Computational Aspects of Stress Wave Problems in Solids

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Scope of the lecture

- History and background theory
- Geometrical, spatial and temporal dispersion
- Waves and finite elements
- Theoretical examples
  - 1D, Love's correction, a wish experiment
  - 2D and 3D Lamb, perpetum mobile by FEA
- Practical examples - percussive rock drilling
- Impact of parallelism
- The art of modelling

1D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c_0^2 \frac{\partial^2 u}{\partial x^2}$$

$$c_0 = \sqrt{\frac{E}{\rho}}$$

$$u = f(c_0 t - x) + F(c_0 t + x)$$

Wave equation 2D - plane stress

$$\frac{\partial \sigma_{ji}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2},$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{E}{1-\mu^2}\left[\frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 v}{\partial x \partial y}\right] + G\left[\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y \partial x}\right],$$

$$\rho \frac{\partial^2 v}{\partial t^2} = \frac{E}{1-\mu^2}\left[\frac{\partial^2 v}{\partial y^2} + \mu \frac{\partial^2 u}{\partial x \partial y}\right] + G\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y}\right].$$

$$u = f(x - c_3 t)$$

$$v = h(y - c_3 t)$$

Longitudinal

dilatational
irrotational
extension

$$c_3 = \sqrt{\frac{E}{\rho(1-\mu^2)}}$$

P … primary

$$u = F(y - c_2 t)$$

$$v = H(x - c_2 t)$$

Transversal

shear
rotational
distortion
equivolumetrical

$$c_2 = \sqrt{\frac{G}{\rho}}$$

S … shear

2D plane strain and 3D

$$c_1 = \sqrt{\frac{\lambda + 2G}{\rho}} = \sqrt{\frac{E}{\rho} \frac{1-\mu}{(1+\mu)(1-2\mu)}}$$

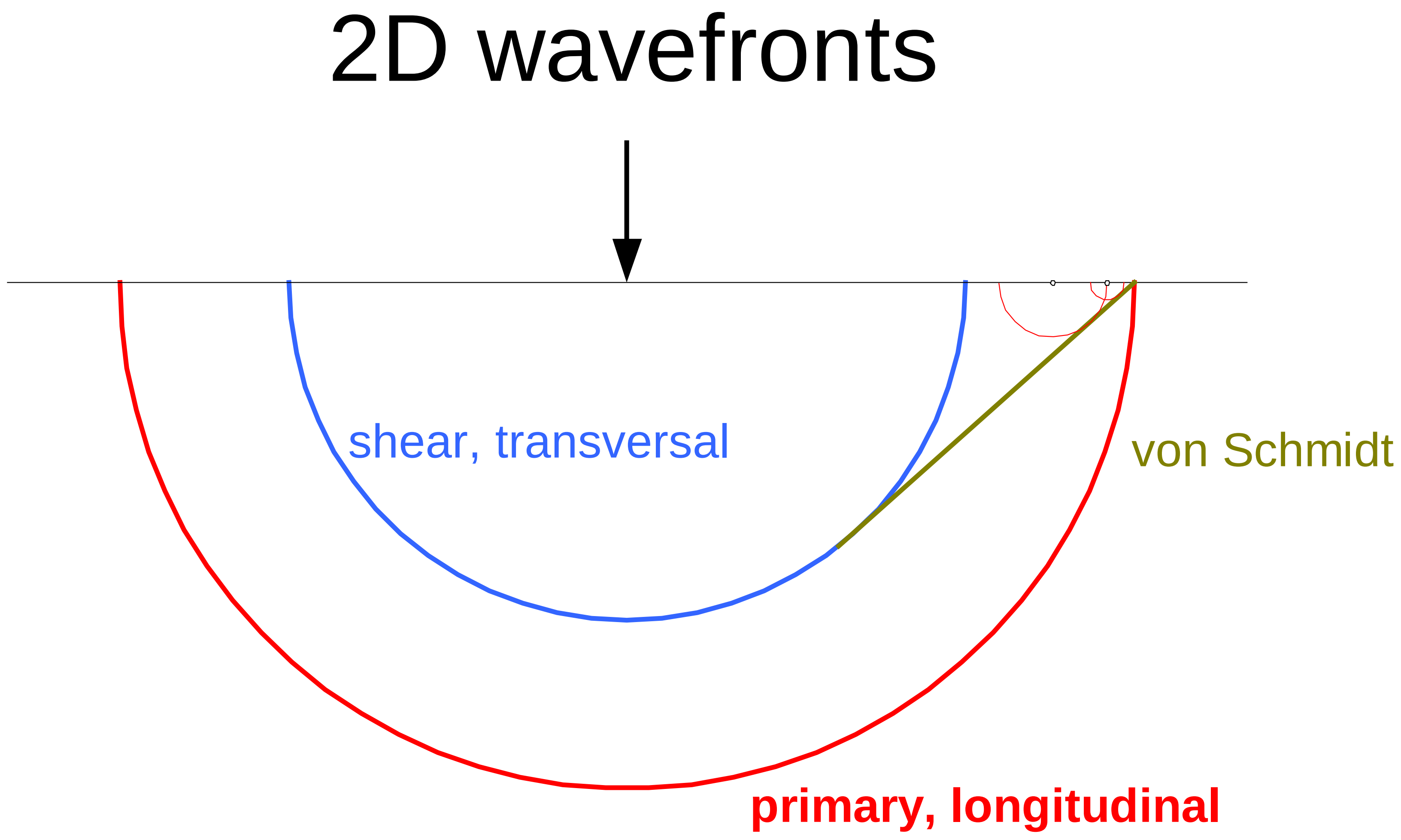
P … primary

$$\lambda = \frac{\mu E}{(1+\mu)(1-2\mu)}$$

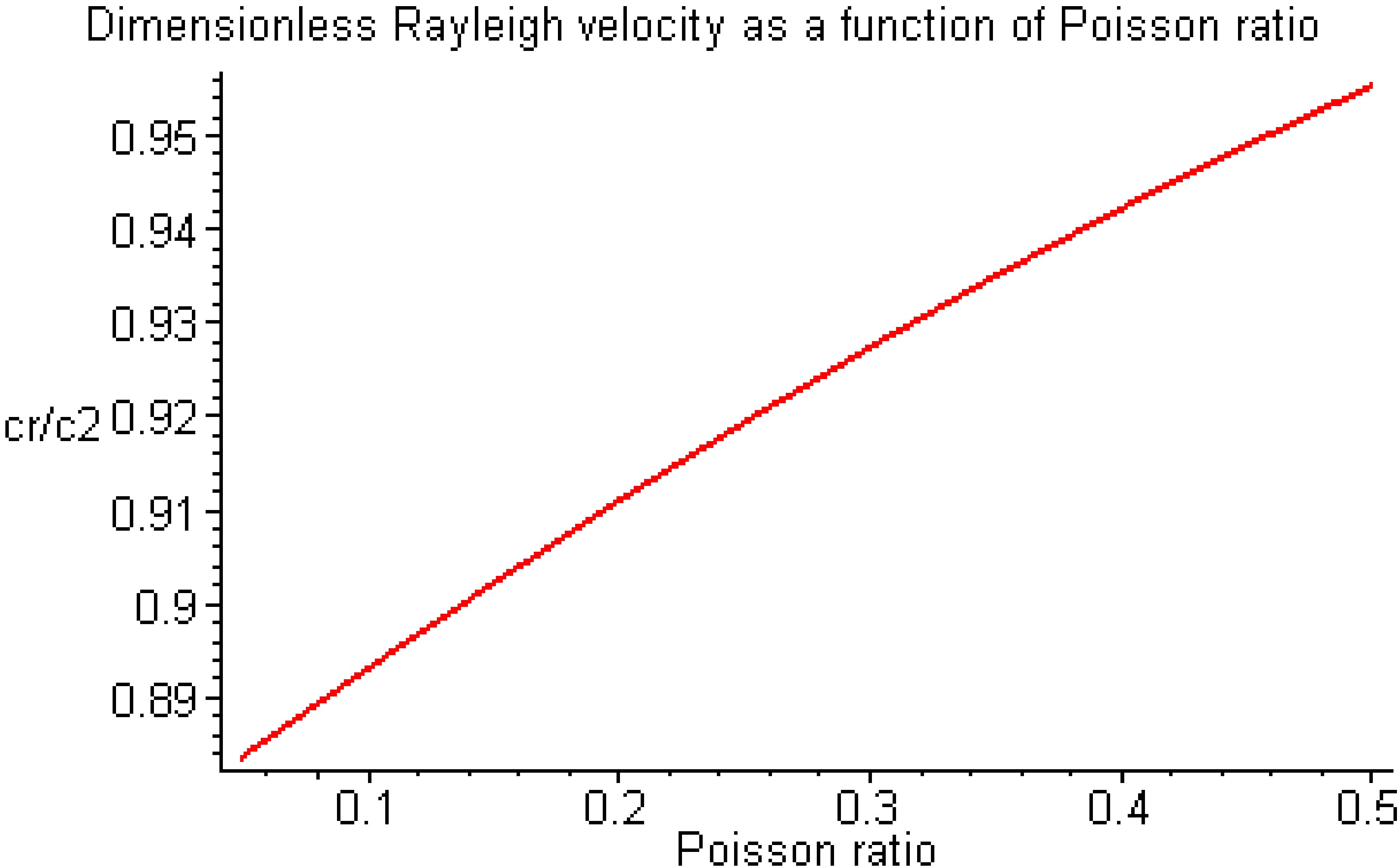
$$G = \frac{E}{2(1+\mu)}$$

Within the scope of linear theory of elasticity, P and S waves are uncoupled.

2D wave fronts - Huygen’s principle



Wave equation on the surface
Rayleigh waves



Typical values for steel in m/s

For E = 2.1e11 Pa, ρ = 7800 kgm^-3, μ = 0.3

$c_0 = \sqrt{E/\rho} = 5189$ … 1D wave, slender bar

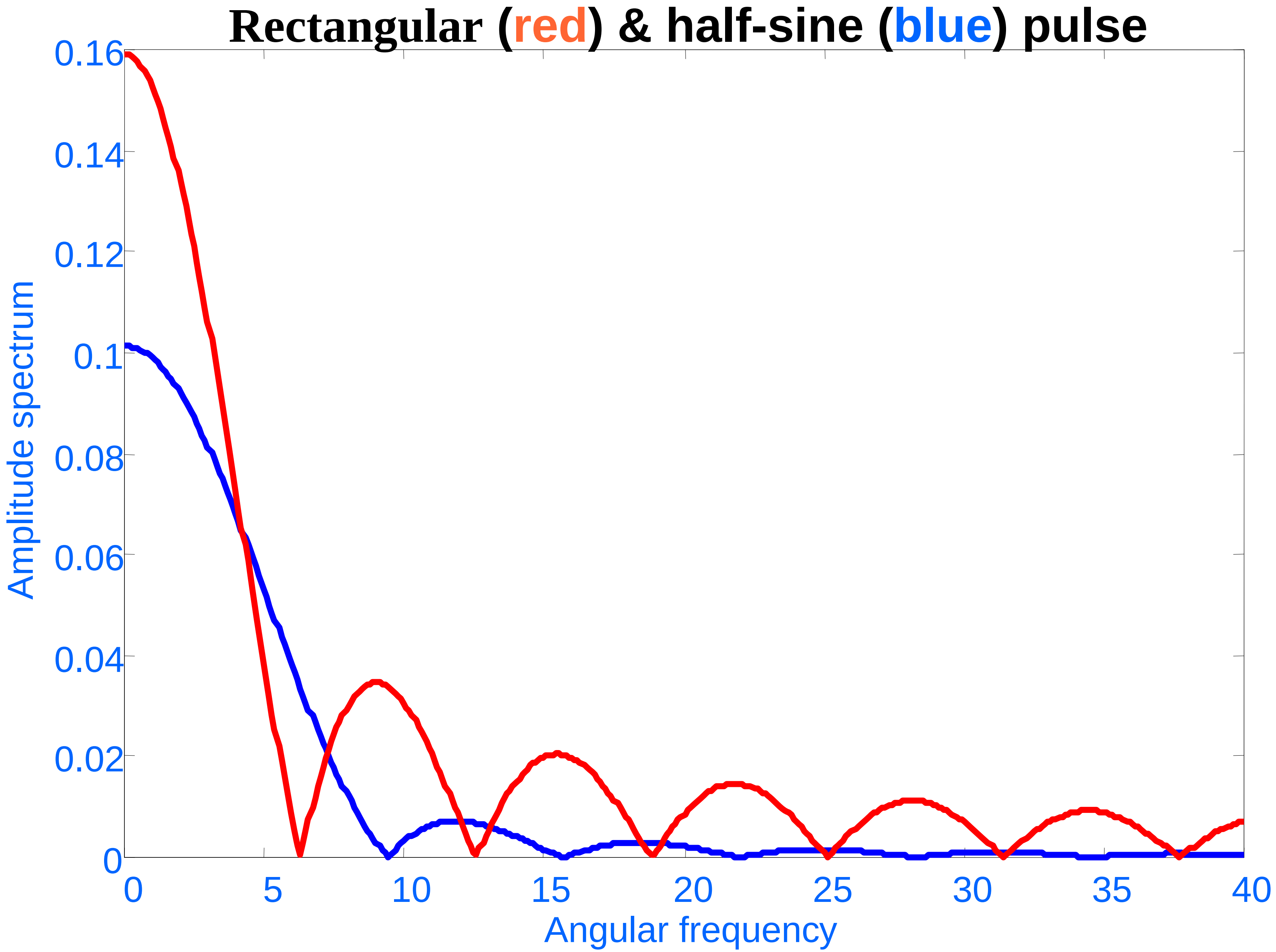
$c_1 = \sqrt{(2G+\lambda)/\rho} = 6020$ … P wave for plane strain, 3D

$c_2 = \sqrt{G/\rho} = 3218$ … S wave, shear

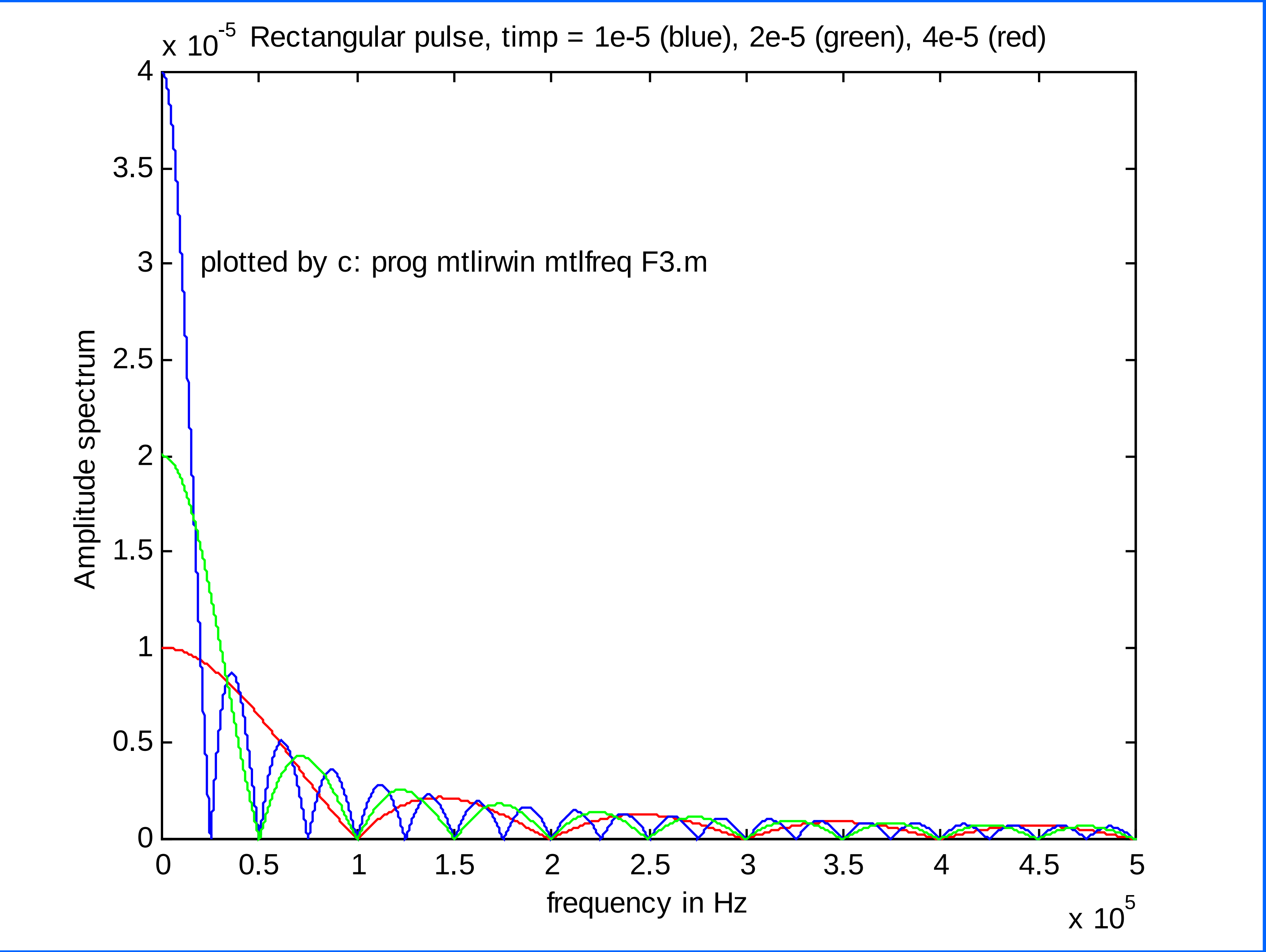
$c_3 = \sqrt{E/(\rho(1-\mu^2))} = 5439$ … P wave for plane stress

$c_\mathrm{R} = 0.9274 c_2 = 2984$ … R wave, Rayleigh for $\mu = 0.3$

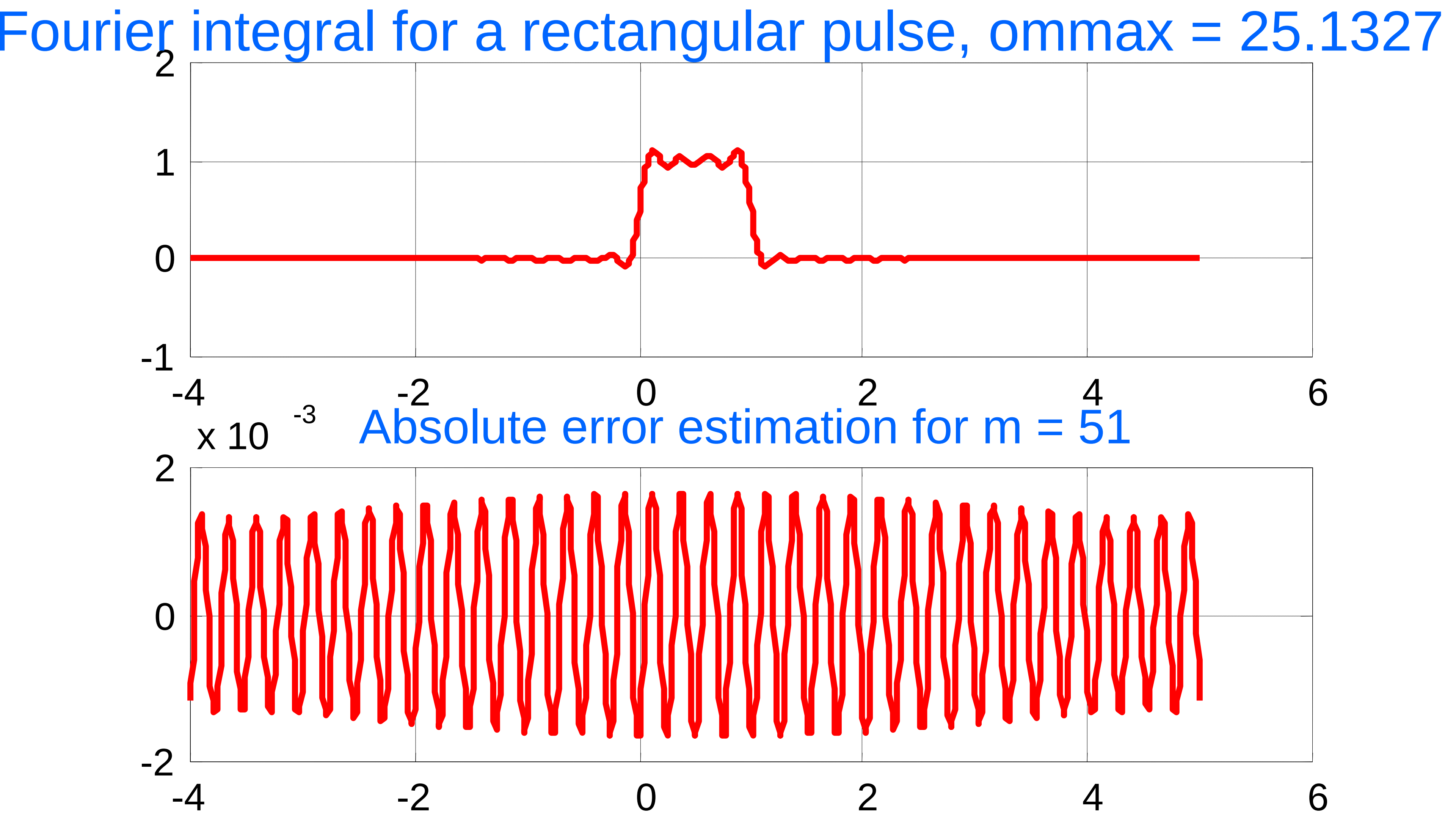
Name of the game is Fourier



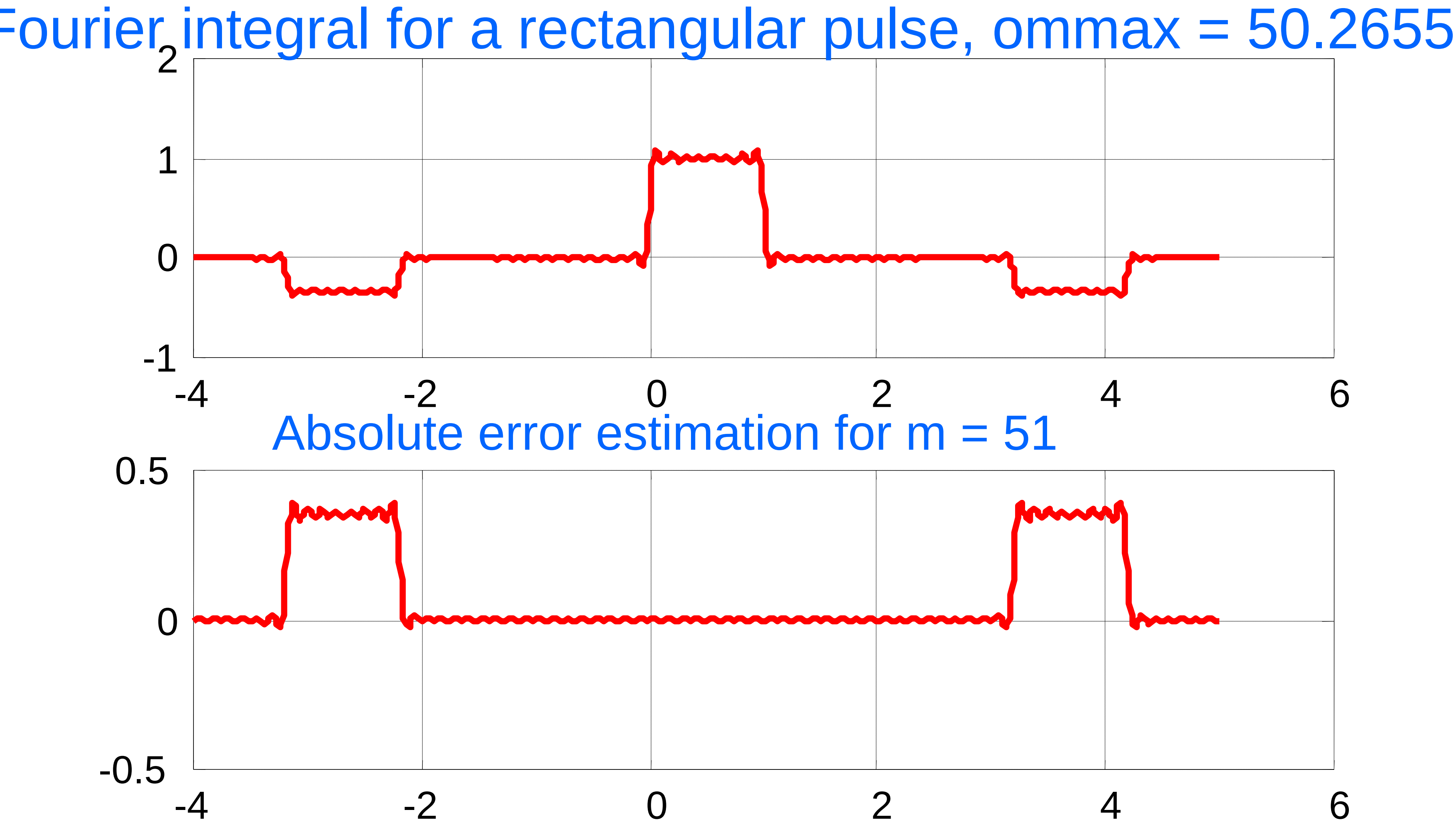
Short, medium and long pulses



Fourier integral of a rectangular pulse, omegamax = 8*pi and m = 51



Fourier integral of a rectangular pulse, omegamax = 16*pi and m = 51



Analytical solution see Graff, Eringen, Brepta, Valeš

For given boundary conditions

- Laplace transform in time
- Fourier transform in space
- Inverse transforms

The result is in the form of infinite series of infinite integrals

when evaluated numerically

- the finite number of terms of series

and

- the finite upper limit of integrals

are taken into account

If you can name a disease, you know what it is … Murphy

$$u(x,t) = Ae^{ik(x-ct)} = Ae^{-i\frac{2\pi}{\lambda}(x-ct)} = Ae^{-i(kx-\omega t)}$$

ω … angular frequency [radians/s]

$f = \omega / 2\pi$ … cyclic frequency [Hz]

$T = 1/f = 2\pi / \omega$ … period [s]

$k = 2\pi / \lambda = \omega / c$ … wavenumber [1/m]

$\lambda = 2\pi / k$ … wavelength [m]

$c = \omega / k$ … phase velocity [m/s]

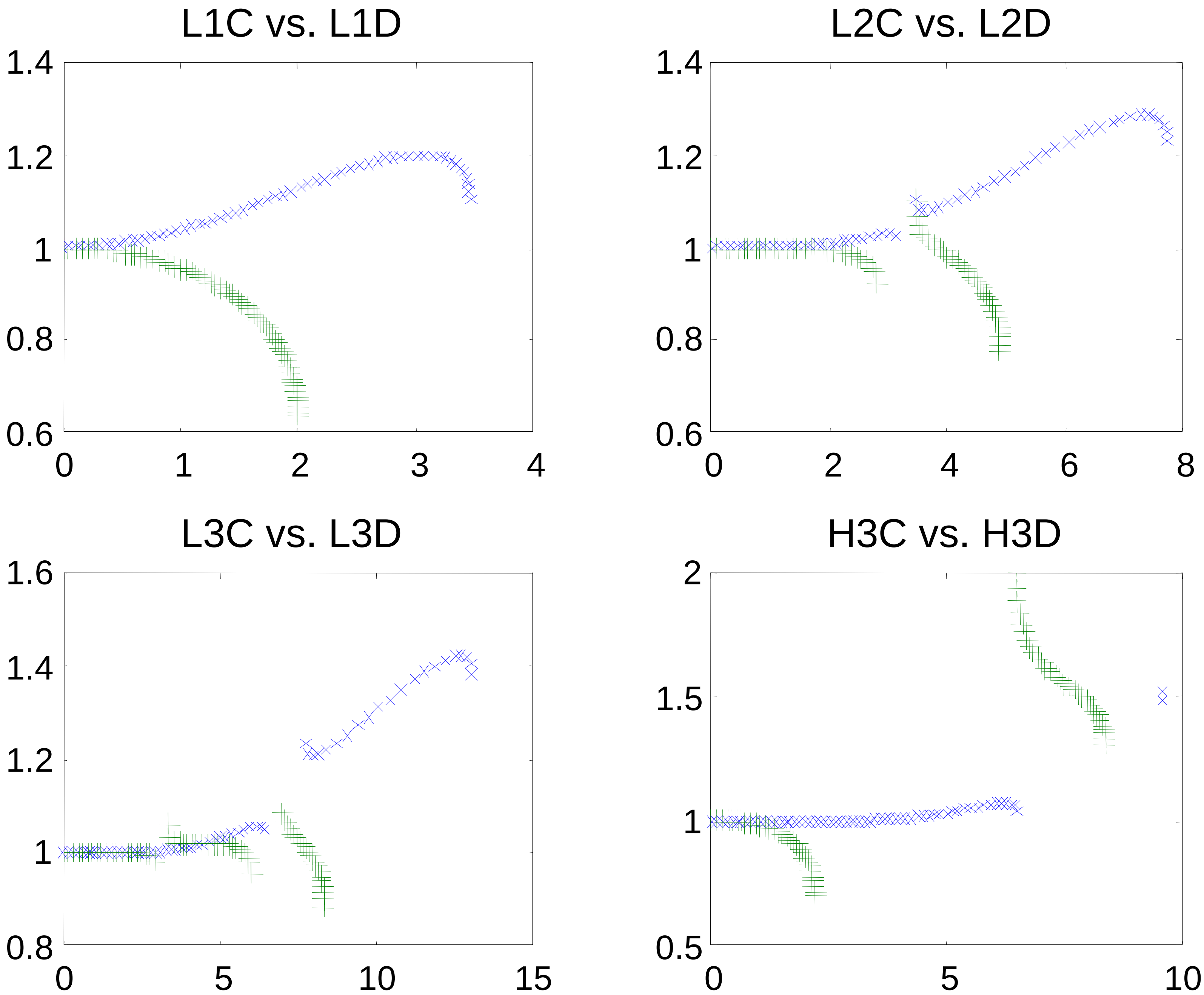
A medium is called non-dispersive
if wavenumber is proportional to frequency
or by other words
if all frequency components propagate by the same speed

Dispersion

- Due to geometry,
- due to space discretization,
- due to time discretization.

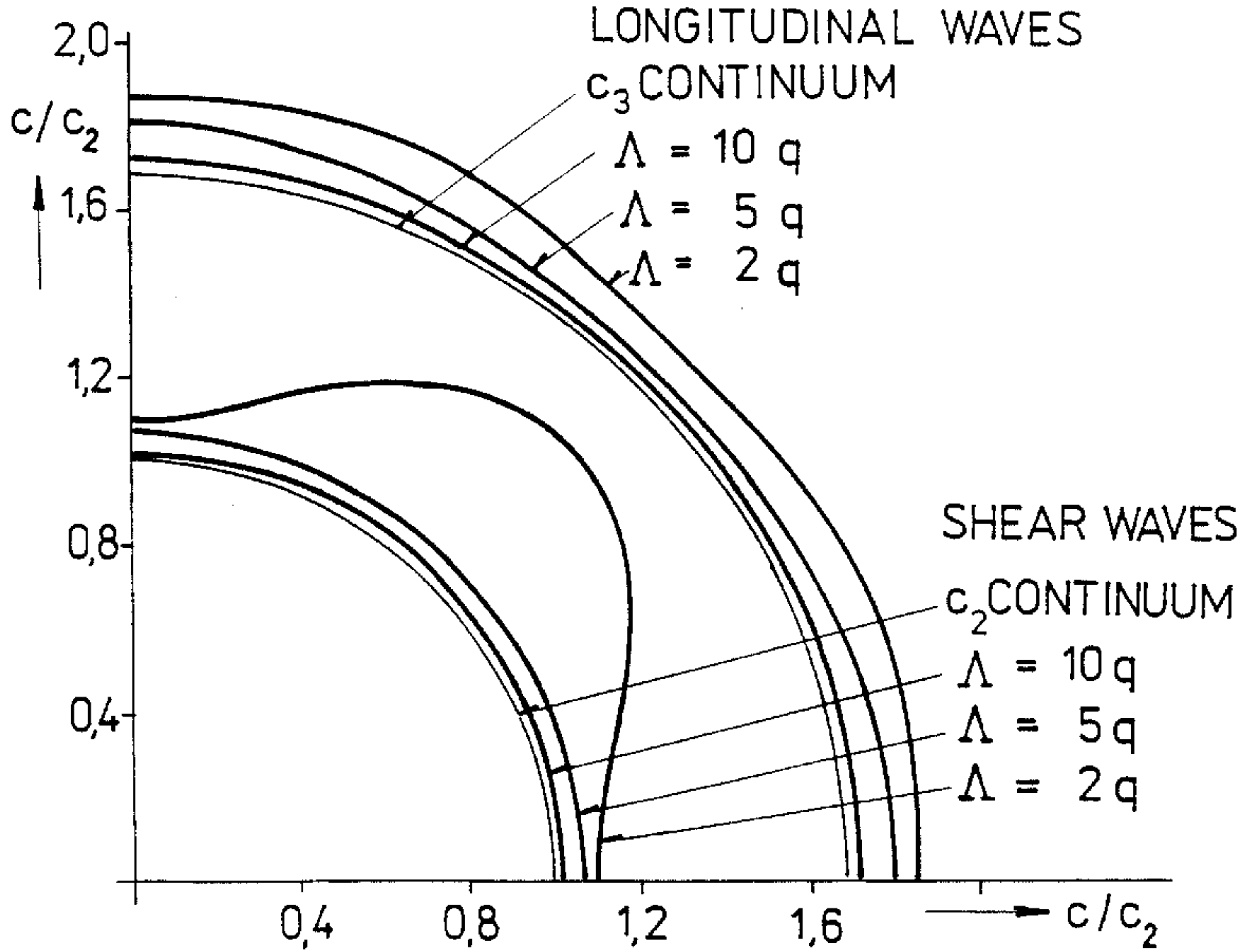
1D elements phase velocity vs. frequency

- Lagrangian
  - Linear polynomial L1
  - Quadratic L2
  - Cubic L3
- Hermitian
  - Cubic H3
- Consistent and diagonal mass formulation

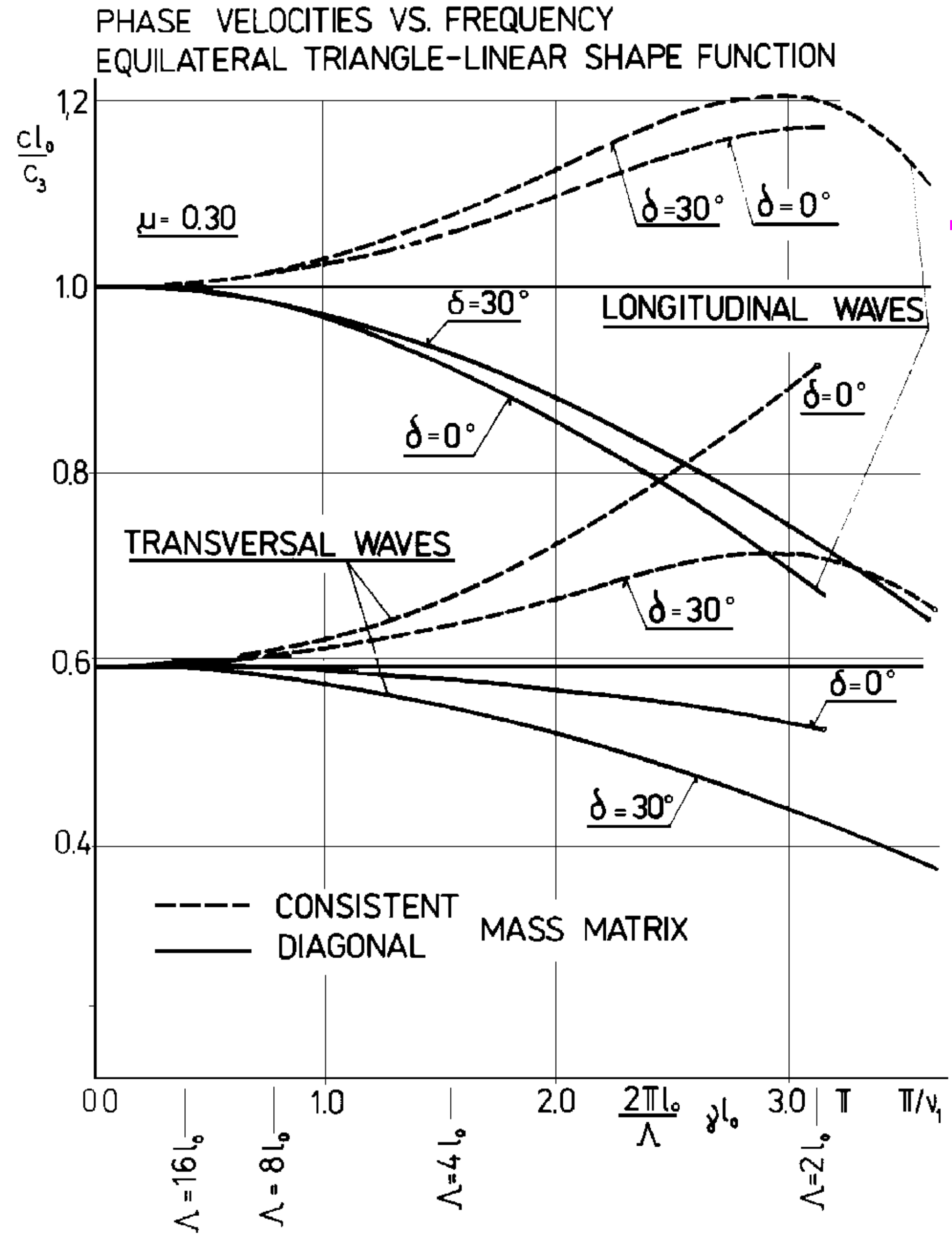

L1C vs. L1D
1.4
1.2
1
0.8
0.6
0
1
2
3
4
L2C vs. L2D
1.4
1.2
1
0.8
0.6
0
2
4
6
8
L3C vs. L3D
1.6
1.4
1.2
1
0.8
0
5
10
15
H3C vs. H3D
2
1.5
1
0.5
0
5
10

Dispersion properties in 2D

- 4-node quadrilateral element
- triangular element

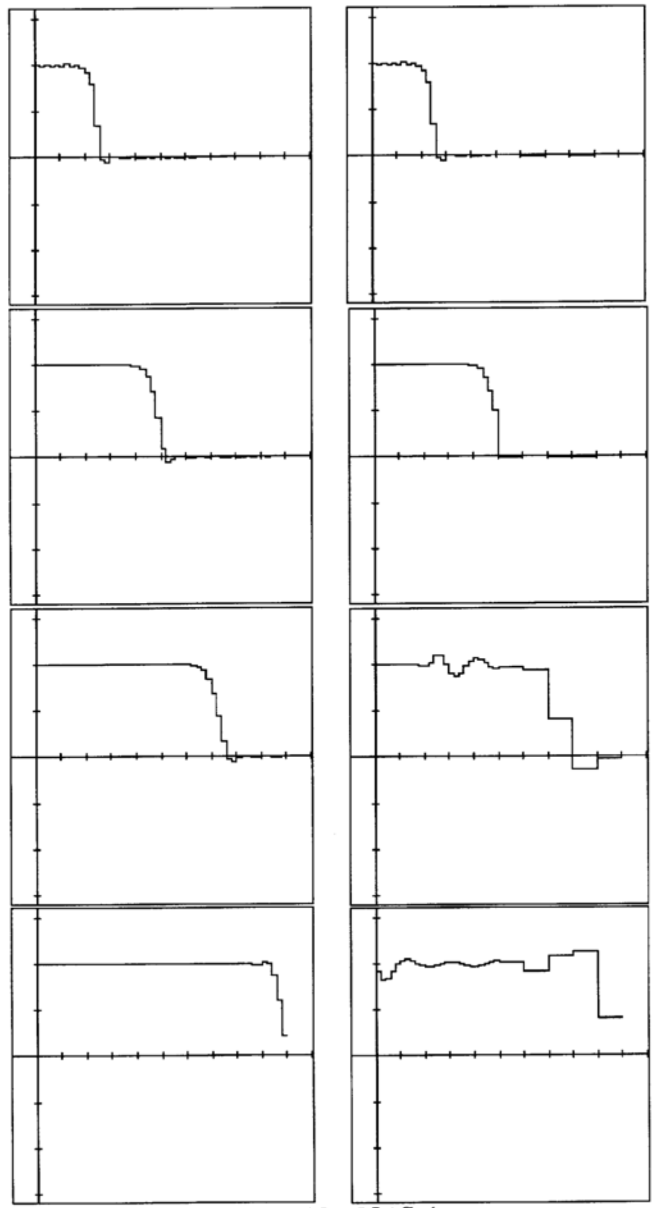


LONGITUDINAL WAVES
c_3 CONTINUUM
Λ = 10 q
Λ = 5 q
Λ = 2 q
SHEAR WAVES
c_2 CONTINUUM
Λ = 10 q
Λ = 5 q
Λ = 2 q
c/c2
2,0
1,6
1,2
0,8
0,4
0,4
0,8
1,2
1,6
c/c2



PHASE VELOCITIES VS. FREQUENCY
EQUILATERAL TRIANGLE-LINEAR SHAPE FUNCTION
$\frac{c l_0}{c_3}$
1,2
1.0
0.8
0.6
0.4
μ = 0.30
δ = 30°
δ = 0°
LONGITUDINAL WAVES
TRANSVERSAL WAVES
CONSISTENT
DIAGONAL
MASS MATRIX
0.0
1.0
2.0
3.0
π
π/ν1
$\frac{2\pi l_0}{\Lambda}$
γl_0
Λ = 16 l_0
Λ = 8 l_0
Λ = 4 l_0
Λ = 2 l_0

THE INFLUENCE OF NONUNIFORM MESH



25 + 25 L1C elements   25 + 5 L1C elements

Newmark, gamma=0.6, Heaviside, Strain vs. length of bar

FE model characteristics

- Dispersive - speed depends on frequency
- Bounded spectrum - finite # of frequencies
- Artificial anisotropy
- Depends on Fourier spectrum of loading

In absolute terms

- 1 mm FE element … 1MHz frequency, for 5000m/s and 5 elements into the wave length of the highest harmonics
- interatomic distance for metals is about 10^(-10) m. Not to be influenced by the corpuscular structure of matter the specimen should be at least 10^4 times bigger, ie. 10^(-6) m. Such an element would be good for 1 GHz.

Elastic waves and finite elements

$$\mathbf{M}\ddot{\mathbf{x}} = \mathbf{F}^{\text{ext}} - \mathbf{F}^{\text{int}}$$

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{P}(t)$$

- Finite element semidiscretization
- Mass matrix formulation
  - consistent
  - diagonal (lumped)
- Time discretization … integration in time
  - explicit — central differences
  - implicit — Newmark family, Houbolt

Correct determination of time step

lmin … — **length of the smallest element**

c … — **speed of propagation**

tmin = lmin/c … — **time through the smallest element**

hmts … — **how many time steps to go through the smallest element**

h = tmin/hmts … — **suitable time step**

hmts < 1 … high frequency components are filtered out, implicit domain

hmts = 1 … stability limit for explicit methods … 2/omegamax,
where omegamax = max(eig(M,K))

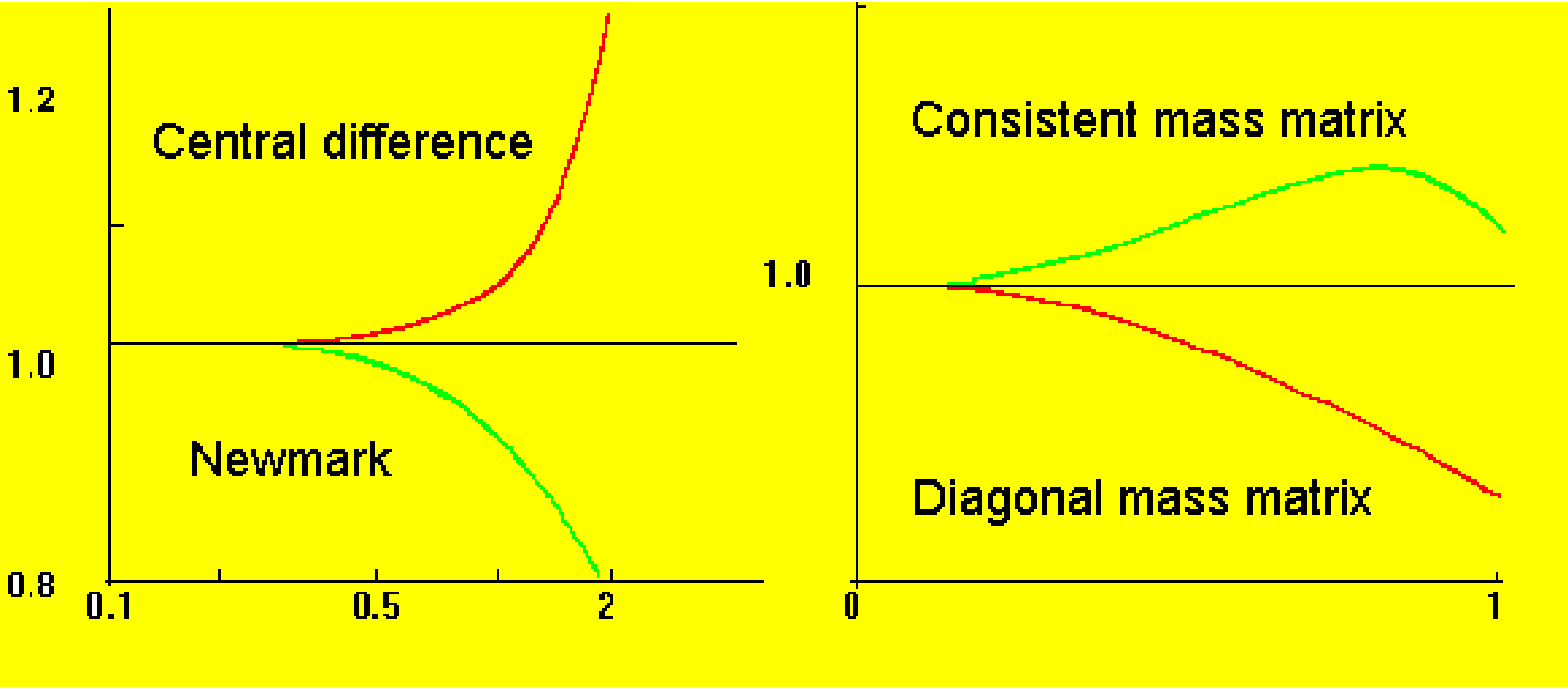
hmts = 2 … my choice

hmts > 2 … the high frequency components, which are wrong, due to time and space dispersion, are integrated ‘correctly’

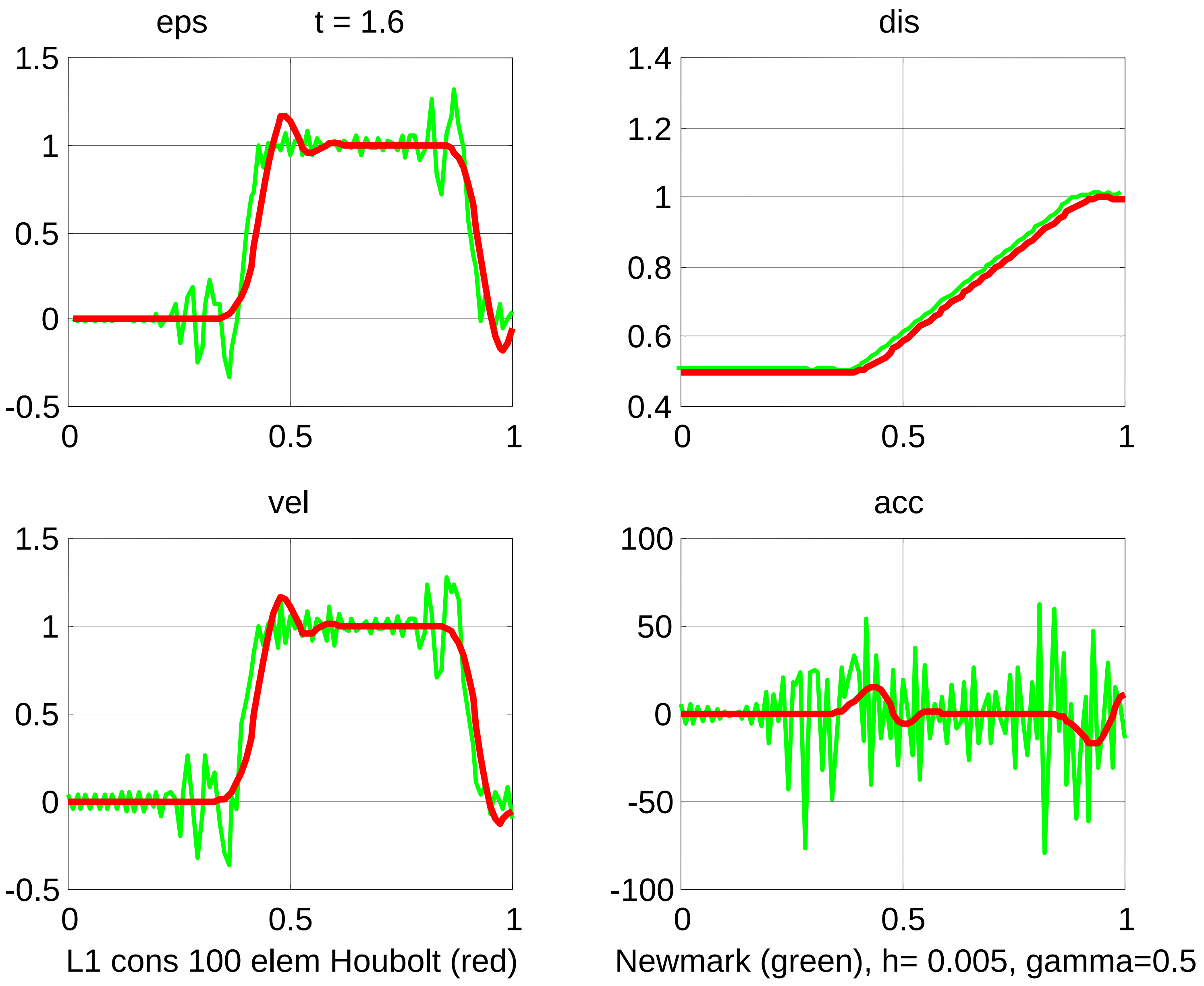
LS Dyna uses

h = 0.9*tmin as a default

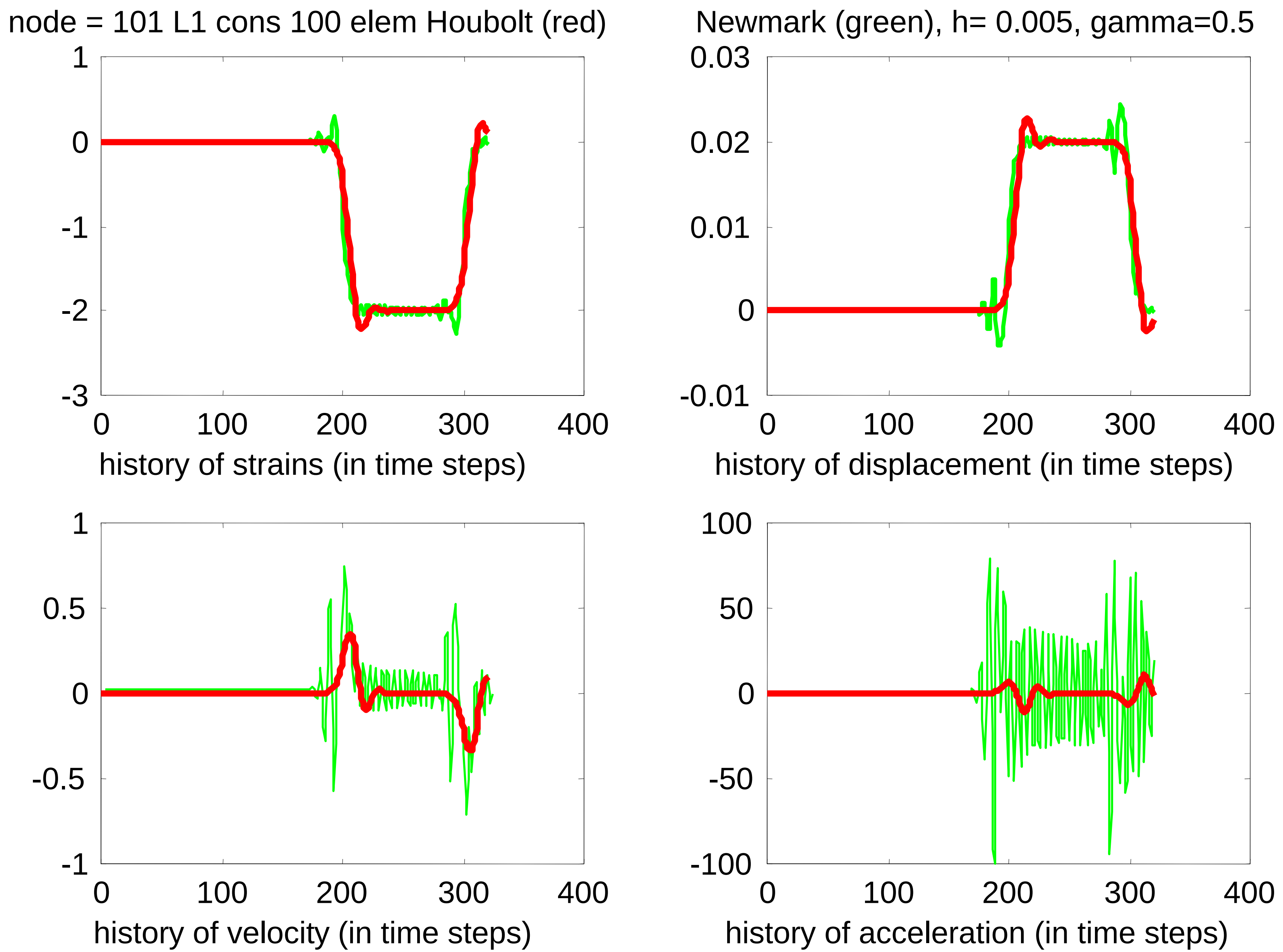
Space and time discretization errors



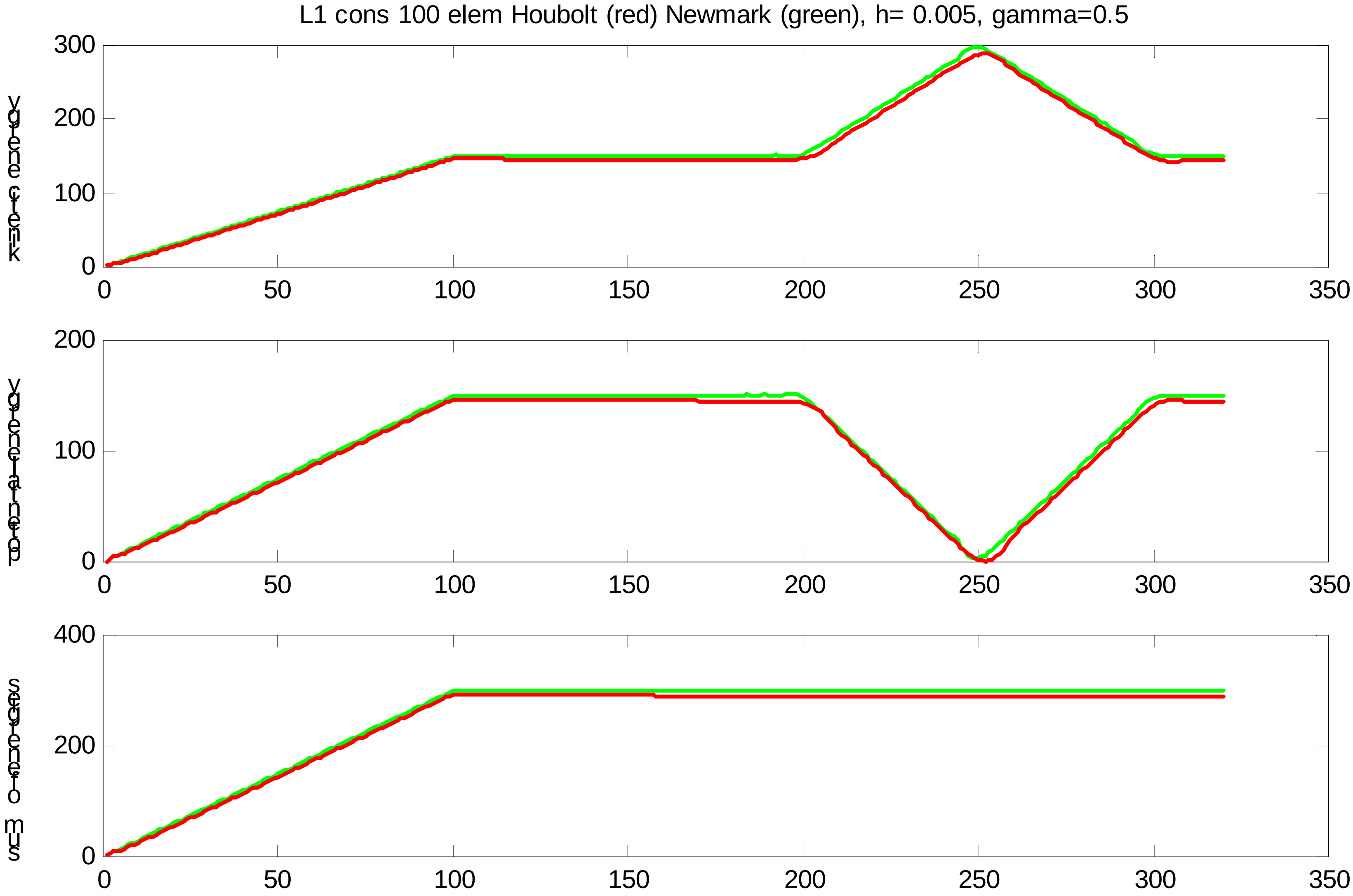
Newmark vs. Houbolt _1



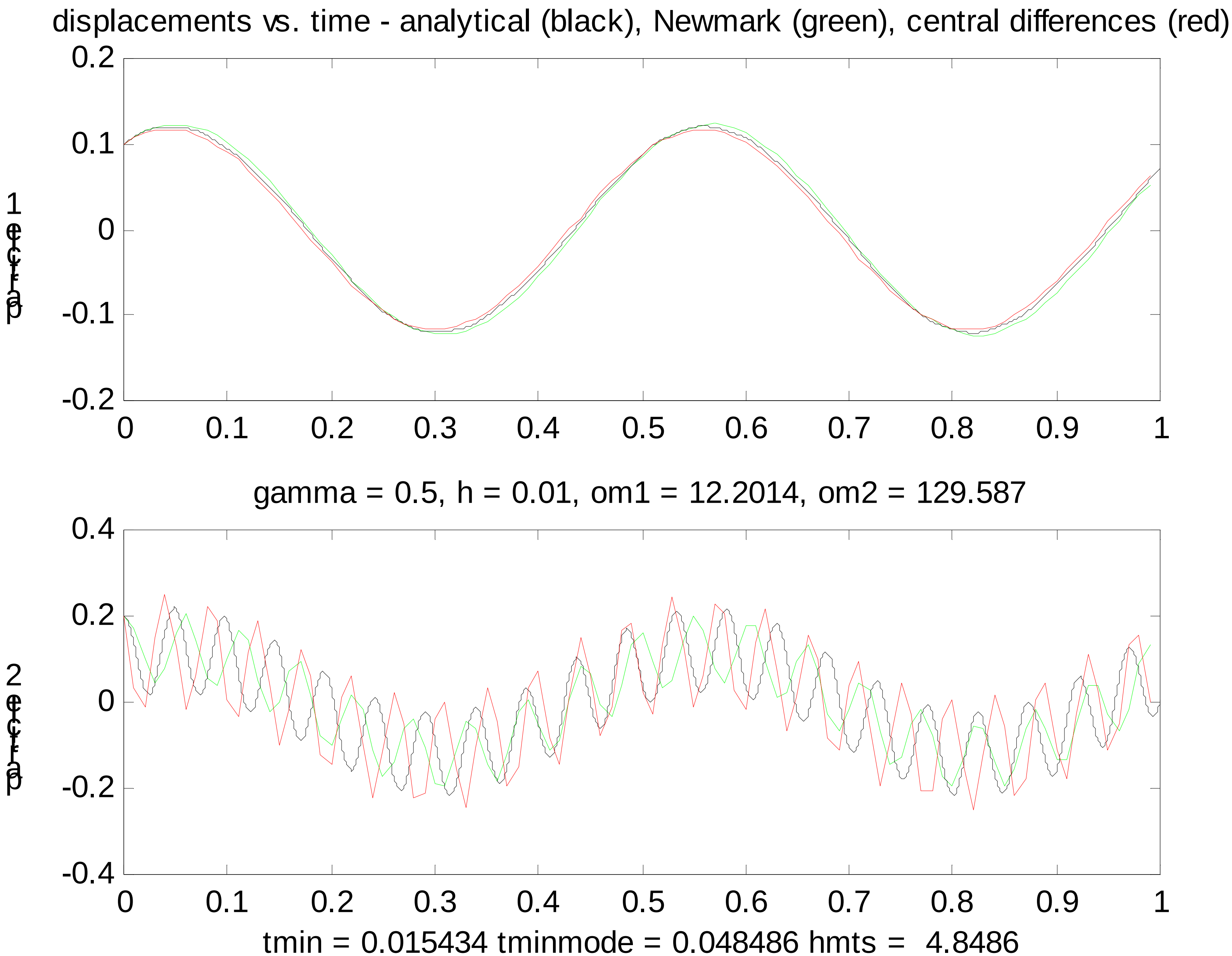
Newmark vs. Houbolt _2



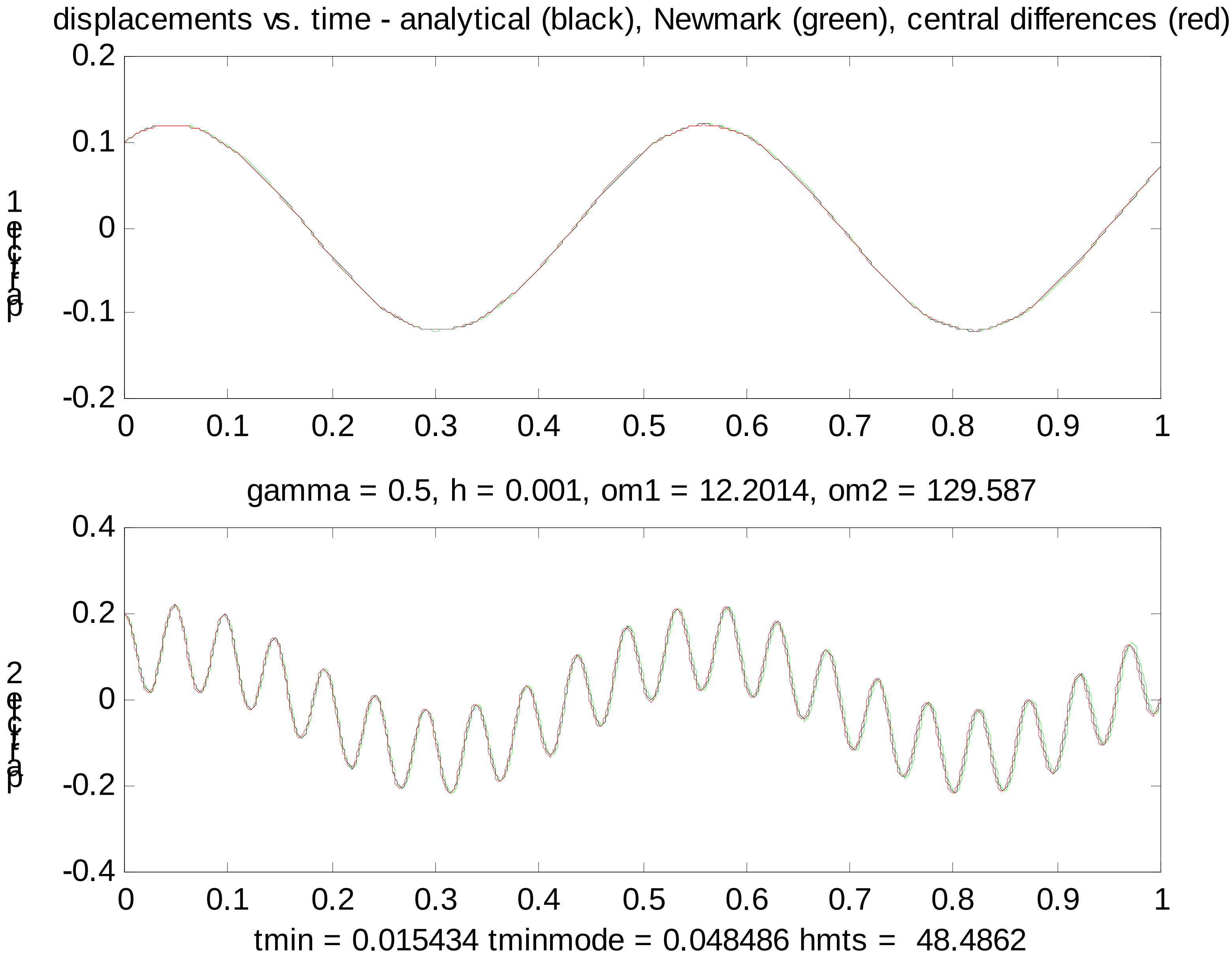
Newmark vs. Houbolt _3



2 dof's -Newmark vs. centr.diff_1



2 dof's -Newmark vs. centr.diff_2



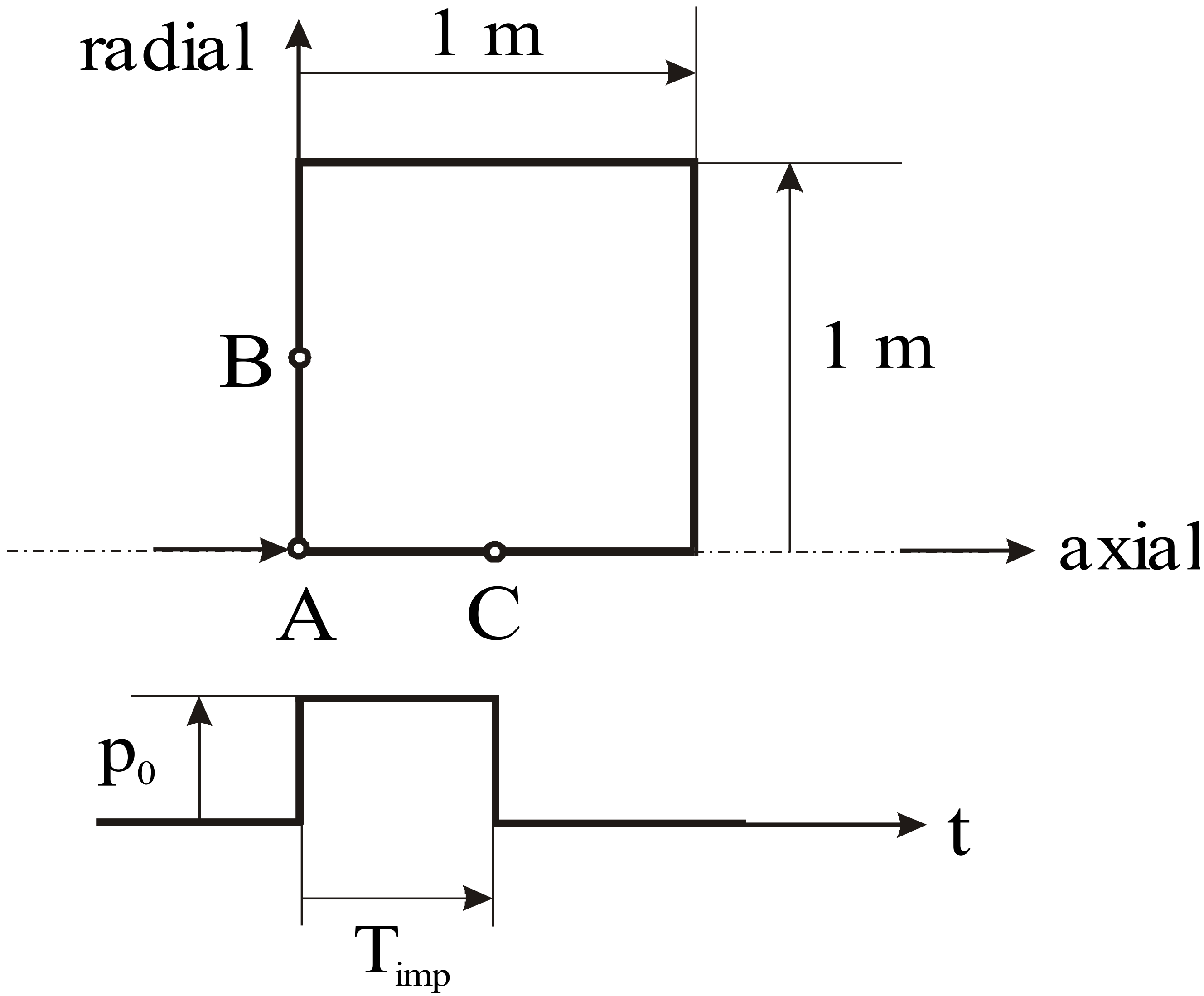
Examples

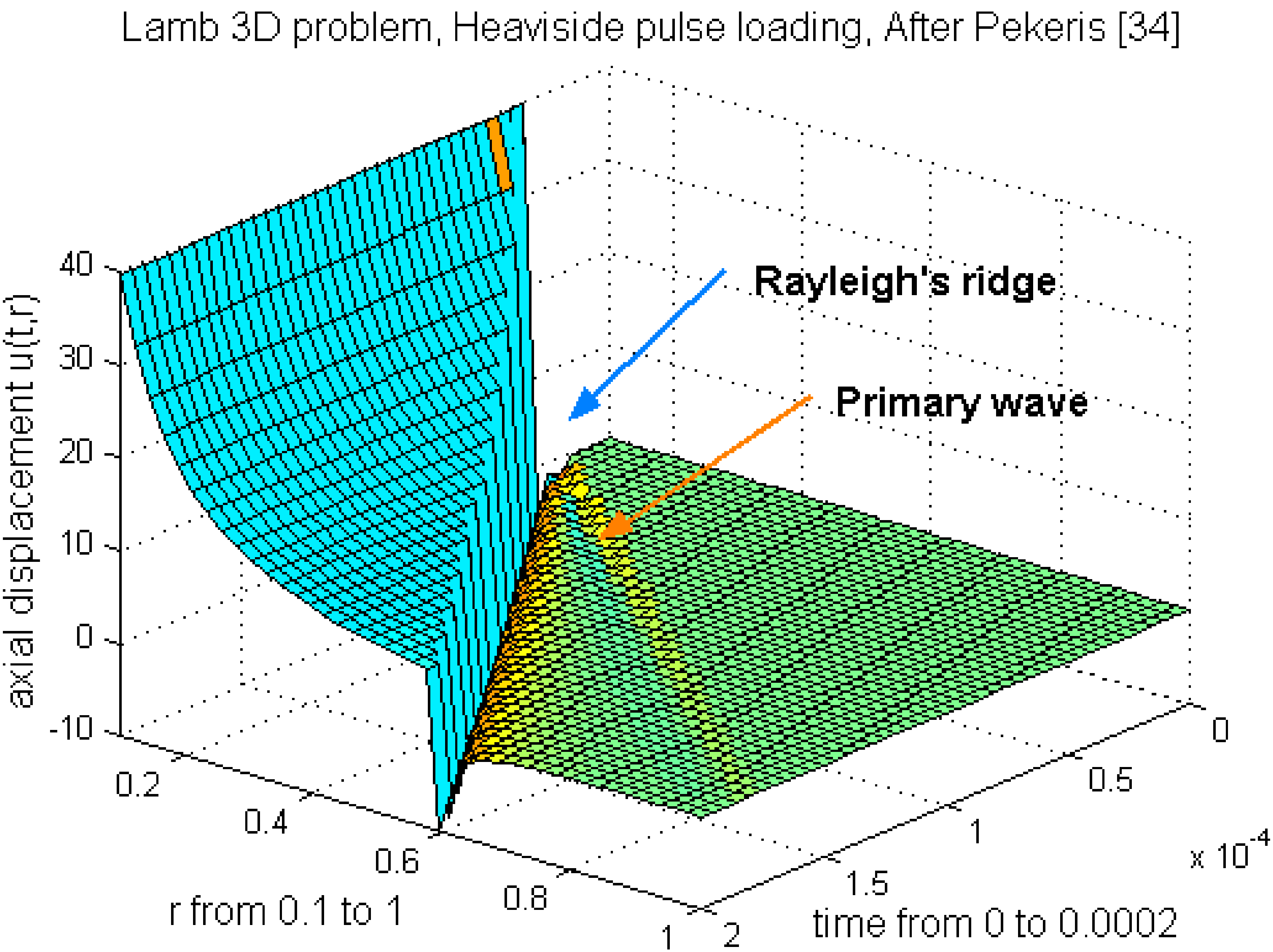
- 1D, solution by Laplace transform
- Love’s correction
- Davies’ experiment
- loading a half-plane
- loading a half-space, 3D Lamb, after Pekeris
- how to create energy making a finer mesh
- drop of a container
- crack propagation

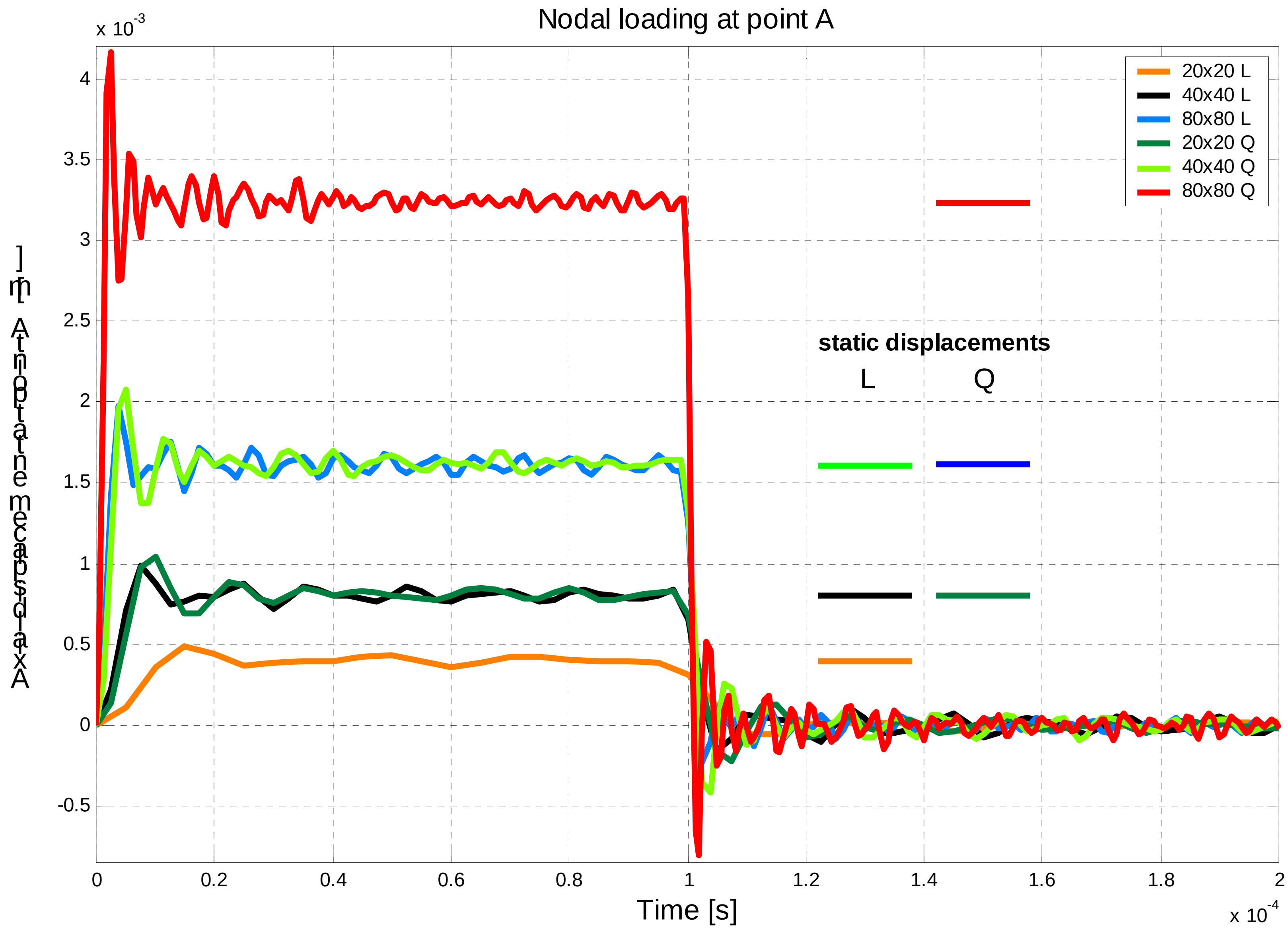
Half space loading by

- Lamb,
- Pekeris,
- Valeš and
- FEA

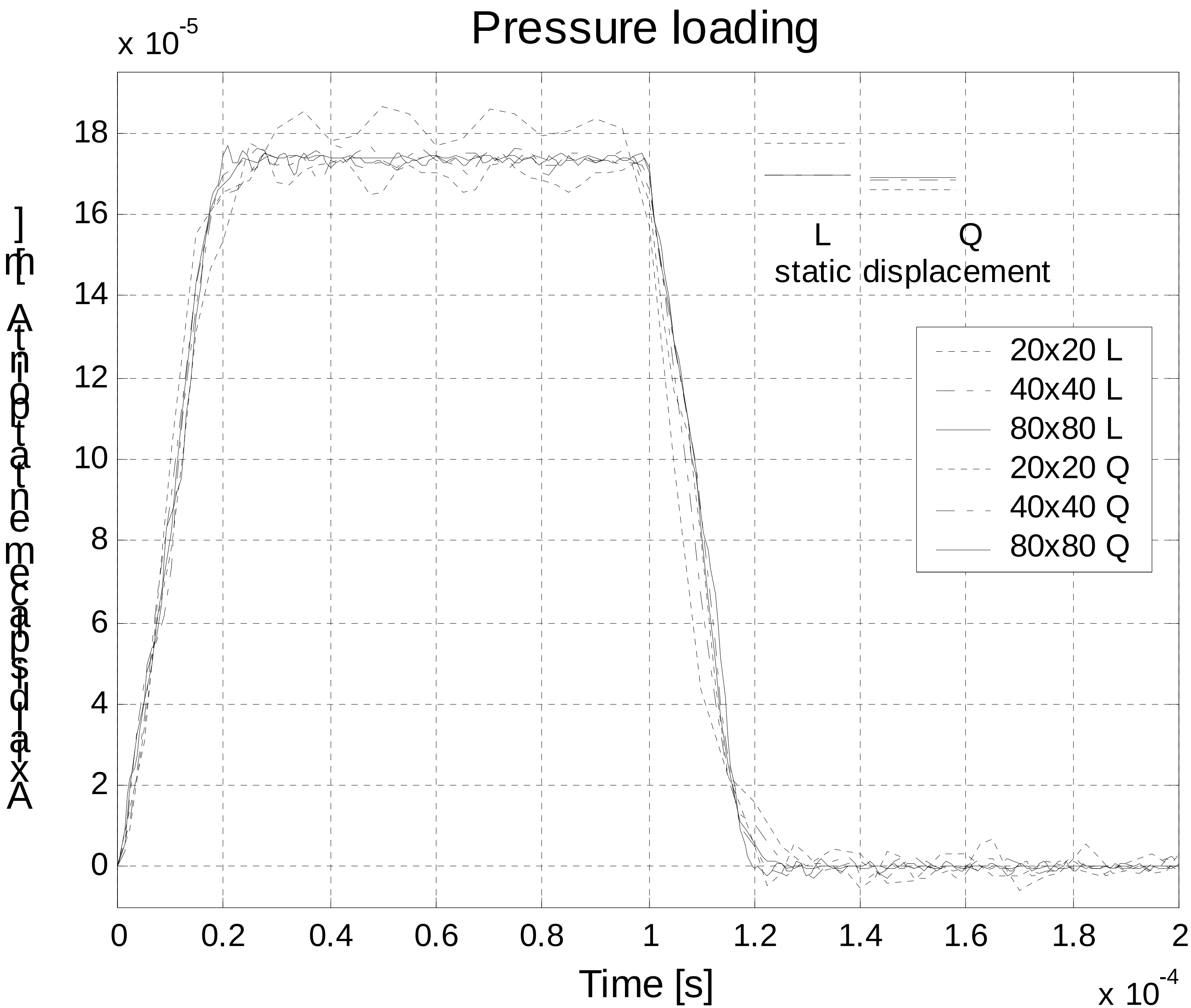
Transient loading of half space - Lamb 3D

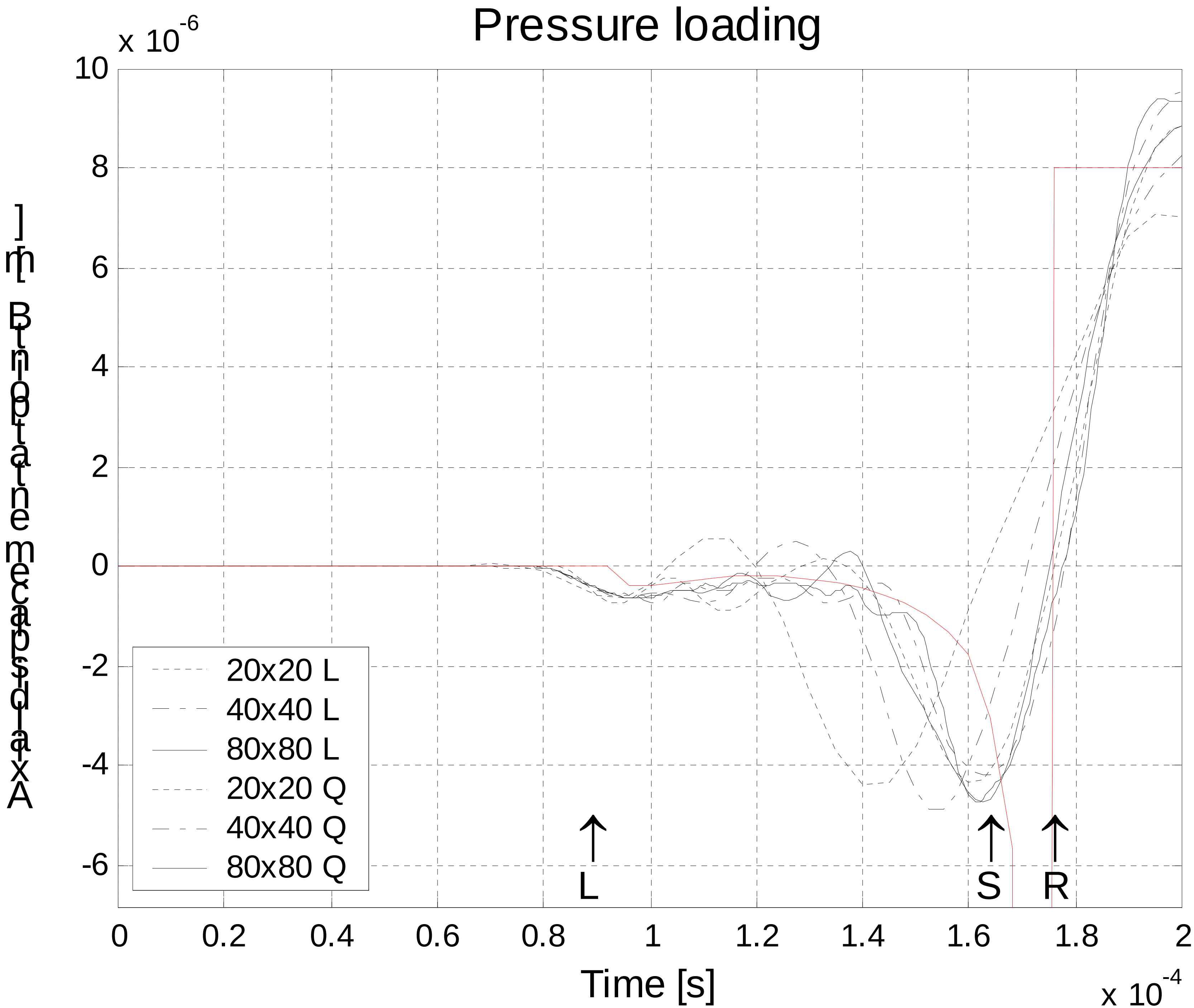



Lamb 3D problem, Heaviside pulse loading, After Pekeris [34]
Rayleigh's ridge
Primary wave
axial displacement u(t,r)
40
30
20
10
0
-10
0.2
0.4
0.6
0.8
1
r from 0.1 to 1
0
0.5
1
1.5
2
x 10^-4
time from 0 to 0.0002

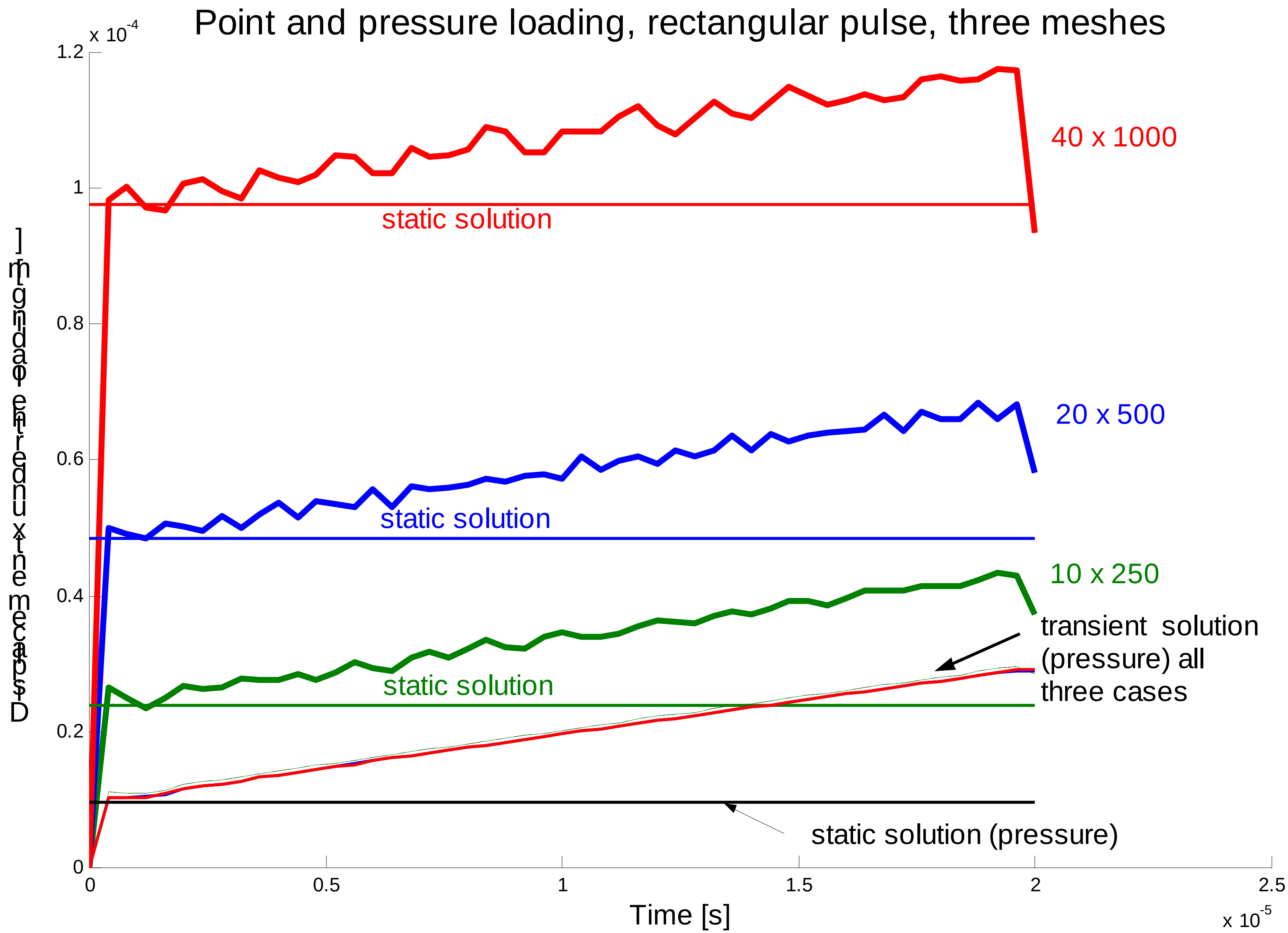


Nodal loading at point A
x 10^-3
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
Axial displacement at point A [m]
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Time [s]
x 10^-4
20x20 L
40x40 L
80x80 L
20x20 Q
40x40 Q
80x80 Q
static displacements
L
Q


Pressure loading
x 10^-5
18
16
14
12
10
8
6
4
2
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Time [s]
x 10^-4
L
Q
static displacement
20x20 L
40x40 L
80x80 L
20x20 Q
40x40 Q
80x80 Q



Pressure loading
x 10^-6
10
8
6
4
2
0
-2
-4
-6
20x20 L
40x40 L
80x80 L
20x20 Q
40x40 Q
80x80 Q
L
S
R
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Time [s]
x 10^-4



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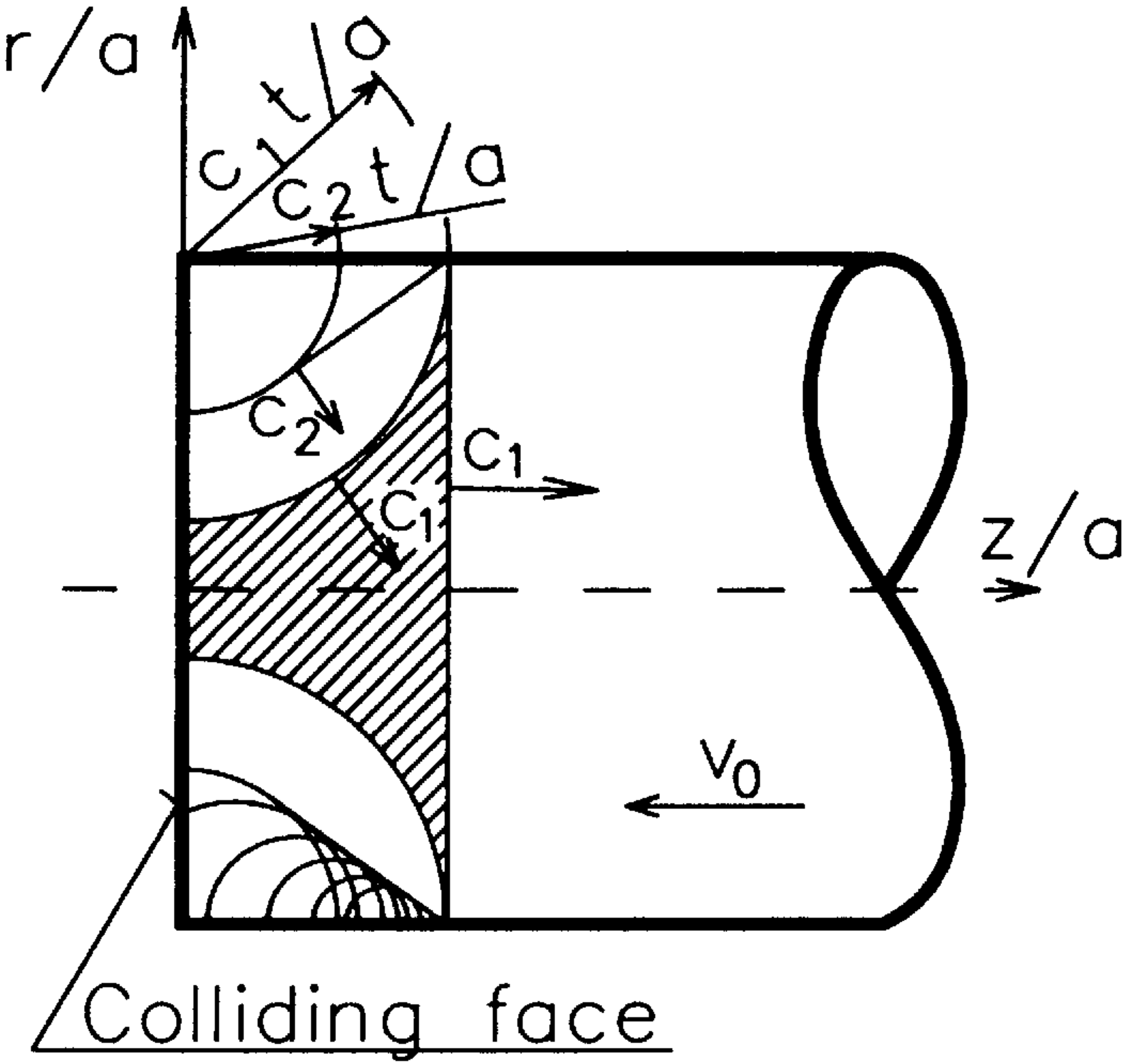
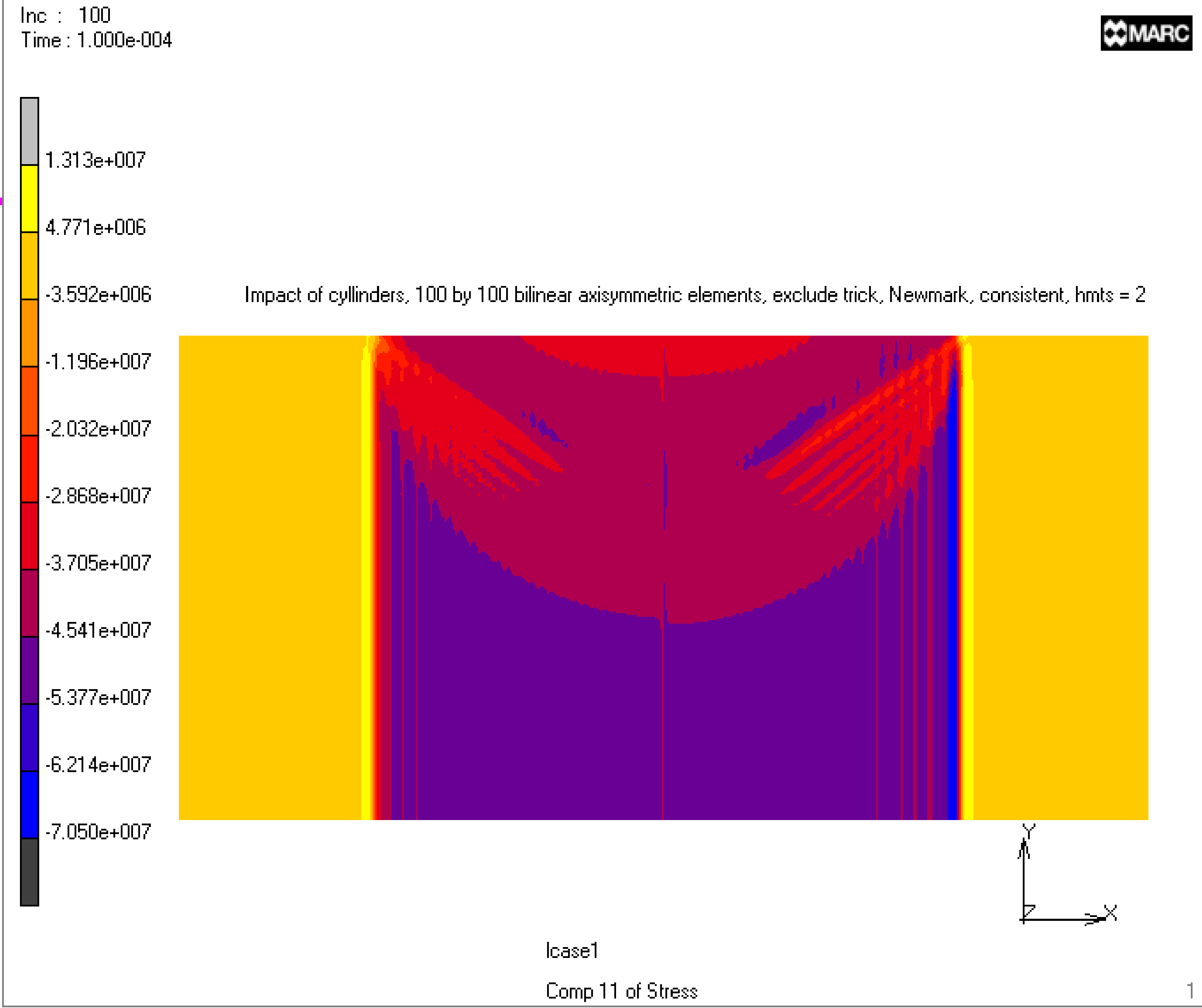
Impact of cylinders

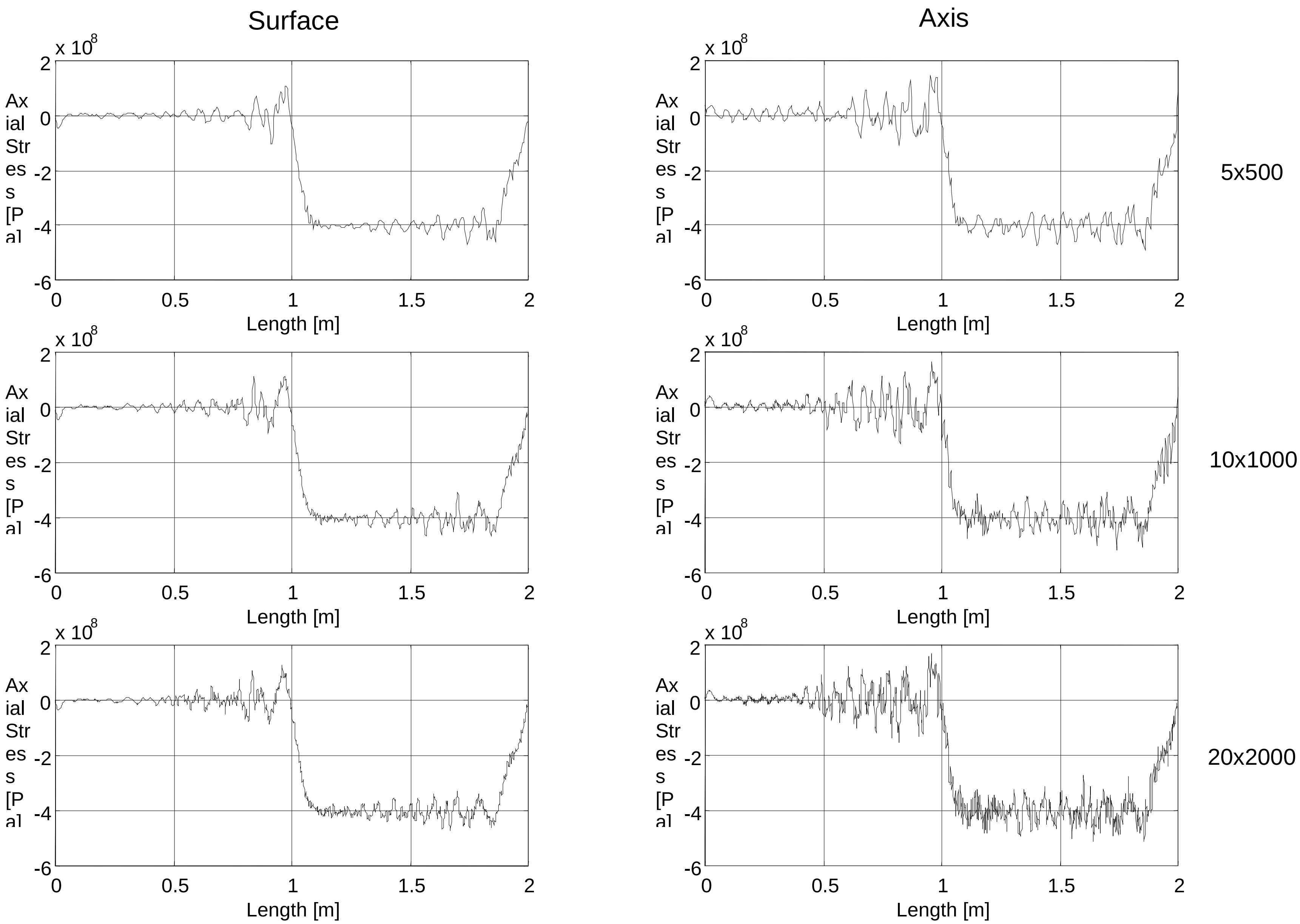
- Analytical solution by Valeš
- FEA
  - 100 by 100 Q4 axisymmetric elements
  - consistent mass formulation
  - Newmark 0.5
  - dirty exclude trick

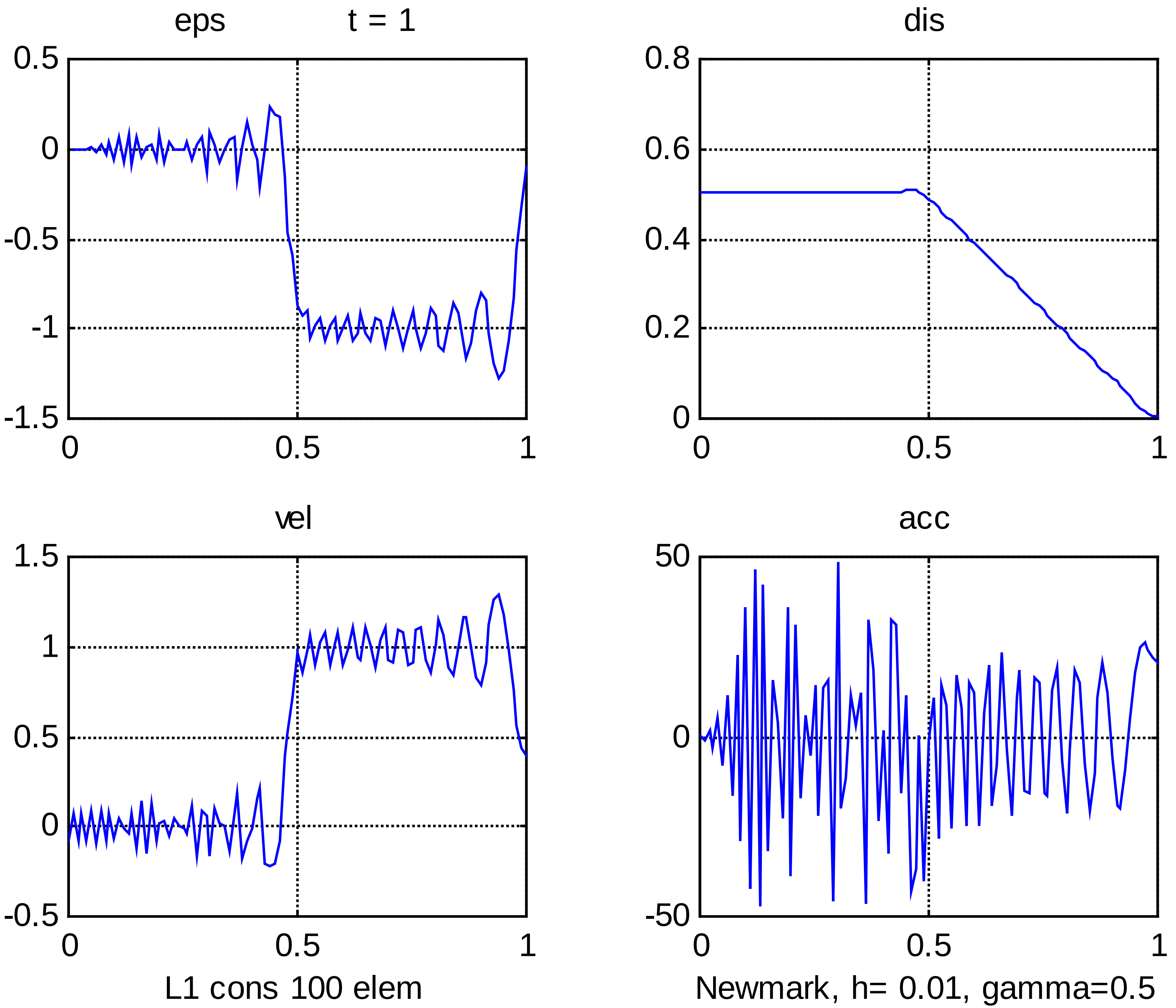
Wave pattern in colliding cylinders




Inc : 100
Time : 1.000e-004
MARC
1.313e+007
4.771e+006
-3.592e+006
-1.196e+007
-2.032e+007
-2.868e+007
-3.705e+007
-4.541e+007
-5.377e+007
-6.214e+007
-7.050e+007
Impact of cyllinders, 100 by 100 bilinear axisymmetric elements, exclude trick, Newmark, consistent, hmts = 2
Y
Z
X
lcase1
Comp 11 of Stress
1

Axisymmetric four-node elements





eps
t = 1
dis
vel
acc
0.5
0
-0.5
-1
-1.5
0.8
0.6
0.4
0.2
1.5
1
50
-50
L1 cons 100 elem
Newmark, h= 0.01, gamma=0.5

Percussive rock drilling_1

- Efficient technological process for rock destruction based on transfer of energy between hammer and drilling rod
- Analytical and FE appraches were used for efficiency computations
- Detailed stress pattern computation
- Time marching operators saving energy

Percussive rock drilling _ 2

- Axial impact loading, the change of moment of momentum due to a spiral grove, with the intention to improve drilling efficiency
- Not available in FE packages - in house PMD, 1mm element

Meshing for a rod with a groove

- 1 mm 8-node almost cubic element
- 483 400 elements
- 505 312 nodes
- 1 515 936 dof‘s

HD memory considerations in MB

- **For implicit operator**
  - stiff. matrices 1165
  - mass matrices 1165
  - topology 186
  - loading 12
  - frontal solver 62 214
  - 1 kin. q. all steps 7200
- **For explicit operator**
  - 1165
  - 1165
  - 186
  - 12
  - global lumped m. 12
  - 7200

Twin extension from the crack tip

- Existing crack, prestressed 2D, plane strain,
- then the specimen is relaxed
- and shock waves by transonic twinning are generated
- local kinetic energies are shown
- in-house molecular dynamics program
- in agreement with experiments for bcc iron

Impact analyses of nuclear waste vessel

- Drop from 9 m
- Stress processes are not important
- Elasto-plastic, rate dependent
- Rigid wall model is not acceptable
- Tuning the FE model, experiment is needed
- History of accelerations and the HMH stresses

FE computation - parallel approach

- LS DYNA, MPP machine,
- car barrier test, 15 mm elements, more than 1 million dof's,
- the results depend on the number of processors used,
- using the same number of processors, the same task gives different results when run at different occasions
- stress patterns are not studied

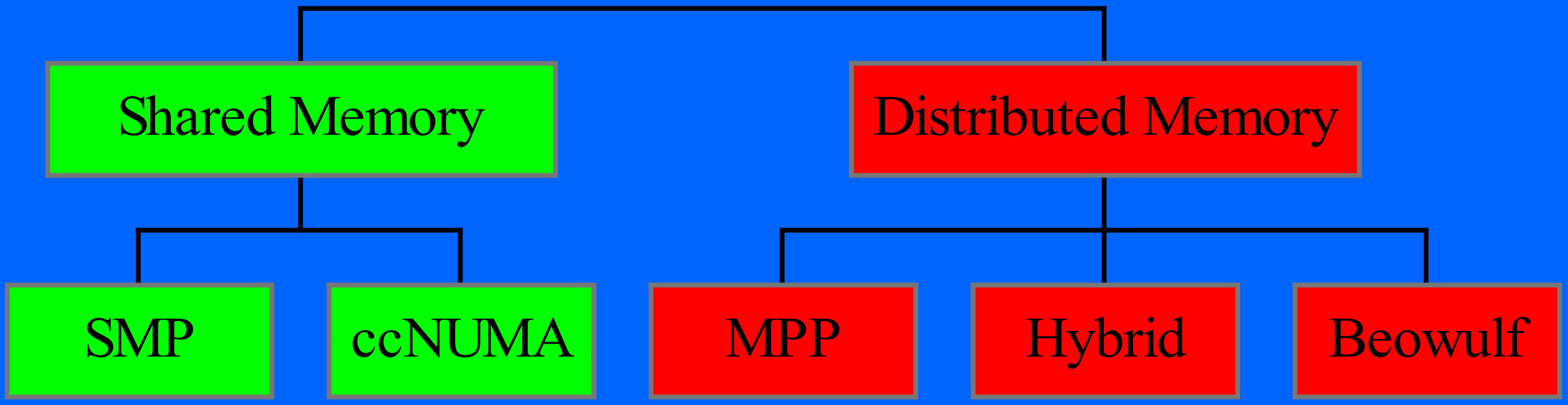
Vector versus parallel approaches - programming considerations

- Sometimes the vector computers are mistakenly considered as the opposites of parallel computers. Actually the vectorization is a form of parallel processing allowing the array elements to be processed by groups. The automatic vectorization of the code secured by vector machine compilers could result in a substantial reduction in computational time.

Vector versus parallel approaches - programming considerations

There is no rivalry between parallel and vector approaches. The future probably belongs to multiprocessor machines (each having hundreds of processors) with huge local memories and with fast and wide communication channels between processors.

Parallel hardware platforms



Software under intensive development, not yet unified

Conclusions - the art of modelling

- **Will man Schweres bewältigen, muss man es sich leicht machen.**

- **If you want to achieve something that is difficult, you must first make it easy.**

Bertold Brecht

The final remark

" ... **to find the point of view from which the subject appears in its greatest simplicity**".

The quotation is due to J.W. Gibbs (1839 - 1903).

In conclusion

- Stress wave computation is a fascinating subject of computational mechanics - usually of XXL size
- State of art
  - solid theoretical foundations, efficient hardware, quickly developing parallel software and people knowing FE craftsmanship
- Goals
  - validity analysis of constitutive equations for non-linear and/or large deformations, mainly by experimental verification, leading to robust and foolproof implementation of NL solvers.

Leftovers

2D wave fronts - Huygen's principle