

## FURTHER REMARKS ON THE NEO-CLASSICAL NAVIER-STOKES EQUATIONS

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The seminal Navier-Stokes equations have been stated yet before creation of principles of thermodynamics and the first and second laws. In the literature there is the common opinion that the Navier-Stokes equations cannot be taken as a thermodynamically correct model of “working fluid” which is able to describe transformation of “heat” into “work” and vice versa. Therefore, in the paper, a new exposition of thermodynamically consistent Navier-Stokes equations is presented. Keeping the line of reasoning, based on the Gyftopoulos and Beretta concept of thermodynamics [1], we will introduce the basic concepts of thermodynamics with the notion of “heating” and “working” fluxes, and will extend the Gyftopoulos and Beretta approach into three-dimensional continuum thermodynamics.

In our approach notion of “energy” and “energy interactions” play a dominant role. The main inconsistencies connected with the internal energy concept as a form of „energy storage” and the transformations of different forms of energy are evaluated. Thus, the balance of energy is finally presented as a sum of internal, kinetic, potential and radiation energies in the system that are compensated by the total energy flux, which consists of working, heating, chemical, electric, magnetic and radiation energy fluxes at the system boundaries. The law of energy conservation can be considered as the most important one which is superior over any other laws of nature.

Let us consider as shortly as possible an example of the Gyftopoulos-Beretta exposition of thermodynamics strictly adopted to the classical Navier-Stokes-Fourier equations. It leads to a neo-classical Navier-Stokes model of working fluid which is a simple viscoelastic, “heat” conducting fluid. The recoverable constitutive relations are expressed only by the intensive state parameters: velocity  $\vec{v}$  (specific density of momentum); specific volume  $v$  and specific entropy  $\eta$ . Specific internal energy is a function of intensive parameters:  $\varepsilon = \varepsilon(v, \eta)$ .

Let start from the primal energy balance in the form of a global equality:

$$\frac{d}{dt}(\mathcal{U} + \mathcal{K} + \Phi) = \oint_{\partial V} (\vec{\mathcal{F}}_{work} + \vec{\mathcal{F}}_{heat}) \cdot \vec{n} dA \quad . \quad (1)$$

Here for our system  $A$  [e.g. a simply connected body  $\mathcal{B}$ ], with a total volume  $V$  and a boundary  $\partial V$ , oriented with a normal vector  $\vec{n}$ , we have denoted internal energy, by:  $\mathcal{U} = \iiint_V \rho \varepsilon dV$ ; kinetic energy, by:  $\mathcal{K} = \iiint_V \rho \kappa dV$ ; potential energy of body forces, by:  $\Phi = \iiint_V \rho \phi dV$ . Equation (1) reads, that total change of stored energy of a body  $\mathcal{B}$  is compensated by energy being supplied (extracted) through the mechanical energy flux:  $\vec{\mathcal{F}}_{work} = \vec{t}\vec{v} + p\vec{h}_v$  and the thermal energy flux:  $\vec{\mathcal{F}}_{heat} = \vec{h}_s\theta$ . Definitions for these fluxes are fundamental for the primal statement of energy balance in the Carnot sense – they contain total, and not only reversible, momentum flux  $\vec{t}$  and total not only diffusive flux of entropy  $\vec{h}_s$  as well as the volume flux  $\vec{h}_v$ .

Even though the mechanical and thermal energy fluxes are applied on disjoint side surfaces  $\partial V = A_{heat} \cup A_{work}$ , there is no mathematical obstacle whatsoever, to employ Stokes' theorem on the divergence. Assuming, that:  $\frac{d}{dt}\rho + \rho \operatorname{div} \vec{v} = 0$  or, in other words – satisfaction of mass balance, equation (1) may be rewritten:

$$\iiint_V [(\varepsilon + \kappa + \phi)(\dot{\rho} + \rho \operatorname{div} \vec{v}) + \rho(\dot{\varepsilon} + \dot{\kappa} + \dot{\phi}) - \operatorname{div}(\vec{t}\vec{v} + \vec{h}_v p + \vec{h}_s \theta)] dV = 0 \quad (2)$$

This equation is always true, regardless the set of governing equations (that is, equations for mass, momentum, volume and entropy balances) is satisfied or not. It will now be shown that, if the fields and parameters of state satisfy conditions laid on by governing equations, equation of energy receives a very important form allowing for further restrictions to describing fields – it means that even though the solutions may satisfy conservation equations for mass, momentum, volume and entropy, they need not to satisfy conservation equation for energy, and in consequence lead to generating energy *ex nihilo*.

In eq. (2), according to Brenner's [2] concept of volume balance, we introduce the following splitting of the volume flux:

$$\vec{h}_v = \vec{v} + \vec{h}'_v \quad , \quad (3)$$

what is consistent with postulate that the volume flux is identical with the volume velocity:  $\vec{h}_v = \vec{v}_v = \vec{v} + \alpha_B \operatorname{grad} p$  [2].

Material derivatives of kinetic and potential energy are  $\dot{\kappa} = \dot{\vec{v}} \cdot \vec{v}$  and  $\dot{\phi} = -\frac{\partial \phi}{\partial \vec{x}} \cdot \dot{\vec{x}} = -\vec{b} \cdot \vec{v}$ , respectively. The following identities also occur:

$\operatorname{div}(\vec{t}\vec{v}) = \operatorname{div} \vec{t} \cdot \vec{v} + \vec{t} \cdot \operatorname{grad} \vec{v}$  and  $\operatorname{div}(\vec{h}_s \theta) = \theta \operatorname{div} \vec{h}_s + \vec{h}_s \cdot \operatorname{grad} \theta$ . According to the well-accepted tradition, the velocity gradient will be denoted by  $\vec{l} = \operatorname{grad} \vec{v}$  and decomposed into symmetric and anti-symmetric parts  $\vec{l} = \vec{d} + \vec{w}$ . Temperature gradient will be denoted by  $\vec{g}_s = \operatorname{grad} \theta$  and pressure gradient by  $\vec{g}_v = \operatorname{grad} p$  [3].

Through addition and subtraction of  $\pm \theta(\rho \dot{\eta} - \rho n_s)$  and  $\pm p(\rho \dot{v} - \rho n_v)$ , the integral of equation (2) becomes:

$$\begin{aligned} & \iiint_V [\rho \dot{\varepsilon} - (\operatorname{div} \vec{t} + \rho \vec{b} - \rho \dot{\vec{v}}) \cdot \vec{v} - (\operatorname{div} \vec{h}_s + \rho n_s - \rho \dot{\eta}) \theta \\ & - (\operatorname{div} \vec{h}_v + \rho n_v - \rho \dot{v}) p - (\varepsilon + \kappa + \phi)(\dot{\rho} + \rho \operatorname{div} \vec{v}) \\ & + \theta(\rho \dot{\eta} - \rho n_s) + p(\rho \dot{v} - \rho n_v) - \vec{h}_s \cdot \vec{g}_s - \vec{h}'_v \cdot \vec{g}_v - \vec{t}' \cdot \vec{l}] dV = 0 \quad (4) \end{aligned}$$

Once the fields  $\rho, \vec{t}, \vec{v}, \vec{h}_s, \eta, \vec{h}_v, v$  satisfy balance equations of mass, momentum, moment of momentum  $\vec{t} = \vec{t}^T$  as well as volume and entropy, including, that  $\vec{t} \cdot \vec{l} = \vec{t} \cdot \vec{d}$ , and  $\vec{t}' = \vec{t} - p \vec{l}$  energy equation (4) reduces to :

$$\iiint_V [\rho \dot{\varepsilon} - \theta(\rho \dot{\eta} - \rho n_s) - p(\rho \dot{v} - \rho n_v) - \vec{h}_s \cdot \vec{g}_s - \vec{h}'_v \cdot \vec{g}_v - \vec{t}' \cdot \vec{d}] dV = 0 \quad (5)$$

Even if an thermo-elastic fluid is expressed via scalar state parameters  $v, \eta$ , what is much simple then in solids, the mechanism of the internal energy storage is more complex, since of multiplicative, not additive, contribution to  $\varepsilon = \varepsilon(\eta, v)$ . The time

material (not the spatial one) rate of internal energy is:  $\dot{\varepsilon} = \frac{\partial \varepsilon}{\partial \eta} \dot{\eta} + \frac{\partial \varepsilon}{\partial v} \dot{v}$ . Therefore, by taking material derivative of the internal energy, we obtain :

$$\iiint_V [\rho(\frac{\partial \varepsilon}{\partial v} - p)\dot{v} + \rho(\frac{\partial \varepsilon}{\partial \eta} - \theta)\dot{\eta} + \rho\theta n_S + \rho p n_V - \vec{h}_S \cdot \vec{g}_S - \vec{h}'_V \cdot \vec{g}_V - \vec{t}' \cdot \vec{d}] dV = 0 . \quad (6)$$

Working fluid will be thermo-elastic in the Carnot sense if, and only if thermodynamic pressure and thermodynamic temperature are connected with internal energy by equations of state  $p = \frac{\partial \varepsilon}{\partial v}$ ;  $\theta = \frac{\partial \varepsilon}{\partial \eta}$ . These two constitutive relations are fundamental for

proving that balance of total energy can be fulfilled in any processes governed by balance of mass, momentum, moment of momentum, volume and entropy. In a case of ideal gas, the internal energy depends on two constitutive coefficients  $c_v$  and  $c_p$  - the specific heat in constant volume and pressure, respectively. From this pair;  $c_v, c_p$ , one

can obtain another pair – Carnot : (1824)  $R = c_p - c_v$  and Poisson (1831):  $\gamma = \frac{c_p}{c_v}$  :

$$\varepsilon(\eta, v) = \frac{1}{\gamma - 1} (v)^{1/(\gamma-1)} \exp\left(\frac{\eta}{c_v}\right) , \quad (7)$$

which leads to the well-established constitutive equations :

$$p = v^{-\gamma} \exp\left(\frac{\eta}{c_v}\right) ; \quad \theta = \frac{1}{\gamma - 1} (v)^{1-\gamma} . \quad (8)$$

Finally, remaining part of the balance of energy is :

$$\mathcal{S}_e = \iiint_V [\rho\theta n_S + \rho p n_V - \vec{h}_S \cdot \vec{g}_S - \vec{h}'_V \cdot \vec{g}_V - \vec{t}' \cdot \vec{d}] dV = 0 . \quad (9)$$

This says that dissipative contribution to energy balance should be self-equilibrated, since dissipation does not mean creation *ex nihilo*. Equation (9) can be also interpreted as an “inner dissipative conversion of energy”. Thus, in some sense eq. (9) could be treated to be definitions of irreversible contribution  $n_V$  (volume production) and  $n_S$  (entropy production). If these contributions would be non-negative  $n_V \geq 0$ ,  $n_S \geq 0$  then remaining parts are also non-negative  $\vec{h}_S \cdot \vec{g}_S \geq 0$ ;  $\vec{h}'_V \cdot \vec{g}_V \geq 0$ ;  $\vec{t}' \cdot \vec{d} \geq 0$ .

The condition of self-equilibrated dissipation (9) can be fulfilled also if we define  $n_V$  and  $n_S$  in form of specific dissipation potential  $\varpi$ . In fluid the momentum flux tensor  $\vec{t}'$  becomes an additive composition of spherical and shape viscous contributions; ea. :  $\vec{t}' = -(\pi - p)\vec{I} + \vec{\tau}$ . In general, the viscous part is traceless:  $\text{tr}(\vec{\tau}) = 0$ , and the volume-like viscous part is a spherical one:  $(\pi - p)\vec{I}$ . Thus, the expression  $\vec{t}' \cdot \vec{d}$  in eq.(9) turns into :  $(\pi - p)\text{div } \vec{v} + \vec{\tau} \cdot \vec{d}$ , and

$$\theta \rho n_S + p \rho n_V = \vec{h}'_V \cdot \vec{g}_V + \vec{h}_S \cdot \vec{g}_S + (\pi - p)\text{div } \vec{v} + \vec{\tau} \cdot \vec{d} . \quad (10)$$

However, the viscous and work and heat diffusion properties depend on the following dissipative potential:

$$\varpi(\vec{g}_V, \vec{g}_S, \vec{d}) = \frac{1}{2} k_\eta \vec{g}_S \cdot \vec{g}_S + \frac{1}{2} \alpha_B \vec{g}_V \cdot \vec{g}_V + \mu \vec{d} \cdot \vec{d} + \left(\frac{2}{3} \mu - \lambda'\right) (\text{tr } \vec{d})^2 , \quad (11)$$

where:  $k_\eta$  is an entropy diffusion coefficient connected with the Fourier conductivity coefficient  $k$  simply as:  $k_\eta = \theta k$ ; and  $\mu$  and  $\lambda'$  are the Stokes shape and volumetric viscosity coefficients and  $\alpha_B$  is the Brenner coefficient. Making use of (10) and the definitions of diffusive fluxes:  $\vec{\tau} = \frac{\partial \varpi}{\partial \vec{d}}$ ,  $\vec{h}_S = \frac{\partial \varpi}{\partial \vec{g}_S}$ ,  $\vec{h}'_V = \frac{\partial \varpi}{\partial \vec{g}_V}$ ,  $\pi - p = \frac{\partial \varpi}{\partial \text{tr} \vec{d}}$  finally we obtain the well-established constitutive equations:

$$\vec{h}_S = k_\eta \vec{g}_S, \quad \vec{h}'_V = \alpha_B \vec{g}_V; \quad \vec{\tau} = 2\mu \vec{d} + \frac{2}{3}\mu(\text{tr} \vec{d})\vec{I}, \quad \pi - p = \lambda' \text{tr} \vec{d}. \quad (12)$$

Additionally, several restrictions are imposed on constitutive coefficients – there are – the Petite inequalities  $c_p > c_v > 0$ ; the Lamè-Kowalski inequality  $k_\eta > 0$ , the Duhem inequalities  $\mu > 0$ ,  $2\mu + 3\lambda' > 0$  and so on. From positivity of dissipation functional it follows also positivity of the Brenner coefficient  $\alpha_B \geq 0$  [3].

Summarizing of the consequences of the Gyftopoulos-Beretta thermodynamics exposition – the presented above mathematical model of continuum – the neo-classical Navier-Stokes - has a thermodynamic consistency, what means that unknown fields assigned from a proper set of governing equations must additionally fulfill the condition of not creating energy from nothing [ $\mathcal{S}_e = 0$ , eq.(9)].

Second consequence is: energy conversion within the working fluid is impossible if state parameters are defined by other than eq. (8) constitutive equations. For multiplicative form of specific energy, like eq.(7) is, the energy conversion is possible without additional assumption, however, for an additive expression of internal energy, like in thermo-elastic solids, or thermo-electric fluids, convertibility is assurance only by additional crossing term in constitutive relations.

Third consequence is the condition of self-equilibrated dissipation which follows from condition of  $\mathcal{S}_e = 0$  and frequently postulated in the literature the principle of mutual “inner conversion” of dissipative energy. Therefore, numerous inequalities [like the Drucker-Prager inequality for granular solids] used in the literature have their sources in the first, not in the second, law of thermodynamics. Finally, it means that the classical Clausius-Duhem inequalities have no deeper foundations and nowadays should be treated as historic no right composition of Clausius’ global entropy inequality with Duhem’s local energy inequality.

Fourth consequence is a new role of vector of entropy flux for which we have proposed an independent constitutive equation (12)<sub>1</sub>. It is helpful solution, since the seminal question of interrelation between thermal energy flux and entropy flux is now eliminated.

## References

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- [3] J. Badur, P. Ziółkowski, Without heat and work – further remarks on the Gyftopoulos-Beretta exposition of thermodynamics, *Int. Journal of Thermodynamics* 2013 (submitted)