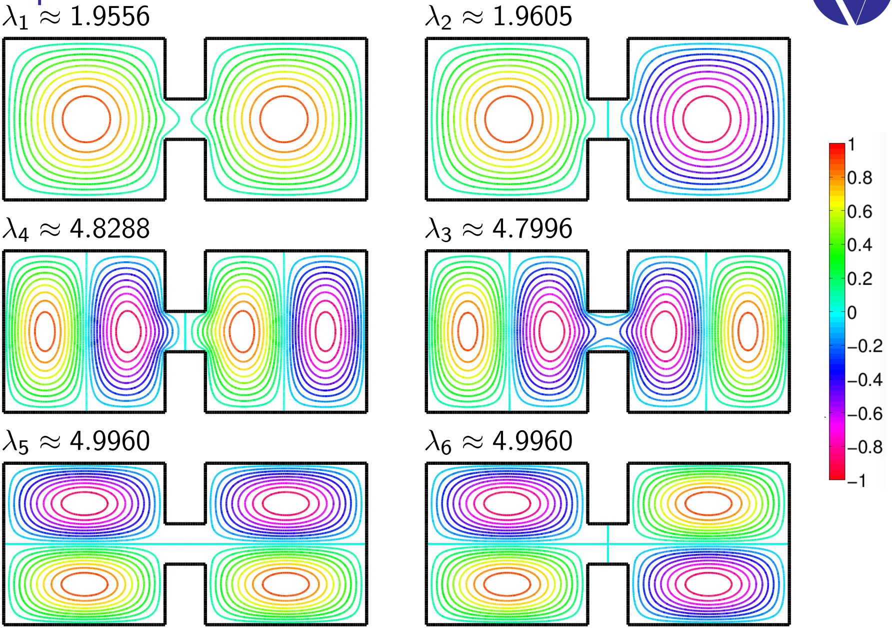


Example: Dumbbell



Lower bounds on eigenvalues

Old problem:
Temple 1928, Weinstein 1937, Kato 1949, Lehmann 1949, 1950, ...

Many results: Hehu Xie, Qun Lin, Jun Hu, Xuefeng Liu, Yidu Yang, Zhimin Zhang, Fubiao Lin, C. Carstensen, J. Gedicke, D. Galistl, G. Barrenechea, M. Plum, J.R. Kuttler, V.G. Sigillito, Y.A. Kuznetsov, S.I. Repin, H. Behnke, F. Goerisch, M.G. Armentano, R.G. Duran, L. Grubišić, ... *many others*

Weinstein's and Kato's bounds

Eigenvalue problem: Find $u_i \in D(A) \setminus \{0\}$ and $\lambda_i \in \mathbb{R}$:

$$Au_i = \lambda_i u_i$$

Setting:

- V ... Hilbert space
- $A : D(A) \to V$ linear, symmetric operator
- $\{u_i\}$ form ON basis in V
- $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$

Weinstein's and Kato's bounds

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Theorem 1 (Weinstein 1937):
Let $u_* \in D(A) \setminus \{0\}$ and $\lambda_* \in \mathbb{R}$ be arbitrary.
Let $\delta = \|Au_* - \lambda_* u_*\| / \|u_*\|$.
Then there exists λ_i such that $\lambda_* - \delta \leq \lambda_i \leq \lambda_* + \delta$.

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Theorem 2 (Kato 1949):
Let $u_* \in D(A) \setminus \{0\}$ be arbitrary and $\lambda_* = \langle Au_*, u_* \rangle / \langle u_*, u_* \rangle$.
Let $\delta = \|Au_* - \lambda_* u_*\| / \|u_*\|$ and $\mu, \nu \in \mathbb{R}$ satisfy

$$\lambda_{i-1} \leq \mu < \lambda_* < \nu \leq \lambda_{i+1} \quad \text{for some } i.$$

Then $\lambda_* - \dfrac{\delta^2}{\nu - \lambda_*} \leq \lambda_i \leq \lambda_* + \dfrac{\delta^2}{\lambda_* - \mu}$.

Proof of Theorem 1

Theorem 1 (Weinstein 1937):
Let $u_* \in D(A) \setminus \{0\}$ and $\lambda_* \in \mathbb{R}$ be arbitrary.
Let $\delta = \|Au_* - \lambda_* u_*\| / \|u_*\|$.
Then there exists λ_i such that $\lambda_* - \delta \leq \lambda_i \leq \lambda_* + \delta$.

Proof:
$$\|Au_* - \lambda_* u_*\|^2 = \sum_{j=1}^{\infty} \langle Au_* - \lambda_* u_*, u_j \rangle^2 = \sum_{j=1}^{\infty} |\lambda_j - \lambda_*|^2 \langle u_*, u_j \rangle^2 \geq \min_j |\lambda_j - \lambda_*|^2 \|u_*\|^2$$

Thus,

$$|\lambda_i - \lambda_*| = \min_j |\lambda_j - \lambda_*| \leq \frac{\|Au_* - \lambda_* u_*\|}{\|u_*\|} = \delta$$

□

Proof of Theorem 2

Theorem 2 (Kato 1949):
Let $u_* \in D(A) \setminus \{0\}$ be arbitrary and $\lambda_* = \langle Au_*, u_* \rangle / \langle u_*, u_* \rangle$.
Let $\delta = \|Au_* - \lambda_* u_*\| / \|u_*\|$ and $\mu, \nu \in \mathbb{R}$ satisfy

$$\lambda_{i-1} \leq \mu < \lambda_* < \nu \leq \lambda_{i+1} \quad \text{for some } i.$$

Then $\lambda_* - \dfrac{\delta^2}{\nu - \lambda_*} \leq \lambda_i \leq \lambda_* + \dfrac{\delta^2}{\lambda_* - \mu}$.

Proof: We have $(\lambda_j - \lambda_i)(\lambda_j - \nu) \geq 0$ for all $j = 1, 2, \ldots$.

$$0 \leq \sum_{j=1}^{\infty} (\lambda_j - \lambda_i)(\lambda_j - \nu)\langle u_*, u_j \rangle^2 = \sum_{j=1}^{\infty} (\lambda_j^2 - (\lambda_i + \nu)\lambda_j + \lambda_i \nu)\langle u_*, u_j \rangle^2 =$$

$$\|Au_*\|^2 - (\lambda_i + \nu)\langle Au_*, u_* \rangle + \lambda_i \nu \|u_*\|^2 = \left(\delta^2 + \lambda_*^2 - (\lambda_i + \nu)\lambda_* + \lambda_i \nu\right) \|u_*\|^2$$

because $\|Au_*\|^2 = (\delta^2 + \lambda_*^2)\|u_*\|^2$. $\square$

Weak form – general setting

Eigenvalue problem: Find $u_i \in V \setminus \{0\}$ and $\lambda_i \in \mathbb{R}$:

$$a(u_i, v) = \lambda_i b(u_i, v) \quad \forall v \in V.$$

Properties:

- $0 < \lambda_1 \le \lambda_2 \le \cdots$
- $b(u_i, u_j) = \delta_{ij}$
- $\|v\|_b^2 = \sum_{j=1}^{\infty} |b(v, u_j)|^2$
- $\|v\|_a^2 = \sum_{j=1}^{\infty} \lambda_j |b(v, u_j)|^2$

Weinstein's bound in the weak form

Theorem 3: Let $u_* \in V \setminus \{0\}$ and $\lambda_* \in \mathbb{R}$ be arbitrary and $w \in V$ be given by

$$a(w, v) = a(u_*, v) - \lambda_* b(u_*, v) \quad \forall v \in V.$$

Then

$$\min_j \frac{|\lambda_j - \lambda_*|^2}{\lambda_j} \leq \frac{\|w\|_a^2}{\|u_*\|_b^2}.$$

Proof:

$$\|w\|_a^2 = \sum_{j=1}^{\infty} \lambda_j |b(w, u_j)|^2 = \sum_{j=1}^{\infty} \frac{|a(w, u_j)|^2}{\lambda_j}$$

$$= \sum_{j=1}^{\infty} \frac{|a(u_*, u_j) - \lambda_* b(u_*, u_j)|^2}{\lambda_j} = \sum_{j=1}^{\infty} \frac{|\lambda_j - \lambda_*|^2}{\lambda_j} |b(u_*, u_j)|^2$$

Thus,

$$\|w\|_a^2 \geq \min_j \frac{|\lambda_j - \lambda_*|^2}{\lambda_j} \sum_{j=1}^{\infty} |b(u_*, u_j)|^2$$

Weinstein's bound in the weak form

Corollary: If

$$\sqrt{\lambda_{i-1}\lambda_i} \le \lambda_* \le \sqrt{\lambda_i \lambda_{i+1}}$$

and

$$\|w\|_a \le \eta$$

then

$$\ell_i \le \lambda_i,$$

where $\ell_i = \dfrac{1}{4\|u_*\|_b^2}\left(-\eta + \sqrt{\eta^2 + 4\lambda_*\|u_*\|_b^2}\right)^2$.

Proof: Clearly,

$$\frac{(\lambda_i - \lambda_*)^2}{\lambda_i} = \min_j \frac{|\lambda_j - \lambda_*|^2}{\lambda_j} \le \frac{\|w\|_a^2}{\|u_*\|_b^2} \le \frac{\eta^2}{\|u_*\|_b^2}$$

and solve for λ_i. □

Kato's bound in the weak form

Theorem 4: Let $u_* \in V \setminus \{0\}$ be arbitrary and let $\lambda_* = \|u_*\|_a^2 / \|u_*\|_b^2$. Let there be $\nu \in \mathbb{R}$ such that

$$\lambda_{i-1} < \lambda_* < \nu \leq \lambda_{i+1}$$

for a fixed index i. Let $\|w\|_a \leq \eta$. Then

$$L_i \leq \lambda_i,$$

where

$$L_i = \lambda_* \left(1 + \frac{\nu}{\lambda_*(\nu - \lambda_*)} \frac{\eta^2}{\|u_*\|_b^2} \right)^{-1}.$$

Complementary upper bound on the residual

Theorem 5: Let $\mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega)$ be such that $-\operatorname{div} \mathbf{q} = \lambda_* u_*$ then

$$\|\nabla w\|_{L^2(\Omega)} \leq \eta = \|\nabla u_* - \mathbf{q}\|_{L^2(\Omega)}.$$

Proof: Let $v \in H_0^1(\Omega)$, then

$$\begin{aligned} a(w, v) &= (\nabla u_*, \nabla v) - \lambda_*(u_*, v) - (\operatorname{div} \mathbf{q}, v) - (\mathbf{q}, \nabla v) \\ &= (\nabla u_* - \mathbf{q}, \nabla v) - (\lambda_* u_* + \operatorname{div} \mathbf{q}, v) \\ &\leq \|\nabla u_* - \mathbf{q}\| \|\nabla v\| \end{aligned}$$

□

[Braess 2007]

Flux reconstruction

- FEM eigenpairs: $\Lambda_{h,i} \in \mathbb{R}$, $u_{h,i} \in V_h$, $\|u_{h,i}\|_{L^2(\Omega)} = 1$, $i = 1, 2, \ldots, m$
- Flux reconstruction: $\mathbf{q}_{h,i} = \sum_{\mathbf{z} \in \mathcal{N}_h} \mathbf{q}_{\mathbf{z},i}$ [Braess, Schöberl 2006]
- Local mixed FEM: $\mathbf{q}_{\mathbf{z},i} \in \mathbf{W}_{\mathbf{z}}$, $d_{\mathbf{z},i} \in P_1^*(\mathcal{T}_{\mathbf{z}})$

$$
\begin{aligned}
(\mathbf{q}_{\mathbf{z},i}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} - (d_{\mathbf{z},i}, \operatorname{div} \mathbf{w}_h)_{\omega_{\mathbf{z}}} &= (\psi_{\mathbf{z}} \nabla u_{h,i}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} \quad && \forall \mathbf{w}_h \in \mathbf{W}_{\mathbf{z}} \\
-(\operatorname{div} \mathbf{q}_{\mathbf{z},i}, \varphi_h)_{\omega_{\mathbf{z}}} &= (r_{\mathbf{z},i}, \varphi_h)_{\omega_{\mathbf{z}}} \quad && \forall \varphi_h \in P_1^*(\mathcal{T}_{\mathbf{z}})
\end{aligned}
$$

where

- $\omega_{\mathbf{z}}$ is the patch of elements around vertex $\mathbf{z} \in \mathcal{N}_h$
- $\mathcal{T}_{\mathbf{z}}$ is the set of elements in $\omega_{\mathbf{z}}$
- $\mathbf{W}_{\mathbf{z}} = \{\mathbf{w}_h \in \mathbf{H}(\operatorname{div}, \omega_{\mathbf{z}}) : \mathbf{w}_h|_K \in \mathbf{RT}_1(K)\ \forall K \in \mathcal{T}_{\mathbf{z}}$ and $\mathbf{w}_h \cdot \mathbf{n}_{\omega_{\mathbf{z}}} = 0$ on $\Gamma_{\omega_{\mathbf{z}}}^{\mathrm{ext}}\}$
- $P_1^*(\mathcal{T}_{\mathbf{z}}) = \begin{cases} \{v_h \in P_1(\mathcal{T}_{\mathbf{z}}) : \int_{\omega_{\mathbf{z}}} v_h \,\mathrm{d}x = 0\} & \text{for } \mathbf{z} \in \mathcal{N}_h \setminus \partial\Omega \\ P_1(\mathcal{T}_{\mathbf{z}}) & \text{for } \mathbf{z} \in \mathcal{N}_h \cap \partial\Omega \end{cases}$
- $r_{\mathbf{z},i} = \Lambda_{h,i} \psi_{\mathbf{z}} u_{h,i} - \nabla \psi_{\mathbf{z}} \cdot \nabla u_{h,i}$

Flux reconstruction

- FEM eigenpairs: $\Lambda_{h,i} \in \mathbb{R}$, $u_{h,i} \in V_h$, $\|u_{h,i}\|_{L^2(\Omega)} = 1$, $i = 1, 2, \ldots, m$
- Flux reconstruction: $\mathbf{q}_{h,i} = \sum_{\mathbf{z} \in \mathcal{N}_h} \mathbf{q}_{\mathbf{z},i}$ [Braess, Schöberl 2006]
- Local mixed FEM: $\mathbf{q}_{\mathbf{z},i} \in \mathbf{W}_\mathbf{z}$, $d_{\mathbf{z},i} \in P_1^*(\mathcal{T}_\mathbf{z})$

$$(\mathbf{q}_{\mathbf{z},i}, \mathbf{w}_h)_{\omega_\mathbf{z}} - (d_{\mathbf{z},i}, \operatorname{div} \mathbf{w}_h)_{\omega_\mathbf{z}} = (\psi_\mathbf{z} \nabla u_{h,i}, \mathbf{w}_h)_{\omega_\mathbf{z}} \quad \forall \mathbf{w}_h \in \mathbf{W}_\mathbf{z}$$
$$-(\operatorname{div} \mathbf{q}_{\mathbf{z},i}, \varphi_h)_{\omega_\mathbf{z}} = (r_{\mathbf{z},i}, \varphi_h)_{\omega_\mathbf{z}} \quad \forall \varphi_h \in P_1^*(\mathcal{T}_\mathbf{z})$$

- Error estimator: $\eta_i = \|\nabla u_{h,i} - \mathbf{q}_{h,i}\|_{L^2(\Omega)}$
- Weinstein's bound: $\ell_i = \left(-\eta_i + \sqrt{\eta_i^2 + 4\Lambda_{h,i}}\right)^2 / 4$

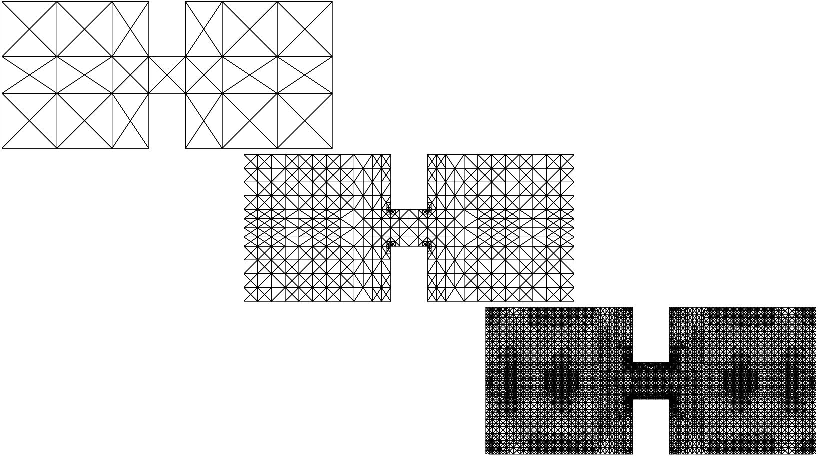
  provided $\Lambda_{h,i} \leq \sqrt{\lambda_i \lambda_{i+1}}$.
- Kato's bound: $L_i = \Lambda_{h,i} \left(1 + \frac{\nu}{\Lambda_{h,i}(\nu - \Lambda_{h,i})} \eta_i^2\right)^{-1}$

  provided $\Lambda_{h,i} < \nu \leq \lambda_{i+1}$.

Example: Dumbbell – convergence

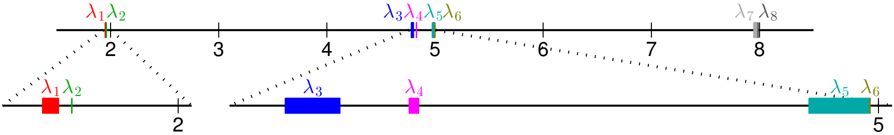
$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega = \text{dumbbell}$$
$$u_i = 0 \quad \text{on } \partial\Omega$$

Adaptively refined meshes:

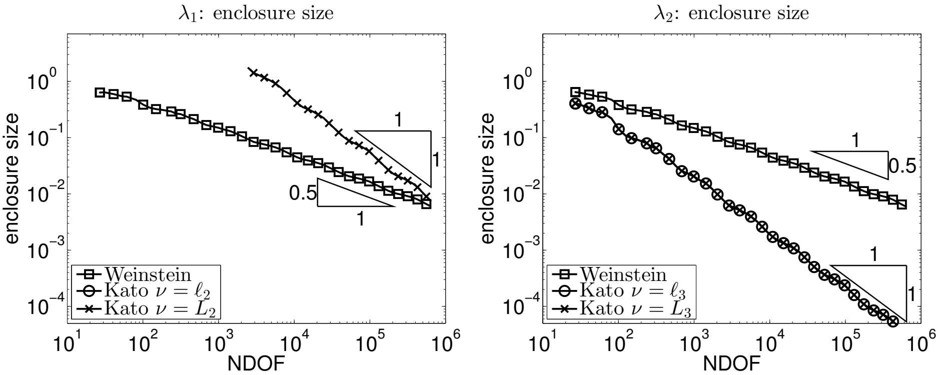


Example: Dumbbell – convergence

Spectrum:

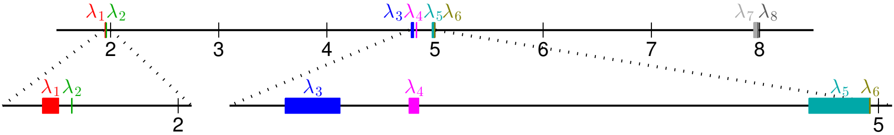


Eigenvalue enclosure sizes:

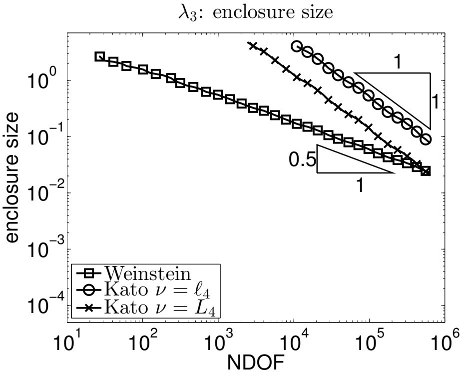


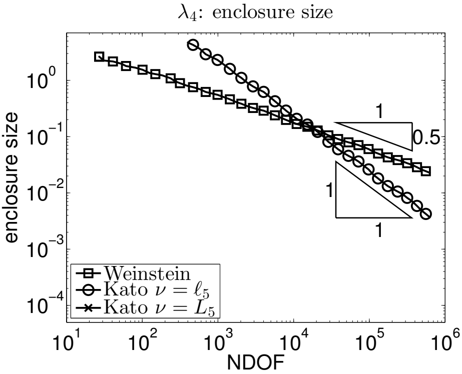
Example: Dumbbell – convergence

Spectrum:



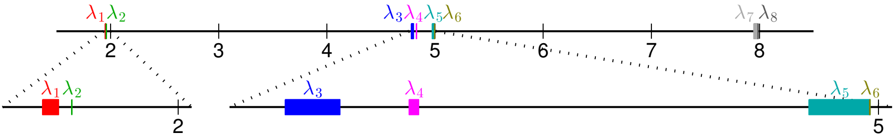
Eigenvalue enclosure sizes:



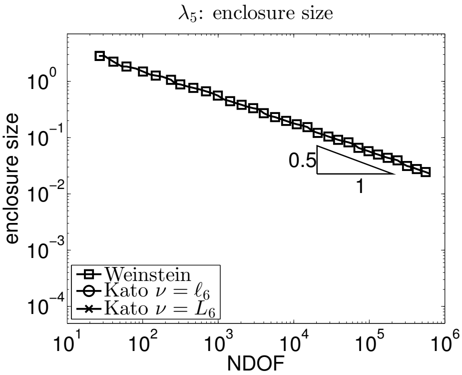


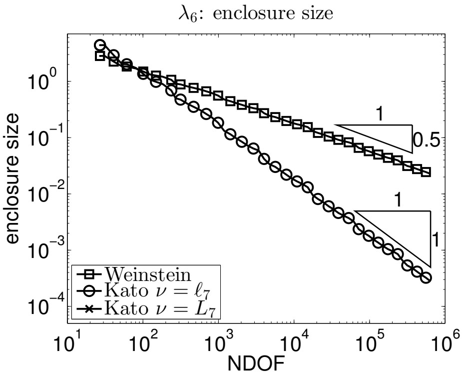
Example: Dumbbell – convergence

Spectrum:



Eigenvalue enclosure sizes:





Closeness condition

Test if

$$\Lambda_{h,i} \leq \sqrt{\underline{\lambda}_i^{\text{best}} \underline{\lambda}_{i+1}^{\text{best}}} \leq \sqrt{\lambda_i \lambda_{i+1}},$$

where $\underline{\lambda}_i^{\text{best}} = \max \left\{ \ell_i, L_i^{\nu=\ell_{i+1}}, L_i^{\nu=L_{i+1}^{\nu=\ell_{i+2}}} \right\}$

	best lower bound	upper bound	$\sqrt{\underline{\lambda}_i^{\text{best}} \underline{\lambda}_{i+1}^{\text{best}}} - \Lambda_{h,i}$
λ_7	$\ell_7 = 7.94671$	$\Lambda_{h,7} = 7.98716$	?
λ_6	$L_6^{\nu=\ell_7} = 4.99667$	$\Lambda_{h,6} = 4.99695$	1.3044
λ_5	$\ell_5 = 4.97426$	$\Lambda_{h,5} = 4.99693$	-0.0115
λ_4	$L_4^{\nu=\ell_5} = 4.82639$	$\Lambda_{h,4} = 4.82999$	0.0698
λ_3	$L_3^{\nu=L_4^{\nu=\ell_5}} = 4.78059$	$\Lambda_{h,3} = 4.80086$	0.0026
λ_2	$L_2^{\nu=\ell_3} = 1.96067$	$\Lambda_{h,2} = 1.96070$	1.1009
λ_1	$\ell_1 = 1.94982$	$\Lambda_{h,1} = 1.95581$	-0.0006

Conclusions

- Good for general symmetric elliptic operators.
- Mixed boundary conditions (e.g. Steklov problem).
- Standard conforming finite element technology.
- Natural for adaptive refinement.
- A priori information on spectrum needed.
- Weinstein – robust, but less accurate.
- Kato – accurate if the spectral gap is large.

Open problems:

- Kato's bound is not suitable for multiple eigenvalues.
- Does exists $k > N_{\mathrm{DOF}}$ such that $|b(u_{h,i}, u_k)| \geq \xi$, where $\xi > 0$ is explicitly given?
- Are there rough lower bounds on λ_i, which are based on i?

Thank you for your attention

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Academy of Mathematics and Systems Science, CAS, Beijing, October 17, 2016