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# Universal equations and constants of turbulent motion

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## Abstract

This paper presents a parameter-free theory of shear-generated turbulence at asymptotically high Reynolds numbers in incompressible fluids. It is based on a two-fluids concept. Both components are materially identical and inviscid. The first component is an ensemble of quasi-rigid dipole-vortex tubes (vortex filaments, excitations) as quasi-particles in chaotic motion. The second is a superfluid performing evasive motions between the tubes. The local dipole motions follow Helmholtz' law. The vortex radii scale with the energy-containing length scale. Collisions between quasi-particles lead either to annihilation (likewise rotation, turbulent dissipation) or to scattering (counterrotation, turbulent diffusion). There are analogies with birth and death processes of population dynamics and their master equations and with Landau's two-fluid theory of liquid helium. For free homogeneous decay the theory predicts the turbulent kinetic energy to follow  $t^{-1}$ . With an adiabatic wall condition it predicts the logarithmic law with von Kármán's constant as  $1/\sqrt{2\pi} = 0.399$ . Likewise rotating couples form localized dissipative patches almost at rest ( $\rightarrow$  intermittency) wherein under local quasi-steady conditions the spectrum evolves into an 'Apollonian gear' as discussed first by Herrmann (1990 *Correlation and Connectivity* (Dordrecht: Kluwer) pp 108–20). Dissipation happens exclusively at scale zero and at finite scales this system is frictionless and reminds of Prigogine's (1947 *Etude Thermodynamique des Phenomenes Irreversibles* (Liege: Desoer) p 143) law of minimum (here: zero) entropy production. The theory predicts further the prefactor of the 3D-wavenumber spectrum (a Kolmogorov constant) as  $\frac{1}{3}(4\pi)^{2/3} = 1.802$ , well within the scatter range of observational, experimental and direct numerical simulation results.

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*The diversity of problems in turbulence should not obscure the fact that the heart of the subject belongs to physics.*

– Falkovich and Sreenivasan (2006) [22].

## 1. Introduction

Many efforts to solve the turbulence problem rest on the idea that the Navier–Stokes equation (NSE) plays the role of a God equation and the application of a certain number of mathematical operations onto NSE could do it. In particular, the Fridman–Keller [42] series expansion of NSE fuelled expectations. Its first-moment element is the Reynolds [66] equation and higher expansion elements for higher moments are subject to various closure hypotheses (for an overview see [82]).

It is often said that closure problems have fundamental character; that they cannot be overcome due to the nonlinearity of NSE. This is even mildly echoed in one of the Millenium-Prize problems of the Clay Mathematical Institute, announced in 2000. However, already physicists of the Renaissance were confronted with strongly nonlinear problems such as celestial (Newton, Kepler and Galileo) and, later, molecular mechanics (Maxwell, Boltzmann)—and solved them. As often, an adequate viewpoint matters. Closure efforts could not answer most elementary questions about turbulence, but they helped to navigate in the jungle. Here, we present an alternative which profited from them<sup>1</sup>.

<sup>1</sup> Everywhere the pronoun 'we' is used in this text, it means the two of us, the dear reader and the author.

A first alternative was already proposed by Prandtl [62] who discussed turbulence in terms of analogies with molecular diffusion, gas kinetics and Brownian motion in the interpretation by Einstein [21]. Prandtl related his mixing length (*Mischungsweg*) to the mean-free path of kinetic gas theory. This concept became popular but detailed questions could not be answered without use of measurements. Although Prandtl has been heavily criticized by Batchelor [6], other authors designed free-hand analogies for turbulence, too, e.g. the early  $K-\omega$  model by Kolmogorov [44], corrected and improved by Saffman [68], further improved by Wilcox [82]. They are today part of a larger set of so-called two-equation turbulence models like  $K-\varepsilon$ ,  $K-K\mathcal{L}$ ,  $K-\tau$ , etc.<sup>2</sup>

Three years before Prandtl, Debye and Hückel [16] based their theory of electrolytes on the assumption that each ion is surrounded by a spherical ‘cloud’ (or screen) of ions of opposite charge, so ‘screening’ the ion. The ideas of Prandtl, Debye and Hückel are early examples of constructive theories for large many-particle ensembles in very different branches of physics, where elementary knowledge of parts and pieces does not suffice to describe the system’s global behavior.

A similar methodical challenge evolved later in the fields of superconductivity and superfluidity for groups around Landau and Feynman (see [3, 23, 49, 50]). They realized that their fundamental equations and principles<sup>3</sup> hold perfectly for the single atom, but properties like electric conductivity (or the gloss of metallic surfaces) *emerge* only when a larger *ensemble* of atoms is put together so that new specific features and interrelations come into play, without violation of fundamental equations and principles, of course.

For superfluidity it was necessary to supplement fundamental equations by the two-fluid theory of Landau (1941)—a not too distant relative of the ion-cloud concept of Debye and Hückel. The interesting point here is that the role of the supplements becomes so strong that they dominate the theory.

In many cases, the formation of ensembles means the breaking of symmetries. So eventually the concept of *emergence* and *broken symmetries* gave physics in the last few years a new and much less reductionistic face than it had until 1923 (see [3, 27, 29, 51]).

Encouraged by the victory of the two-fluid concept in superfluidity, Liepmann [54]) and later on Spiegel [74] made steps to test it for classical fluid turbulence. Along this road, which we follow here too, Spiegel was the very first who used terms such as *excitations*, *quasi-particles* and *two fluids* in the context of classical fluid turbulence, followed later by Spalding [73].<sup>4</sup>

These authors were at least fully aware of the principle, deep-rooted and almost philosophical problems with reductionistically NSE-based approaches. Spiegel underlined recently his thorough conviction again in his closing remarks of the report on the final session of the *Turbulence Colloquium (Marseille 2011)*.<sup>5</sup>

Besides ancient precursors of these ideas like René Descartes (1596–1650), who spoke about ‘tourbillons’

forming the universe, and Lord Kelvin, who coined in 1867 the notion ‘vortex atoms’ (loc. cit. Saffman [70]) we have to mention Marmanis [56] who possibly was the first to propose vortex *dipoles* as *the* fundamental quasi-particles of turbulence. Finally, the numerical Monte-Carlo eddy-collision methods with various vortex-filament primitives indicate that ideas developed by practitioners are not too far away from our theoretical views (e.g. [1]).

We elaborate below the two-fluid concept in greater detail; not exhaustively because the number of potential applications and side-problems is huge. However, we show that, in an idealized sense, turbulence in an incompressible and inviscid continuum fluid ( $Re \rightarrow \infty$ ) can be understood as a statistical many-body ensemble—a tangle of vortex-dipole tubes (or filaments) taken as interacting *discrete particles*. We will answer a number of open questions of turbulence without use of empirical parameters.

## 2. Broken symmetry and irreversibility

In his anti-reductionistic article summarizing experiences of condensed-matter physics, Anderson [2] states that ‘more is different’. The turbulence theory presented below is an extreme example. While the case of *one* quasi-particle (an almost frictionless vortex dipole of zero net circulation) in a volume is non-dissipative and symmetric with respect to time and circulation, already the presence of *two* quasi-particles in a common volume has the potential to break the symmetry and to give rise to the emergence of turbulent dissipation: according to the laws of Helmholtz, dipoles are always in motion, may thus collide and may—depending on the occasional collision angle—form likewise-rotating and thus unstable couples. Their centers of mass stay almost at rest and evolve into a spectrum of smaller and smaller vortices until their kinetic energy is converted into heat at scales of size zero. In a sense, we have here a most simple many-body problem because already the transition from  $\mathcal{N} = 1$  to 2 suffices to break the symmetry and allow the *emergence* of dissipation.

Setting  $Re \rightarrow \infty$  in NSE means vanishing viscosity such that only the Euler equation remains. However, the latter has non-unique solutions, already visible in the trivial case of the inviscid Burgers equation. Additional information is needed to achieve uniqueness, i.e. with  $Re \rightarrow \infty$  we first delete physics (viscosity) in NSE, to be forced then to *add* (from outside) reasonable physics to finally reinstate uniqueness. However, this is not yet all to be overcome. In addition, we have the problem of *localization* and integrability of a solution. Classical weak vortex solutions of the Euler equation extend into the full volume and are thus not integrable. However, real-world vortex ensembles exhibit individually localized vortices and finite scales.

Those contradictions are easily resolved if we introduce localized quasi-particles, in a sense embedded in the two-fluid concept initiated earlier by Landau, Tisza, Feynman, Liepmann, Spiegel and Spalding, resting all on Debye and Hückel.

<sup>2</sup> Comprehensive overviews on the history of turbulence theory can be found in the works of Davidson *et al* [14, 15].

<sup>3</sup> Schrödinger equation, Pauli principle, etc.

<sup>4</sup> See [www.archive.org/details/nasa/\\_techdoc/\\_19860008217](http://www.archive.org/details/nasa/_techdoc/_19860008217).

<sup>5</sup> See <http://turbulence.ens.fr/>.

### 3. Vortex tubes and dipoles: Batchelor couples and von-Kármán couples

#### 3.1. Potential vortex

The classical potential vortex around a closed vortex line (e.g. the centerline of a smoke ring) represents an exact weak solution of the Euler equation. The vortex line has infinitely thin diameter, infinitely high angular velocity, but finite circulation,  $c = \pi\rho^2\omega < \infty$ , where  $\rho$  and  $\omega$  are the radial coordinate and vorticity. The fluid outside the vortex line is inviscid and irrotational and the radial velocity  $v = \omega\rho \sim 1/\rho$  decreases with distance  $\rho$  from the centerline and extends into space.

The potential vortex is not applicable here as a model body of a ‘vorticon’, also not its relative, the Rankine vortex. They both are filling the space.

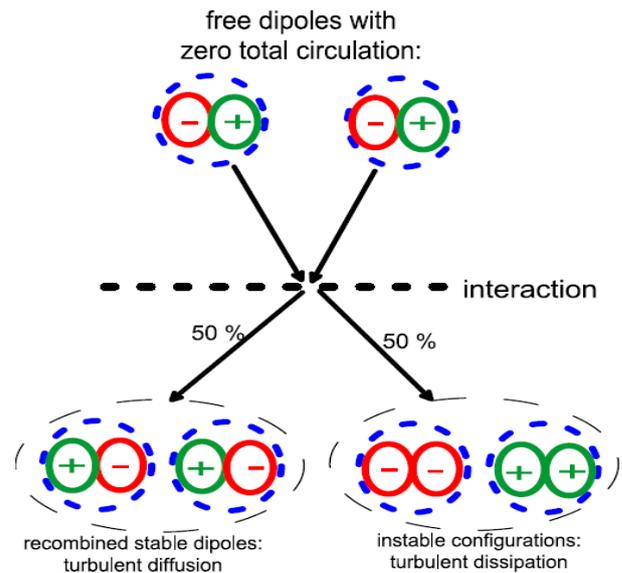
#### 3.2. Vortex tube

Instead of the above we use the vortex-tube concept, which is a Rankine vortex without sticking condition between its forced and free parts, i.e. its radial velocity increases linearly from center to tube radius  $r$  (like in a rigid body), but for  $\rho > r$  there is an inviscid and independent potential flow governed by volume conservation. Thus vorticity is confined to the interior of a spaghetti-like tube as described e.g. by Lugt [55] and in greater detail by Pullin and Saffman [65] who quote papers by Kuo and Corrsin [46], and Brown and Roshko [12], all discussing tubes as dominating characteristic structures. A newer simulation study was presented in 2011 by Wilczek [83] in the form of instructive movies on vortex ensembles in motion on the internet.<sup>6</sup>

The centerlines of our vortex tubes form either closed loops or they are attached to boundaries. The problem of stability of the tubes is discussed further below.

#### 3.3. Batchelor couple

This is a vortex *dipole* made up of two anti-parallel vortices [52] and symbolized here by  $(+ \uparrow -)$  or  $(- \uparrow +)$  where plus and minus mean the signs of vorticity within the vortices and the arrow the direction of motion of the couple or dipole. In classical interpretation, the flow field of one vortex moves the other vortex and *vice versa*. The total circulation of a dipole is zero because the vorticities carry opposite signs. In practice, such couples are stable over moderate propagation times. In our idealized image, they are made up of vortex tubes, move frictionless with local center-of-gravity velocity  $u = \omega r$  and conserve all their properties except their position in space because they propagate with their local  $u$  in the same direction as the fluid between the two tubes. In a dense ensemble of Batchelor couples, their trajectories are no longer straight lines due to mutual interactions. A couple’s kinetic energy density is  $u^2/2 = r^2\omega^2/2$ .



**Figure 1.** Collisions of two dipoles: the left pathway is ‘diffusive’; it is a recombination of dipole elements and chaotically scatters the trajectories (turbulent diffusion). The right pathway is ‘dissipative’; it evolves into an unstable vortex configuration which decays ‘somehow’ into heat.

#### 3.4. Von-Kármán couple

It is a counterpart of the Batchelor couple, made up likewise of rotating vortex tubes and symbolized by  $(+ \parallel +)$  or  $(- \parallel -)$ . Its total circulation does not vanish. Such a couple is long since known to be fundamentally unstable. Its kinetic energy is eventually dissipated into heat (e.g. [47]). Further below we discuss details of this process.

### 4. Dipole chemistry in reaction–diffusion approximation

For two-dimensional (2D) vortex trajectories it has been found by Aref and Eckhardt [4, 20] that the trajectories are chaotic so that for very high  $Re$  and 3D motions of tube-like vortex dipoles in the form of a dense 3D tangle we may assume also chaotic motions where collisions cannot be excluded. Let us consider an asymptotically high tube density so that the chaotic trajectories between collisions are short and locally homogeneous (in a statistical sense), e.g. in the theory of Brownian motion. Then the total dynamic process of the tangle may be described as [25] ‘...conservative dynamics punctuated by dissipative events...’: some ‘elastic’ collisions lead to energy-conserving reorganizations of Batchelor couples, resembling turbulent diffusion, whereas other collisions are dissipative or ‘inelastic’ and lead to the formation of fundamentally unstable von-Kármán couples whose energy moves to smaller and smallest scales where it decays, resembling turbulent dissipation or dipole annihilation.

Figure 1 shows the two equally possible results of a dipole–dipole collision. For symmetry reasons the two pathways have identical probabilities of  $\frac{1}{2}$ .

The simplest mathematical structure describing the statistics of the above two irreversible processes of (turbulent) diffusion and (turbulent) dissipation of dipoles is given by

<sup>6</sup> <http://pauli.uni-muenster.de/tp/menu/forschen/ag-friedrich/mitarbeiter/wilczek-michael.html>.

the following reaction–diffusion equation, which may be understood also as a special case of the Oregonator (see e.g. chapter 9 in [29]):

$$\frac{\partial \mathcal{N}}{\partial t} + \frac{\partial}{\partial \vec{x}} \left( \vec{U} \mathcal{N} - \nu \frac{\partial \mathcal{N}}{\partial \vec{x}} \right) = \mathcal{F} - \beta \mathcal{N}^2. \quad (1)$$

Here,  $\mathcal{N}$  is the volume density of dipole tubes (or filaments),  $\nu$  is turbulent diffusivity,  $\beta$  is a constant,  $\mathcal{F}$  is a source term describing the generation of dipole tubes and set equal to zero for the moment.  $\nu$  and  $\beta$  are unknown so far. We only know for sure that the exponent of  $\mathcal{N}$  in the last term of (1) is really *two* because it needs *two* colliding dipoles to generate (with probability 1/2) unstable configurations which dissipate energy. But if energy is gone, a whole dipole is gone as our dipole tubes as quasi-particles differ from their inviscid potential-flow environment only by their kinetic energy. However, the fluid occupied by the ‘dead’ particle is still there.

The presence of a mean flow and the corresponding advection of dipoles with the flow is sketched in (1) for reasons of completeness by the term with mean-flow vector  $\vec{U}$ . For simplicity we always set further  $\vec{U} \equiv 0$ .

One may view (1) from a simplistic point as a pure analogy with chemical reaction–diffusion processes. But one may also derive this equation over many pages from scratch, beginning with a master equation for the probabilities, as sketched by Baumert ([7], chapter 5.6). Fortunately, this was already done by other authors with the greatest care many years ago for whole classes of such processes, and went into the many textbooks on stochastic-dynamic systems, physical kinetics and other fields ([29, 30, 45, 76, 77]). We refer the more technically interested reader to the literature.

## 5. Two fluids, dressed and naked tubes

### 5.1. Two fluids

We consider a circulation-free volume filled with two different forms of a materially uniform incompressible fluid:

- inviscid, quasi-rigid and incompressible but deformable vortex tubes as quasi-particles with finite radius  $r$  and vorticity module  $\omega$ , vorticity bundled within the tube; in our analogy with the kinetic theory of gases, dipole tubes are the particles.
- the inviscid fluid *between* the dipole tubes; in our analogy with the kinetic theory of gases it is the vacuum.

Fluid (b) behaves like a super-fluid and receives no force from the moving quasi-rigid dipole tubes (d’Alembert’s paradox). The fluids (a, b) differ only in their state of motion. While the tubes rotate around their (in general curvilinear) axes and move locally relative to the volume according to Helmholtz’ laws, the fluid between the tubes performs corresponding evasive motions according to the principle of volume conservation.

### 5.2. Naked tubes

Above we have used the concept of vortex dipoles made up of ideal vortex tubes immersed in an inviscid fluid and

exchanging no energy with it. However, this concept is only applicable if the vortex is a rigid body. If it is a fluid, the problem of stability arises because the quasi-rigid vortex tube as an exact solution of the Euler equation is accompanied by the following pressure head as a consequence of inertial (centrifugal) forces:

$$p = p_0 + \frac{\rho}{2} \times \omega^2 r^2. \quad (2)$$

Here,  $p_0$  is the background pressure of a laminar reference flow, e.g. in the ocean the depth-depending hydrostatic pressure. If the pressure outside the vortex were simply  $p_0$ , then, due to the action of the outward-directed pressure force given by the second term in (2), the vortex would lose stability against small perturbations.

### 5.3. Dressed tubes

This contradiction can be explained by the consideration of ensemble effects. It has been noted in the introduction of this paper that in many-body problems like turbulence the phenomenon of *emergence* deserves special attention. This means that in a (local) volume element with a larger number ( $\mathcal{N} \gg 1$ ) of similar vortex tubes (a local ‘cloud’) the tube ensemble itself generates the background pressure (2) which eventually keeps all the individual tubes—at least in the center of the cloud—in a ‘sufficiently stable’ state.

In thermodynamically open systems like turbulent flows such a cloud is in a quasi-steady state (*Fließgleichgewicht*, in the sense of Bertalanffy [10]), i.e. the processes of dipole generation (see below) and their annihilation by collisions almost compensate each other. Any quasi-particle (dipole) has thus only a limited statistical lifetime. Therefore ‘sufficient stabilization’ of a dressed dipole by a cloud is stability in a statistical sense during its (statistical) lifetime. Here, Kelvin waves at the surface of a tube play a secondary role because they are not accompanied by friction, do not contribute to dissipation.

## 6. Particle number, turbulent kinetic energy (TKE) and rms vorticity frequency

Consider a small volume element  $\delta V$  populated by an ensemble of  $j = 1 \dots \mathcal{N}$  dipoles with individual effective vortex radii,  $r_j$ , and rms vorticity moduli,  $\omega_j$ . The latter can be interpreted as rms values of individual dipoles  $j$  as follows:

$$\omega_j^2 = \frac{1}{2} [(-\hat{\omega}_j)^2 + (+\hat{\omega}_j)^2] = \hat{\omega}_j^2, \quad (3)$$

where  $+\hat{\omega}_j$  and  $-\hat{\omega}_j$  are the individual vorticities of the two vortex tubes forming the dipole.

The dipole energy is conserved as long as dissipative events do not take place. The volume density of dipoles is  $\mathcal{N}/\delta V$ . The total TKE within  $\delta V$  is the sum of the kinetic energies of the individual dipoles:

$$K_{\delta V} = \frac{1}{\delta V} \sum_{j \in \delta V} \frac{1}{2} r_j^2 \omega_j^2 = \frac{\mathcal{N}}{\delta V} \bar{k}, \quad (4)$$

$$\omega_{\delta\mathcal{V}}^2 = \frac{1}{\delta\mathcal{V}} \sum_{j \in \delta\mathcal{V}} \omega_j^2 = \frac{\mathcal{N}}{\delta\mathcal{V}} \bar{\omega}^2. \quad (5)$$

Multiplication of (4), (5) with  $\delta\mathcal{V}$  gives

$$K = \delta\mathcal{V} \cdot K_{\delta\mathcal{V}} = \sum_{j \in \delta\mathcal{V}} \frac{1}{2} r_j^2 \omega_j^2 = \mathcal{N} \bar{k}, \quad (6)$$

$$\omega^2 = \delta\mathcal{V} \cdot \omega_{\delta\mathcal{V}}^2 = \sum_{j \in \delta\mathcal{V}} \omega_j^2 = \mathcal{N} \bar{\omega}^2. \quad (7)$$

$K_{\delta\mathcal{V}}$ ,  $\omega_{\delta\mathcal{V}}$  are local volume densities of TKE and rms vorticity magnitude, respectively. They and  $K$ ,  $\omega$  are extensive variables by definition, i.e. they scale with the dipole number in  $\delta\mathcal{V}$ .  $\bar{k}$  and  $\bar{\omega}$  are ensemble averages and, as such, intensive variables which do not change when new particles with average properties are added to  $\delta\mathcal{V}$ :

$$\bar{k} = \frac{1}{\mathcal{N}} \sum_{j \in \delta\mathcal{V}} \frac{1}{2} r_j^2 \omega_j^2, \quad (8)$$

$$\bar{\omega}^2 = \frac{1}{\mathcal{N}} \sum_{j \in \delta\mathcal{V}} \omega_j^2. \quad (9)$$

We turn now to equation (1) where, according to (6) and (7), we replace  $\mathcal{N}$  with  $K/\bar{k}$  and  $\omega/\bar{\omega}$ , respectively, to get eventually balance equations for the extensive variables  $K$  and  $\Omega = \omega/2\pi$ , provided that the intensive variables  $\bar{k}$  and  $\bar{\omega}$  vary sufficiently weakly in time and space compared with the dipole number  $\mathcal{N}$ :

$$\frac{\partial K}{\partial t} + \frac{\partial}{\partial \vec{x}} \left( \vec{U} K - \nu \frac{\partial K}{\partial \vec{x}} \right) = \mathcal{F}_K - \beta_K K^2, \quad (10)$$

$$\frac{\partial \Omega}{\partial t} + \frac{\partial}{\partial \vec{x}} \left( \vec{U} \Omega - \nu \frac{\partial \Omega}{\partial \vec{x}} \right) = \mathcal{F}_\Omega - \beta_\Omega \Omega^2. \quad (11)$$

Here,  $\Omega = 1/T$  is the ordinary<sup>7</sup> vorticity frequency and related to  $\omega$  by the often used constant  $\kappa$  which further below appears to be von-Kármán's constant:

$$\omega = 2\pi \Omega = \Omega/\kappa^2, \quad (12)$$

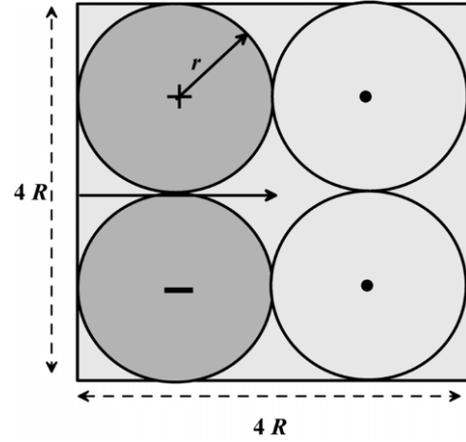
$$\kappa = (2\pi)^{-1/2} \approx 0.399. \quad (13)$$

We see that

$$\beta_K = \beta/\bar{k}, \quad \beta_\Omega = 2\pi\beta/\bar{\omega}, \quad (14)$$

$$\mathcal{F}_K = \bar{k} \mathcal{F}, \quad \mathcal{F}_\Omega = \bar{\omega} \mathcal{F}/2\pi. \quad (15)$$

<sup>7</sup> While  $\omega = 2\pi/T$  is an angular frequency,  $\Omega = 1/T$  is an ordinary frequency.



**Figure 2.** Local cross section through a dipole tangle far from boundaries at maximum dipole density ( $Re \rightarrow \infty$ ). The dark gray dipole in the left half square labels the initial situation, the right half square a situation after a translation motion from left to right into its collision position, where it is either annihilated or scattered. A free dipole obviously occupies statistically a cross sectional area of  $(4\bar{r})^2$ .

## 7. Mixing length and eddy viscosity

Prandtl [61] hypothesized that his mixing length has a twofold meaning. It should be ‘considered as the diameter of the masses of fluid moving as a whole’ (in our picture: as a characteristic vortex radius) or ‘as the distance traversed by a mass of this type before it becomes blended in with neighboring masses ...’ (in our picture: as a mean free path); i.e. he stated a tight relation between the radius and the free path of a dipole in a dense ensemble. Indeed, for a dipole tangle this can be shown (figure 2). We first define the ensemble average,  $\bar{r}$ , of the vortex radii as an average over a volume element weighted by vorticity as follows:

$$\bar{r}^2 = \frac{\sum r_j^2 \omega_j^2}{\sum \omega_j^2} = \frac{2K}{\omega^2} = \frac{2K}{(2\pi\Omega)^2}. \quad (16)$$

We now determine the eddy viscosity,  $\nu$ , in analogy to Einstein’s theory of Brownian motion in terms of a mean free path,  $\lambda$ , and a mean free flight time,  $\tau = \lambda/\bar{u}$ :

$$\nu = \lambda^2/2\tau. \quad (17)$$

Here,  $\bar{u}$  is the mean velocity of a dipole,

$$\bar{u}^2 = 2K = \sum u_j^2 = \sum r_j^2 \omega_j^2. \quad (18)$$

For a turbulent tangle of dipole tubes at  $Re = \infty$  in a quasi-steady state far from solid boundaries the mean free path will clearly be short but cannot vanish because there is an equilibrium of dipole generation, annihilation and motions that lead to the latter. This implies that [7]

$$\lambda = 2\bar{r}. \quad (19)$$

Algebra finally gives the following:

$$\nu = K/\pi\Omega. \quad (20)$$

This relation is called the Prandtl–Kolmogorov relation.

## 8. Parameters $\beta_K$ , $\beta_\Omega$ and $\zeta$

We consider an initial-value problem ( $\mathcal{F}_K = 0$ ,  $\mathcal{F}_\Omega = 0$ ) in a spatially homogeneous volume where  $\partial K/\partial \vec{x} = 0$  and  $\partial \Omega/\partial \vec{x} = 0$  such that equations (10), (11) reduce to

$$\frac{dK}{dt} + \beta_K K^2 = 0, \quad (21)$$

$$\frac{d\Omega}{dt} + \beta_\Omega \Omega^2 = 0. \quad (22)$$

For large  $t$  it follows that

$$K(t) = (\beta_K t)^{-1}, \quad (23)$$

$$\Omega(t) = (\beta_\Omega t)^{-1}, \quad (24)$$

and, according to (20), for eddy viscosity holds

$$\nu(t) = \frac{K(t)}{\pi \Omega(t)} = \frac{\beta_\Omega}{\pi \beta_K} = \text{const.} \quad (25)$$

Equation (23) coincides with the results of fairly general similarity analyses of NSEs by Oberlack [58] and with experimental results by Dickey and Mellor [17].

We now use the definition of the dissipation rate,  $\varepsilon$ , which is the last term on the right-hand side of (10) and carries the units of TKE (density) per time, i.e.  $\varepsilon \sim K/T \sim K\Omega$ , ( $\text{m}^2 \text{s}^{-3}$ ). For reasons of convenience we write this variable as follows and introduce a still unknown auxiliary variable,  $\zeta$ :

$$\varepsilon = \beta_K K^2 = \zeta \frac{K\Omega}{\pi} = \zeta \frac{K^2}{\nu\pi^2}, \quad (26)$$

where we made use of (20). This implies

$$\beta_K = \zeta/\nu\pi^2. \quad (27)$$

When the dipoles behave in general like in free decay then we may use (25) which gives then the expressions

$$\beta_\Omega = \zeta/\pi, \quad (28)$$

$$\beta_K = \zeta/\nu\pi^2. \quad (29)$$

With these results we may rewrite (10) and (11) as follows:

$$\frac{\partial K}{\partial t} + \frac{\partial}{\partial \vec{x}} \left( \vec{U} K - \nu \frac{\partial K}{\partial \vec{x}} \right) = \mathcal{F}_K - \frac{\zeta K^2}{\nu\pi^2}, \quad (30)$$

$$\frac{\partial \Omega}{\partial t} + \frac{\partial}{\partial \vec{x}} \left( \vec{U} \Omega - \nu \frac{\partial \Omega}{\partial \vec{x}} \right) = \mathcal{F}_\Omega - \frac{\zeta}{\pi} \Omega^2, \quad (31)$$

with  $\nu$  given by (20) and  $\zeta \equiv 1$  as shown further below.

## 9. TKE and vorticity generation by mean-flow shear

### 9.1. Generation of TKE

To further complete equations (30) and (31), we have, besides  $\zeta$ , to specify the source terms  $\mathcal{F}_K$ ,  $\mathcal{F}_\Omega$ . The specification of  $\mathcal{F}_K$  for shear flows is trivial because it is given by the

classical losses of the mean-flow energy balance and can therefore be copied from textbooks (e.g. Wilcox [82] or Schlichting–Gersten [71]):

$$\begin{aligned} \mathcal{F}_K &= - \sum_{i,j=1}^3 \langle u'_i u'_j \rangle \frac{\partial U_i}{\partial x_j} \\ &= \nu S^2 - \frac{2}{3} K \sum_{i,j=1}^3 \delta_{ij} \frac{\partial U_i}{\partial x_j}, \end{aligned} \quad (32)$$

where  $-\langle u'_i u'_j \rangle$  is the Reynolds stress tensor defined as

$$-\langle u'_i u'_j \rangle = 2\nu S_{ij} - \frac{2}{3} \delta_{ij} \cdot K \quad (33)$$

and  $S_{ij}$  is the rate of strain tensor defined as

$$S_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right). \quad (34)$$

$S^2$  is the total instantaneous shear squared,

$$S^2 = \sum_{i,j=1}^3 \left( \frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_j} \right) \frac{\partial U_i}{\partial x_j}, \quad (35)$$

$U_i$  is the  $i$ th component of the mean-flow velocity vector  $\vec{U} = (U_1, U_2, U_3)^T$  and  $\delta_{ij}$  is the Kronecker symbol which is zero for  $i \neq j$  and unity for  $i = j$ . In the simple case of a plane horizontal flow with vertical shear like wind over plane terrain or flow in a plane wide channel we have  $i = 1$  and  $j = 3$  such that (32) reads as follows:

$$\mathcal{F}_K = \nu S^2. \quad (36)$$

### 9.2. Generation of vorticity

We use a fundamental macroscopic argument by Tennekes [80]. It was first cast into mathematical form by Baumert and Peters [8, 9] and carefully discussed by Kantha [39, 40], Kantha *et al* [38] and Kantha and Clayson [41]. Tennekes hypothesized that, in a neutrally stratified homogeneous shear flow, an energy-containing turbulent length scale,  $\mathcal{L}$ , cannot, on dimensional grounds, depend on the ambient shear.

Further, also on dimensional grounds we have  $L \sim K^{1/2} \Omega^{-1}$  and thus  $L^2 \sim K \Omega^{-2}$  such that the time evolution of the length scale follows

$$\frac{1}{L^2} \frac{dL^2}{dt} \sim \frac{1}{K} \frac{dK}{dt} - 2 \frac{1}{\Omega} \frac{d\Omega}{dt}. \quad (37)$$

We replace  $dK/dt$  with  $\mathcal{F}_K$  according to (32) and find

$$\frac{1}{L^2} \frac{dL^2}{dt} \sim \frac{S^2}{\pi \Omega} - 2 \frac{1}{\Omega} \frac{d\Omega}{dt}. \quad (38)$$

Tennekes' hypothesis means that the evolution of  $L$  cannot be controlled by  $S$ , which means  $dL^2/dt = 0$  and implies that

$$\frac{S^2}{\pi \Omega} = \frac{2\mathcal{F}_\Omega}{\Omega}, \quad (39)$$

where we replaced  $d\Omega/dt$  with  $\mathcal{F}_\Omega$ . Algebra gives

$$\mathcal{F}_\Omega = S^2/2\pi \quad (40)$$

and for the simple case of a plane horizontal flow with vertical shear we may complete with some algebra equations (30) and (31) as follows:

$$\frac{\partial K}{\partial t} + \frac{\partial}{\partial \bar{x}} \left( \bar{U} K - \nu \frac{\partial K}{\partial \bar{x}} \right) = \nu (S^2 - \zeta \Omega^2), \quad (41)$$

$$\frac{\partial \Omega}{\partial t} + \frac{\partial}{\partial \bar{x}} \left( \bar{U} \Omega - \nu \frac{\partial \Omega}{\partial \bar{x}} \right) = \frac{1}{\pi} \left( \frac{S^2}{2} - \zeta \Omega^2 \right) \quad (42)$$

with  $\nu$  again given by (20).

## 10. Turbulent boundary layer

### 10.1. Boundary conditions

Consider a horizontally homogeneous flow and its stationary boundary layer close to a plane solid wall at  $x_3 = z = 0$  where for convenience  $z$  is introduced here as the only relevant coordinate, pointing orthogonal from the wall into the fluid. Thus the shear is

$$S = |dU/dz|. \quad (43)$$

This special case means that in (41) and (42) the horizontal and time derivatives vanish and it remains

$$-\frac{d}{dz} \left( \nu \frac{dK}{dz} \right) = \nu (S^2 - \zeta \Omega^2), \quad (44)$$

$$-\frac{d}{dz} \left( \nu \frac{d\Omega}{dz} \right) = \frac{1}{\pi} \left( \frac{S^2}{2} - \zeta \Omega^2 \right). \quad (45)$$

The diffusive TKE flux into the viscous sublayer at  $z = z_0$  has to vanish in the sense of an adiabatic boundary condition,

$$(\nu dK/dz)_{z=z_0} = 0, \quad (46)$$

such that also the flux divergence on the left-hand side of (44) vanishes, giving  $K = K_0$  and  $\nu (S^2 - \zeta \Omega^2) = 0$  or

$$\Omega = S/\sqrt{\zeta}. \quad (47)$$

### 10.2. Logarithmic law of the wall

We insert (47) into (45). The unknown  $\zeta$  cancels out and we have to solve the following equation for  $S = S(z)$ ,

$$2K_0 \frac{d}{dz} \left( \frac{1}{S} \frac{dS}{dz} \right) = S^2, \quad (48)$$

which gives

$$S(z) = \frac{dU}{dz} = \frac{\sqrt{2K_0}}{z}. \quad (49)$$

Integration of (49) gives the logarithmic law of the wall.

In boundary layer theory the bottom shear stress is defined in terms of the squared friction velocity,  $u_f^2$ ,

$$u_f^2 = \nu \frac{dU}{dz} = \frac{K_0}{\pi \Omega} S, \quad (50)$$

and with (47) it follows that

$$K_0 = \pi u_f^2 / \sqrt{\zeta}. \quad (51)$$

This allows (49) to be rewritten as follows:

$$\frac{dU}{dz} = \frac{u_f}{\tilde{\kappa} S z}, \quad (52)$$

with  $\tilde{\kappa}$  as a modified von-Kármán constant defined with respect to (13) through

$$\tilde{\kappa} = \kappa \zeta^{1/4}. \quad (53)$$

Integration of (52) provides us with

$$U(z) = \frac{u_f}{\tilde{\kappa}} \ln \left( \frac{z}{z_0} \right). \quad (54)$$

### 10.3. Mixing length $L$

Consider the definition of the effective (statistically averaged) dipole radius of an ensemble through (16). We solve this equation for  $K$  and express the TKE in terms of  $\bar{r}$  and  $\Omega$  as follows:

$$K = 2\pi^2 \bar{r}^2 \Omega^2. \quad (55)$$

Following now Hinze ([33], p. 279, equation 5-2), in present notation Prandtl's mixing length  $L$ , which is also termed the 'energy-containing length scale' in the literature, is defined in terms of eddy viscosity and shear as follows:

$$\nu = L^2 \left| \frac{dU}{dz} \right| = L^2 S. \quad (56)$$

Due to the eddy-viscosity formula (20), relation (56) gives

$$L^2 = \frac{K}{\pi \Omega^2 \zeta^{1/2}}, \quad (57)$$

so that in the neighborhood of a solid wall we get with (47) the following result:

$$K = \pi L^2 \Omega^2 \sqrt{\zeta}. \quad (58)$$

Comparing (55) with (58) gives

$$L = \bar{r} / \tilde{\kappa}. \quad (59)$$

The physical meaning of  $L$  can be understood as follows (see figure 3). If we may set  $\zeta = 1$  then (59) gives together with (13) and (53) the following relation:

$$L^2 = 2 \times (\pi \bar{r}^2). \quad (60)$$

It means that  $L$  is the length of a square with an area equal to the cross-sectional area of a dipole (in a statistically averaged sense) because  $\pi \bar{r}^2$  is the cross-sectional area of one vortex tube. In an asymptotic sense this case corresponds to the *maximum deformation* of a dipole and justifies setting  $\zeta \equiv 1$ .

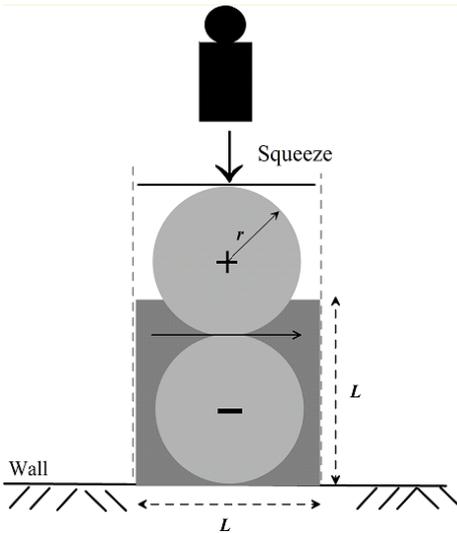
While Prandtl's mixing-length concept was applicable only in the vicinity of solid boundaries so that it attracted respectful criticism (Wilcox [82]), our concept generalizes Prandtl and works also far from boundaries, even in the free stream of stratified fluids where  $L$  may approach the Thorpe scale and/or the Ozmidov scale, depending on the conditions [9].

We summarize this section as follows:

$$\nu = u_f L, \quad (61)$$

$$L = \kappa z, \quad (62)$$

$$z = L/\kappa = \bar{r}/\kappa^2. \quad (63)$$



**Figure 3.** Cross section through a dipole sheet at a solid boundary. In this figure, we have  $r = \bar{r}$ , i.e. the sketched quasi-particle is to be understood as an ensemble average.

## 11. Spectra, dissipative patches and spectral constants

### 11.1. Spectra

Up to now our discussion was concentrated on only a few scales like  $\bar{r}$  and  $L$  interrelated by  $\kappa$ , and on  $T$  interrelated with  $\Omega$  and  $\omega$  etc. However, the reality of turbulence exhibits a whole range of scales at which fluctuations occur. They have relatively stable spectral distributions. This problem attracted early attention by Richardson [67] and Taylor [79]. Kolmogorov [43] found on dimensional grounds that the kinetic energy spectrum as a function of wavenumber, wherein energy flows steadily from the large (energy-containing) scales to the much smaller dissipative scales, may be written as follows,

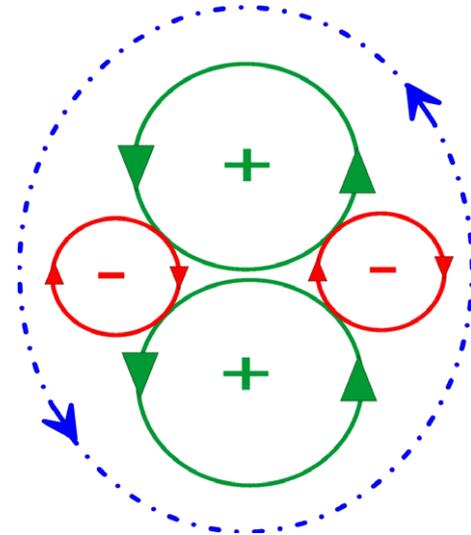
$$dK = \alpha_1 \varepsilon^{\alpha_2} k^{-\alpha_3} dk. \quad (64)$$

Here,  $k = 2\pi/\Lambda$  is the wave number and  $\Lambda$  the wavelength.  $\varepsilon$  is the dissipation rate of TKE. Dimensional arguments force that  $\alpha_2 = 2/3$  and  $\alpha_3 = 5/3$ , in agreement with the oceanographic observations by Grant *et al* [28] in a tidal inlet with  $Re \approx 10^8$  and a depth of about 100 m. A value for  $\alpha_1$  is derived below.

### 11.2. Devil's gear

Our view of the above Kolmogorov–Richardson cascade has been filled with life through a numerical simulation study by Herrmann [31] who demonstrated that Kolmogorov's value for  $\alpha_3$  corresponds numerically to the data of a space-filling bearing (see also Herrmann *et al* [32]). The latter is the densest non-overlapping (Apollonian) circle packing in the plane, with side condition that the circles are pointwise in contact but able to rotate freely, without friction or slipping (a 'devil's gear' *sensu* Poepe [60]). The contact condition for two different 'wheels' with indices 1 and 2 of the gear reads

$$u = \omega_1 r_1 = \omega_2 r_2, \quad (65)$$



**Figure 4.** Cross section through the first developmental stage of a dissipative patch, i.e. of an unstable pair of likewise rotating vortices resulting from a dipole–dipole collision (right pathway in figure 1). The green circles represent primary energy-containing vortices. They do *not* touch each other due to spontaneously formed secondary vortices which initiate a whole vortex cascade. The broken blue line and the arrows symbolize the slow rotation of the patch around its center of mass.

where  $u$  is necessarily constant throughout the gear and governed by the energy of the decaying (initially energy-containing) vortex pair as  $u = \sqrt{2K}$ . It follows that

$$\omega_2 = \omega_1 \frac{r_1}{r_2}, \quad (66)$$

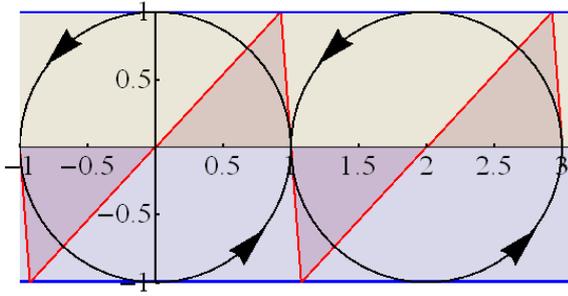
and for very small  $r_2$  the frequency  $\omega_2$  may become high, even acoustically relevant.

### 11.3. The dissipative patch

If the above gear is frictionless then the question arises where energy can be dissipated. In a real fluid with non-zero viscosity, dissipation happens at all scales, mainly but not exclusively where the velocity gradient is highest, here: at a scale vanishing with  $Re \rightarrow \infty$  to the size zero. Our dissipative patch (figure 4 shows the first stage of its formation) is thus 'almost frictionless' and therefore a Hamiltonian clockwork, excepting scales of size zero.

The formation of a fully developed spectrum of 'wheels' from figure 4 deserves certain perturbations 'from the sides', a condition which is guaranteed by the random reconnection/recombination and scatter processes sketched in the left half of figure 1 and also by the incomplete mutual pressure compensation of the vortices in our vortex ensemble. In a quasi-steady state these perturbations initiate roll-up instabilities at the boundaries of the respective larger vortices so that eventually and in a statistical sense a patch like in figure 4 is formed and evolves steadily into a fully developed gear.

Figure 1 illustrates the possible results of a dipole–dipole collision. While the left pathway shows the recombination of counter-rotating vortices from counter-rotating vortices, the right shows the formation of a couple of *likewise* rotating vortices from counter-rotating vortices. The latter then revolve



**Figure 5.** Outer limits of a dissipative patch (see figure 4). The maximum wavelength is obviously equal to  $\Lambda_0 = 2\bar{r}$ .

around a common center of mass, which remains nearly at rest (figure 4). Such a couple is fundamentally unstable [72], a quasi-steady dissipative patch evolving into a full gear in the sense of the mechanisms discussed in the last section. This picture lets us expect that dissipation should exhibit spatially *patchy* behavior, which we may also call *intermittency*. This problem has been studied extensively by various authors from other points of view (see e.g. Frisch [25]) and cannot yet be discussed here from our new viewpoint in greater detail.

#### 11.4. Kolmogorov's constant $\alpha_1$

This constant belongs to the wavenumber spectrum and deserves an idea about the outer limits and inner structure of an unstable, dissipative patch as sketched in figure 5 for the beginning of the cascade process evolving into a structure like the one given in figure 4.

The most important message of figure 5 is that the *longest* or energy-containing wavelength of the dissipative patch equals  $\Lambda_0 = 2\bar{r}$ . The wavelength in a dipole is  $4\bar{r}$ . The dipole performs chaotic trajectories in a white-noise sense and forms no patch or spectrum. This difference between the two configurations is essential. We use our  $\Lambda_0$  as a lower integration limit for the spectral energy distribution. It is important to underline that  $\Lambda_0$  labels the upper wavelength limit (the longest wavelike motions) in a dissipative patch. This limit is actually not influenced by the formation details of the spectrum.

We integrate (64) over the dissipative patch in the sense sketched in figures 4 and 5 and obtain

$$K = \alpha_1 \varepsilon^{2/3} \int_{k_0}^{\infty} k^{-5/3} dk = \alpha_1 \frac{3}{2} \left( \frac{\varepsilon}{k_0} \right)^{2/3}, \quad (67)$$

where  $k_0 = 2\pi/\Lambda_0$  characterizes the lower end of the turbulence spectrum in the wavenumber space. We loosely assign the wavenumber range  $k = 0, \dots, k_0$  to the mean flow which may basically be resolved in numerical models. The dissipation rate  $\varepsilon$  in (67) can be expressed as follows:

$$\varepsilon = K/\tau, \quad (68)$$

with  $\tau$  being the lifetime of a dissipative patch. Inserting (68) in (67) and rearranging gives the following:

$$\alpha_1 = \frac{2}{3} (2\pi)^{2/3} K^{1/3} \left( \frac{\tau}{2\bar{r}} \right)^{2/3}. \quad (69)$$

In a local quasi-equilibrium sense for a dense vortex ensemble the marching dipoles can occupy only those places which are simultaneously ‘emptied’ through dipole annihilation or dissipative patches by decay. This means that the lifetime of a dissipative patch,  $\tau = K/\varepsilon$ , should equal the time of ‘free flight’ of a dipole over a distance  $2\bar{r}$ :

$$\tau = K/\varepsilon = 2\bar{r}/u. \quad (70)$$

Here, we used the scalar dipole velocity  $u$ ,

$$u = \omega r = \sqrt{2K}. \quad (71)$$

After some algebra we get the pre-factor of the 3D wavenumber spectrum as follows:

$$\alpha_1 = \frac{1}{3} (4\pi)^{2/3} = 1.802. \quad (72)$$

The corresponding value of an ideal 1D spectrum is one third of the above, i.e. 0.60.

## 12. Discussion

### 12.1. Equations of turbulent motion

The results of our considerations can be summarized as follows:

$$\frac{\partial K}{\partial t} + \frac{\partial}{\partial \bar{x}} \left( \vec{U} K - \nu \frac{\partial K}{\partial \bar{x}} \right) = \nu (S^2 - \Omega^2), \quad (73)$$

$$\frac{\partial \Omega}{\partial t} + \frac{\partial}{\partial \bar{x}} \left( \vec{U} \Omega - \nu \frac{\partial \Omega}{\partial \bar{x}} \right) = \frac{1}{\pi} \left( \frac{S^2}{2} - \Omega^2 \right), \quad (74)$$

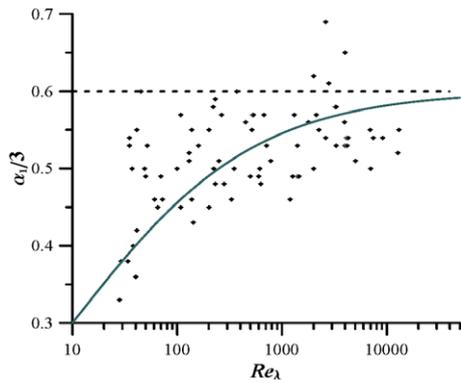
$$\nu = \frac{K}{\pi \Omega}. \quad (75)$$

These equations are structurally identical with the  $K$ - $\omega$  closure model discussed by Wilcox. There are only slight differences in the pre-factors of the terms.

This theory applies exclusively to locally homogeneous, isotropic and moderately unsteady high-Reynolds number flows. Extreme non-stationarities and/or sharp spatial gradients like in shockwaves are possibly not covered. As a rule, temporal changes of the mean flow should happen on time scales sufficiently long compared with  $T = 1/\Omega$  because otherwise spectral universality (64) has possibly not enough time to become well enough established.

Our equations reproduce the logarithmic law of the wall and predict the universal von-Kármán's constant as  $\kappa = 1/\sqrt{2}\pi = 0.399 \approx 0.4$ , where the latter value counts as an internationally accepted standard (see also [34]; we do not follow the controversy of logarithmic versus power-law boundary layer *sensu* Barenblatt [5]). The value corresponds nicely to measurements under *favorable* pressure gradients. Similar support comes from Jimenez and Moser [37] on the basis of an extensive review. They state: *The Kármán constant  $\kappa \approx 0.4$  is approximately universal.*

However, Landau and Lifshitz ([50], p 173) wrote that *... $\kappa$  (is) a numerical constant, the von-Kármán constant, whose value cannot be calculated theoretically and must be determined experimentally. It is found to be  $\kappa = 0.4$ .*



**Figure 6.** Experimental and observational results for  $\alpha_1/3$  measured, collected from the literature, and analyzed by Sreenivasan [75]. The solid line follows our somewhat arbitrary approximation  $0.6 \times \sqrt{Re_\lambda} / (\sqrt{Re_*} + \sqrt{Re_\lambda})$  wherein  $\alpha_1/3 = 0.6$  is our theoretically derived asymptotic value. Here, we took  $Re_* = 10$ .

With respect to the measurability of  $\kappa$  even the opposite might be true: if ‘physics disappears’ (when  $Re \rightarrow \infty$ ), only the laws of an ‘inert geometry’ (Euler and volume conservation) remain,  $\kappa$  can possibly no longer be understood as a classical physical quantity. For an ideal liquid it represents pure geometry.

Our equations describe the free decay of turbulence following  $K \sim t^{-m}$  with  $m = 1$ , in agreement with Dickey and Mellor’s [17] high- $Re$  laboratory experiments and with Oberlack’s [58] theoretical result for the NSE. Today, it is still not clear why some decay experiments lead to  $m > 1$ . Possibly it is a matter of initial conditions [36]: at high  $Re$  viscosity is comparatively small so that its regularizing effect in the relaxation process towards a fully self-similar spectrum will take more time than at lower  $Re$ . In some cases this time may even exceed the lifetime of turbulence. This underlines the necessity of deeper experimental work.

### 12.2. Kolmogorov’s constant

The rounded numerical values  $\alpha_1 = 1.8$  or  $\alpha_1/3 = 0.6$  predicted by our theory for Kolmogorov’s universal constant are situated well within the error bars of many high- $Re$  observations, NSE- and renormalization group-based analytical approximations, laboratory and direct numerical simulation of Navier–Stokes (DNS) experiments.

Based on observations, Tennekes and Lumley [81] gave in 1972 the value  $\alpha_1 = 1.62$ , but still with some uncertainty. The 1995 study by Sreenivasan [75] (see our figure 6) is possibly the most comprehensive review of experimental and observational values for the number  $\alpha_1/3$  until now. Yeung and Zhou [86] reported in 1997 a value of  $\alpha_1 = 1.62$  based on high-resolution DNS studies with up to  $512^3$  grid points. In 2010 a study by Donzis and Sreenivasan [19] on a DNS grid of  $4096^3$  reported  $\alpha_1 \approx 1.58$ .

Much higher Reynolds numbers than in DNS were found in oceanic measurements of Lagrangian frequency spectra by Lien and D’Asaro [53]. These authors stated for the prefactor  $\beta_1$  in the Lagrangian frequency spectrum that ... since the present uncertainty is comparable to that between high quality estimates of the Eulerian one-dimensional

longitudinal Kolmogorov constant measured by many dozen investigators over the last 50 years, large improvements in the accuracy of the estimate of  $\beta_1$  seem unlikely.

There are other theoretical efforts to calculate the universal constants, technically complex and mostly of singular character. Beginning with an initiating work by Forster *et al* [24], systematic analytical approximations using RG methods and related techniques for NSE gave rise to some estimates. For example, Yakhot and Orszag [84, 85] found  $\alpha_1 \approx 1.62$ , whereas McComb and Watt [57] derived  $\alpha_1 = 1.60 \pm 0.01$  and Park and Deem [59] obtained  $\alpha_1 = 1.68$ .

### 12.3. Coda

Saffman ([69], loc. cit. Davidson [14], p. 107) feared that ... in searching for a theory of turbulence, perhaps we are looking for a chimera. This has recently been enforced by Hunt [35]: ... But there are good reasons why the answer to the big question that Landau and Batchelor raised about whether there is a general theory of turbulence is probably ‘no’.

In a most general sense these authors are possibly right. But a comment is necessary. Turbulence as a physical phenomenon is strange for more than one reason. There is its finite lifetime during which it may change its face. There is the extraordinary high scatter in the measurements of universal constants of turbulence, compared with the precision of fundamental constants of physics like, e.g., the mass of the proton [26]. There is further the strange concept of a fluid as a continuum where control volumes may be sub-divided into sub-volumes *ad infinitum*—in contrast to real-world physical fluids, which are made of molecules and carry absolute scales, so that the idea  $Re \rightarrow \infty$  appears at first glance as nonsense.

Turbulence in our sense ( $Re \rightarrow \infty$ ) is thus a reasonable concept only within the limits of fluids as continua. Only within these limits are our universal equations of turbulent motion (73)–(75), the universal Kolmogorov spectrum (64) and our universal von-Kármán (13) and Kolmogorov constants (72) accurate approximations of real-world physical fluids.

The experiments with ultra-high Reynolds numbers in Princeton [87], Oregon [18] and in the European CICLoPE [78] might go far beyond these limitations.

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