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Abstract

We investigate the pointwise asymptotic behavior of solutions to the stationary Navier-Stokes equation in $\mathbb{R}^n$ $(n \geq 3)$. We show the existence of a unique solution $\{u, p\}$ such that $|\nabla^j u(x)| = O(|x|^{1-n-j})$ and $|\nabla^k p(x)| = O(|x|^{-n-k})$ $(j, k = 0, 1, \ldots)$ as $|x| \to \infty$, assuming the smallness of the external force and the rapid decay of its derivatives. The solution $\{u, p\}$ decays more rapidly than the Stokes fundamental solution.

1 Introduction

We study decay properties of solutions to the stationary Navier-Stokes equation in $\mathbb{R}^n$ with $n \geq 3$:

$$(1.1) \qquad \begin{cases} -\Delta u + u \cdot \nabla u + \nabla p = \operatorname{div} F & \text{in } \mathbb{R}^n, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^n, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

Here $u = (u_1, \cdots, u_n)$ and p denote, respectively, the unknown velocity and pressure of a viscous incompressible fluid, while $F = (F_{ij})_{i,j=1}^n$ is a given tensor with $\operatorname{div} F = (\sum_{i=1}^n \partial_{x_i} F_{ij})_{j=1}^n$ denoting the external force.

It is well known that for every $F \in L^2(\mathbb{R}^n)$ there exists at least one weak solution u to (1.1) with finite Dirichlet integral ([9, 3]). Decay properties of the weak solution in [9, 3] are still open problems in spite of their importance in the study of, for instance, uniqueness and stability. In this paper we are especially interested in the pointwise decay at infinity of solutions to (1.1). Frehse-Růžička[2] proved, in the case $n = 3$, that (1.1) possesses a weak solution u decaying like $|x|^{-1+\delta}$ $(0 < \delta < 1)$ without restricting the size of

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the external force with compact support. Novotny-Padula[13] and Borchers-Miyakawa[1] established the existence of a unique solution u such that

$$|u(x)| = O(|x|^{2-n}), \quad |\nabla u(x)| = O(|x|^{1-n}) \quad \text{as } |x| \to \infty, \tag{1.2}$$

assuming that both F and ∇F are small in an appropriate sense, see also [5]. The decay rates (1.2) coincide with those of the Stokes fundamental solution and thus look natural. In three-dimensional exterior problems, (1.2) are optimal decay rates in the sense that solutions which decay as $|x| \to \infty$ more rapidly than (1.2) exist only under a special situation ([6, 1, 7]), however, this is not the case with the whole space problem (1.1). Miyakawa[11] constructed a unique solution $\{u, p\}$ of (1.1) satisfying

$$|\nabla^j u(x)| = O(|x|^{1-n-j}) \ \ (j = 0, 1, 2), \quad |\nabla^k p(x)| = O(|x|^{-n-k}) \ \ (k = 0, 1)$$

as $|x| \to \infty$, under stronger conditions on the smallness and decay of F and its derivatives than [13, 1]. The solution in [11] decays more rapidly than the Stokes fundamental solution, and it is not known whether its decay rate is optimal.

In this paper, we slightly extend the result of [11]. We shall show that if F is sufficiently small in a sense and $\nabla^j F$ decays rapidly at infinity for all $j = 0, 1, \dots, m$ $(m \geq 1)$, then there exists a unique solution $\{u, p\}$ of (1.1) such that

$$\begin{aligned} |\nabla^j u(x)| &= O(|x|^{1-n-j}) \quad (j = 0, 1, \dots, m), \\ |\nabla^k p(x)| &= O(|x|^{-n-k}) \quad (k = 0, \dots, m-1), \end{aligned} \tag{1.3}$$

as $|x| \to \infty$. Our result can describe the decay rates of higher derivatives of the solution and the case $m \geq 3$ is not covered by [11]. It should be emphasized that in our result we assume the smallness of only F, while, as far as the author checked, the size of ∇F as well as F is restricted in the proof of [11, Theorem 1.1 (i)] although it is not mentioned in the paper. Furthermore, we allow F and its derivatives to decay more slowly than [11], see Remark 2.2 below.

The proof is based on the analysis of the representation formula of solutions via the Stokes fundamental solution. The existence of a solution u decaying like $|x|^{1-n}$ can be obtained in the same way as [11] and thus we mainly study the decay rates of its derivatives. In order to obtain the decay property (1.3) without assuming the smallness of $\nabla^j F$ $(j \geq 1)$, we use the important property of the Stokes fundamental solution that its second derivative is the Calderón-Zygmund kernel ([14]). With this property and basic estimates for the fundamental solution in hand, we show slow decay estimates for derivatives of u and then employ the bootstrap argument to get the desired decay property (1.3).

2 Main result

Before stating our result, we introduce some function spaces. In what follows, we adopt the same symbols for vector and scalar function spaces as long as there is no confusion. For $1 \leq q \leq \infty$, we denote by $L^q(\mathbb{R}^n)$ the usual Lebesgue space with norm $\|\cdot\|_q$. Let

$C_{0,\sigma}^{\infty}(\mathbb{R}^n)$ be the set of smooth solenoidal vector fields with compact support in $\mathbb{R}^n$ and we define the homogeneous Sobolev space $\dot{H}_{0,\sigma}^{1}(\mathbb{R}^n)$ by the completion of $C_{0,\sigma}^{\infty}(\mathbb{R}^n)$ in the norm $\|\nabla \cdot\|_2$. For $\mu > 0$ we introduce the space X_μ defined by

$$X_\mu := \{u \in L^\infty(\mathbb{R}^n);\ \sup_{x\in\mathbb{R}^n} (|x|+1)^\mu |u(x)| < \infty\}$$

with norm

$$\|u\|_{X_\mu} := \sup_{x\in\mathbb{R}^n} (|x|+1)^\mu |u(x)|.$$

It is easy to check that $X_{\mu_1} \subset X_{\mu_2}$ if $\mu_2 < \mu_1$. Furthermore, for $f \in X_{\mu_3}$ and $g \in X_{\mu_4}$ we have $fg \in X_{\mu_3+\mu_4}$. We also need some notation for derivatives. Let us set $D_i u(x) := \partial_{x_i} u$. By $\alpha = (\alpha_1, \dots, \alpha_n)$ we denote a multiindex of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $D^\alpha u(x) := D_1^{\alpha_1} \cdots D_n^{\alpha_n} u$.

Recall the Stokes fundamental solution $E = (E_{ij})_{i,j=1}^n$ and $Q = (Q_i)_{i=1}^n$ with components

$$E_{ij}(x) = \frac{1}{2\omega_n}\left(\frac{\delta_{ij}}{n-2}|x|^{2-n} + \frac{x_i x_j}{|x|^n}\right), \quad Q_i(x) = \frac{x_i}{\omega_n |x|^n}.$$

Here ω_n is the surface area of the unit sphere in $\mathbb{R}^n$. We consider (1.1) in the form of the integral equation

$$u(x) = \int_{\mathbb{R}^n} \nabla E(x-y)(F - u\otimes u)(y)\, dy, \tag{2.1}$$

where $u \otimes u = (u_i u_j)_{i,j=1}^n$. Note that, under suitable decay conditions on u and F, a solution u of (2.1) can be written as

$$u(x) = \int_{\mathbb{R}^n} E(x-y)(\mathrm{div}\ F - u\cdot\nabla u)(y)\, dy.$$

The associated pressure p, uniquely determined up to addition of constants, is given by

$$p(x) = \int_{\mathbb{R}^n} Q(x-y)\cdot(\mathrm{div}\ F - u\cdot\nabla u)(y)\, dy.$$

The main result of this paper is stated in the following theorem.

Theorem 2.1. *Let m be a positive integer and $0 < \delta < 1$. Suppose*

$$F \in X_{n+\delta} \quad \text{and} \quad \nabla^j F \in X_{n+j} \quad \text{for all } j = 1, \dots, m. \tag{2.2}$$

If F is sufficiently small in $X_{n+\delta}$, then (2.1) admits a solution $u \in X_{n-1}$ such that

$$\nabla^j u \in X_{n-1+j} \quad \text{for all } j = 1, \dots, m, \tag{2.3}$$

and

$$\nabla^{m+1} u \in L^{1,\infty}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \quad \text{for } 1 < q < \infty,$$

where $L^{1,\infty}(\mathbb{R}^n)$ is the weak-L^1 space. The solution u is unique in the class of solutions $v \in X_{n-1}$ with $\nabla v \in X_n$ to (2.1). Furthermore, the associated pressure p satisfies

$$\nabla^k p \in X_{n+k} \quad \text{for all } k = 0, \dots, m-1. \tag{2.4}$$

Remark 2.1. The natural class of F seems to be the space X_n. However, the proof of Theorem 2.1 does not work if we have only $F \in X_n$, since the case $\mu = n$ is not contained in Lemma 3.1 below. We introduced the constant $0 < \delta < 1$ and assumed $F \in X_{n+\delta}$ so that Lemma 3.1 is applicable and F is integrable in $\mathbb{R}^n$.

Remark 2.2. Our classes of F and its derivatives in (2.2) are larger than those in [11]. Indeed, Miyakawa[11] assumed $\nabla^j F \in X_{2n-2+j}$ $(j = 0, 1, 2)$ and we note that $X_{2n-2} \subset X_{n+\delta}$ and $X_{2n-2+k} \subset X_{n+k}$ $(k = 1, 2)$.

Remark 2.3. The fast decay property of the solution in Theorem 2.1 enables us to apply some uniqueness criteria such as [4, 10, 8, 12], and the uniqueness holds in larger classes of solutions, see the proof of Theorem 2.1 below. (The results of [4, 10, 8, 12] are concerned with the exterior problems, however, they are still valid for the whole space problem (1.1).) In particular, as a consequence of Theorem 2.1 and [4, 10], we can deduce that if F satisfies (2.2) and is sufficiently small in $X_{n+\delta}$, then every weak solution $u \in \dot{H}^1_{0,\sigma}(\mathbb{R}^n)$ with the energy inequality $\|\nabla u\|_2^2 \le -\int_{\mathbb{R}^n} F \cdot \nabla u \, dx$ of (1.1) in the sense of [9, 3] satisfies $|\nabla^j u(x)| = O(|x|^{1-n-j})$ as $|x| \to \infty$ for all $j = 0, 1, \ldots, m$.

3 Proof of Theorem 2.1

We begin with the estimates of weakly singular integrals. The estimates below must be more or less well-known, however, we give the proof for the reader's convenience. In what follows, we denote by C various constants and note, in particular, that all constants C appearing in this paper are independent of $x \in \mathbb{R}^n$.

Lemma 3.1. *Let $0 < \lambda < n$ and $\mu > 0$ with $\lambda + \mu > n$. There exist constants $C > 0$ depending only on n, λ and μ such that*

$$\int_{\mathbb{R}^n} \frac{dy}{|x-y|^\lambda (|y|+1)^\mu} \le \begin{cases} C(|x|+1)^{n-\lambda-\mu} & \text{if } 0 < \mu < n, \\ C(|x|+1)^{-\lambda} & \text{if } \mu > n. \end{cases}$$

Proof. We first consider the case $0 < \mu < n$. If $|x| \ge 1/5$, by [4, II, Lemma 7.2] we have

$$\int_{\mathbb{R}^n} \frac{dy}{|x-y|^\lambda (|y|+1)^\mu} \le C|x|^{n-\lambda-\mu} \le 6^{\lambda+\mu-n} C(|x|+1)^{n-\lambda-\mu}.$$

For $0 \le |x| < 1/5$, we write

$$\begin{aligned} &\int_{\mathbb{R}^n} \frac{dy}{|x-y|^\lambda (|y|+1)^\mu} \\ &= \int_{|x-y|<\frac{|x|+1}{2}} \frac{dy}{|x-y|^\lambda (|y|+1)^\mu} + \int_{|x-y|>\frac{|x|+1}{2}} \frac{dy}{|x-y|^\lambda (|y|+1)^\mu} \\ &=: I_1 + I_2. \end{aligned}$$

Since $|x-y| < (|x|+1)/2$ implies $(|x|-1)/2 < |y| < (3|x|+1)/2$ and since $\lambda < n$, we obtain

$$I_1 \le \frac{2^\mu}{(|x|+1)^\mu} \int_{|x-y|<\frac{|x|+1}{2}} \frac{dy}{|x-y|^\lambda} \le C(|x|+1)^{n-\lambda-\mu}. \tag{3.1}$$

For $0 \le |x| < 1/5$ and $|x-y| > (|x|+1)/2$ there holds

$$|x| + |y| \ge |x-y| > (|x|+1)/2 > 3|x|,$$

so that $|y|/2 > |x|$ and

$$I_2 \le 2^\lambda \int_{|x-y|>\frac{|x|+1}{2}} \frac{dy}{|y|^\lambda (|y|+1)^\mu} \le 2^\lambda \int_{\mathbb{R}^n} \frac{dy}{|y|^\lambda (|y|+1)^\mu}.$$

By the assumptions on λ and μ, the integral on the right-hand side converges. Thus

$$I_2 \le C \le \left(\frac{6}{5}\right)^{\lambda+\mu-n} C(|x|+1)^{n-\lambda-\mu}. \tag{3.2}$$

The desired estimate for $0 \le |x| < 1/5$ follows from (3.1) and (3.2).

Next, we assume $\mu > n$. Since the estimate (3.1) for I_1 is valid for all $x \in \mathbb{R}^n$, we have only to estimate I_2. The assumption $\mu > n$ leads us to

$$\begin{aligned} I_2 &\le \frac{2^\lambda}{(|x|+1)^\lambda} \int_{|x-y|>\frac{|x|+1}{2}} \frac{dy}{(|y|+1)^\mu} \\ &\le \frac{2^\lambda}{(|x|+1)^\lambda} \int_{\mathbb{R}^n} \frac{dy}{(|y|+1)^\mu} \\ &\le C(|x|+1)^{-\lambda}. \end{aligned}$$

Therefore, we deduce

$$\int_{\mathbb{R}^n} \frac{dy}{|x-y|^\lambda (|y|+1)^\mu} \le C\left\{(|x|+1)^{n-\lambda-\mu} + (|x|+1)^{-\lambda}\right\} \le C(|x|+1)^{-\lambda}.$$

The proof is complete. □

In order to show the decay property (2.3) of a solution $u \in X_{n-1}$ to (2.1), we need slow decay estimates for the derivatives. Following the argument due to Miyakawa [11, Theorem 1.1 (iii)], we prove the slow decay property in the next lemma. The property of the Stokes fundamental solution that $\nabla^2 E$ is the Calderón-Zygmund kernel ([14]) plays a crucial role to get information on the class of $\nabla^j u$ $(j \ge 1)$ without assuming the smallness of $\nabla^j F$.

Lemma 3.2. *Let m be a positive integer and $0 < \delta < 1$. Suppose $F \in X_{n+\delta}$ and $u \in X_{n-1}$ is a solution of* (2.1). *If $\nabla^j F \in X_{n+\delta}$ for all $j = 1, \dots, m$, then $\nabla^j u \in X_{n-1}$ for all $j = 1, \dots, m$.*

Proof. We prove by induction on m. The case $m = 1$ follows from the argument below, by letting $m = 0$ formally, and thus we omit the proof for $m = 1$. Assume, in addition to $\nabla^j F \in X_{n+\delta}$ for all $j = 1, \ldots, m$, that $\nabla^{m+1} F \in X_{n+\delta}$ and $\nabla^j u \in X_{n-1}$ for all $j = 1, \ldots, m$. We observe that $\nabla^{m+1} u \in L^\infty(\mathbb{R}^n)$. Let α be any multiindex of order $m+1$. Since $m + 1 \geq 2$, there holds $\alpha_i \geq 1$ for some $1 \leq i \leq n$ and, without loss of generality, we may assume $\alpha_1 \geq 1$. Put $\tilde{\alpha} = (1, 0, \ldots, 0)$ and we write

$$D^\alpha u(x) = \int_{\mathbb{R}^n} D^{\tilde{\alpha}} \nabla E(x-y) D^{\alpha - \tilde{\alpha}} (F - u \otimes u)(y)\, dy.$$

Since $\nabla^2 E$ is the Calderón-Zygmund kernel and $D^{\alpha-\tilde{\alpha}}(F - u \otimes u) \in X_{n+\delta}$, we see $D^\alpha u \in L^q(\mathbb{R}^n)$ for all $1 < q < \infty$. This implies $u \otimes D^\alpha u \in L^r(\mathbb{R}^n)$ for all $1 \leq r < \infty$. For $1 \leq i \leq n$ we also write

$$D_i D^\alpha u(x) = \int_{\mathbb{R}^n} D_i \nabla E(x-y) D^\alpha (F - u \otimes u)(y)\, dy$$

and the same argument as above yields $D_i D^\alpha u \in L^q(\mathbb{R}^n)$ for all $1 < q < \infty$. The index i and the multiindex α of order $m+1$ are arbitrary, and we thus deduce $\nabla^{m+1} u, \nabla^{m+2} u \in L^q(\mathbb{R}^n)$ for all $1 < q < \infty$. Therefore, it follows from the Gagliardo-Nirenberg inequality that

$$\|\nabla^{m+1} u\|_\infty \leq C \|\nabla^{m+1} u\|_r^{1-n/r} \|\nabla^{m+2} u\|_r^{n/r}$$

for $n < r < \infty$.

Since $\nabla^{m+1} u \in L^\infty(\mathbb{R}^n)$ and $|\nabla E(x-y)| \leq C|x-y|^{1-n}$, by Lemma 3.1 we see

$$\begin{aligned}
D^\alpha u(x) &= \int_{\mathbb{R}^n} \nabla E(x-y) D^\alpha (F - u \otimes u)(y)\, dy \\
&\leq \int_{\mathbb{R}^n} \frac{C}{|x-y|^{n-1}} \Bigg(|\nabla^{m+1} F| + \sum_{\ell=1}^{m} |\nabla^\ell u| |\nabla^{m+1-\ell} u| \\
&\qquad\qquad + |u| |\nabla^{m+1} u| \Bigg)(y)\, dy \\
&\leq \int_{\mathbb{R}^n} \frac{C}{|x-y|^{n-1}} \Bigg\{ \frac{\|\nabla^{m+1} F\|_{X_{n+\delta}}}{(|y|+1)^{n+\delta}} + \sum_{\ell=1}^{m} \frac{\|\nabla^\ell u\|_{X_{n-1}}}{(|y|+1)^{n-1}} \\
&\qquad\qquad \times \frac{\|\nabla^{m+1-\ell} u\|_{X_{n-1}}}{(|y|+1)^{n-1}} + \frac{\|u\|_{X_{n-1}} \|\nabla^{m+1} u\|_\infty}{(|y|+1)^{n-1}} \Bigg\}\, dy \\
&\leq C\{(|x|+1)^{1-n} + (|x|+1)^{2-n}\} \\
&\leq C(|x|+1)^{2-n}.
\end{aligned}$$

The multiindex α of order $m+1$ is arbitrary and we thus obtain $\nabla^{m+1} u \in X_{n-2}$. We repeat the calculation above using $\nabla^{m+1} u \in X_{n-2}$, instead of $\nabla^{m+1} u \in L^\infty(\mathbb{R}^n)$, to get $\nabla^{m+1} u \in X_{n-1}$ for $n \geq 4$. In the case $n = 3$, we use $|u||\nabla^{m+1} u| \in X_3 \subset X_{5/2}$ in the calculation above to obtain $\nabla^{m+1}u \in X_{3/2}$. Hence $|u||\nabla^{m+1}u| \in X_{7/2}$ and repeating the calculation once again yields $\nabla^{m+1}u \in X_2$. This completes the induction on m and, as a conclusion, we derive

$$\nabla^j u \in X_{n-1} \quad \text{for all } j = 1, \dots, m,$$

under the assumption of the lemma. □

Now we give the proof of Theorem 2.1.

Proof of Theorem 2.1. The existence of a solution $u \in X_{n-1}$ to (2.1) is proved essentially in [11]. The proof is based on the estimate

$$\left\| \int_{\mathbb{R}^n} \nabla E(x-y)(F - v \otimes v)(y)\, dy \right\|_{X_{n-1}} \leq C(\|F\|_{X_{n+\delta}} + \|v\|_{X_{n-1}}^2)$$

for $v \in X_{n-1}$, which follows from Lemma 3.1. Here the constant C depends only on n and δ. Then the typical argument via the contraction mapping principle yields a solution $u \in X_{n-1}$ to (2.1) provided that F is sufficiently small in $X_{n+\delta}$, see [11, Theorem 1.1 (i)] for details. We note that uniqueness of the solution $u \in X_{n-1}$ in the class of small solutions in X_{n-1} is also shown in [11, Theorem 1.1 (i)].

We prove by induction on m the decay property (2.3) of the solution $u \in X_{n-1}$ obtained above. We can check that the case $m = 1$ follows from the argument below, by letting $m = 0$ formally, and thus we omit the proof for $m = 1$. Assume, in addition to (2.2), that $\nabla^{m+1}F \in X_{n+m+1}$ and $\nabla^j u \in X_{n-1+j}$ for all $j = 1, \dots, m$. According to Lemma 3.2, we have $\nabla^{m+1}u \in X_{n-1}$. Let α be any multiindex of order $m+1$. We write

$$\begin{aligned}
D^\alpha u(x) &= \int_{\mathbb{R}^n} \nabla E(x-y) D^\alpha (F - u \otimes u)(y)\, dy \\
&= \int_{|x-y|<\frac{|x|+1}{2}} \nabla E(x-y) D^\alpha (F - u \otimes u)(y)\, dy \\
&\quad + \int_{|x-y|>\frac{|x|+1}{2}} \nabla E(x-y) D^\alpha (F - u \otimes u)(y)\, dy \\
&=: I_3 + I_4.
\end{aligned}$$

Since $\nabla^{m+1}u \in X_{n-1}$, we have

$$
\begin{aligned}
|I_3| &\le \int_{|x-y|<\frac{|x|+1}{2}} \frac{C}{|x-y|^{n-1}} \Bigg(|\nabla^{m+1}F| + \sum_{\ell=1}^{m} |\nabla^\ell u||\nabla^{m+1-\ell}u| \\
&\qquad\qquad + |u||\nabla^{m+1}u| \Bigg)(y)\,dy \\
&\le \int_{|x-y|<\frac{|x|+1}{2}} \frac{C}{|x-y|^{n-1}} \Bigg\{ \frac{\|\nabla^{m+1}F\|_{X_{n+m+1}}}{(|y|+1)^{n+m+1}} + \sum_{\ell=1}^{m} \frac{\|\nabla^\ell u\|_{X_{n-1+\ell}}}{(|y|+1)^{n-1+\ell}} \\
&\qquad\qquad \times \frac{\|\nabla^{m+1-\ell}u\|_{X_{n+m-\ell}}}{(|y|+1)^{n+m-\ell}} + \frac{\|u\|_{X_{n-1}}\|\nabla^{m+1}u\|_{X_{n-1}}}{(|y|+1)^{2n-2}} \Bigg\} dy \\
&\le C \int_{|x-y|<\frac{|x|+1}{2}} \frac{dy}{|x-y|^{n-1}(|y|+1)^{\gamma}} \qquad (\gamma := \min\{n+m+1, 2n-2\}) \\
&\le \frac{C}{(|x|+1)^{\gamma}} \int_{|x-y|<\frac{|x|+1}{2}} \frac{dy}{|x-y|^{n-1}} \\
&\le C(|x|+1)^{1-\gamma}.
\end{aligned} \tag{3.3}
$$

Concerning the estimate for I_4, we integrate I_4 by parts for $m+1$ times to get

$$
|I_4| \le I_{41} + I_{42},
$$

where

$$
\begin{aligned}
I_{41} &:= \int_{|x-y|>\frac{|x|+1}{2}} |\nabla^{m+2}E(x-y)||(F - u\otimes u)(y)|\,dy, \\
I_{42} &:= \sum_{\ell=1}^{m+1} \int_{|x-y|=\frac{|x|+1}{2}} |\nabla^{\ell}E(x-y)| \Bigg(|\nabla^{m+1-\ell}F| \\
&\qquad\qquad + \sum_{i=0}^{m+1-\ell} |\nabla^i u||\nabla^{m+1-\ell-i}u| \Bigg)(y)\,dS_y.
\end{aligned}
$$

Recalling that $|\nabla^j E(x-y)| \le C|x-y|^{2-n-j}$, we obtain

$$
\begin{aligned}
I_{41} &\le \frac{C}{(|x|+1)^{n+m}} \int_{|x-y|>\frac{|x|+1}{2}} \left\{ \frac{\|F\|_{X_{n+\delta}}}{(|y|+1)^{n+\delta}} + \frac{\|u\|^2_{X_{n-1}}}{(|y|+1)^{2n-2}} \right\} dy \\
&\le C(|x|+1)^{-n-m}.
\end{aligned} \tag{3.4}
$$

Since $|x-y| = (|x|+1)/2$ implies $(|x|-1)/2 \le |y| \le (3|x|+1)/2$, we see

$$
\begin{aligned}
I_{42} &\le \sum_{\ell=1}^{m+1} \int_{|x-y|=\frac{|x|+1}{2}} \frac{C}{|x-y|^{n-2+\ell}} \left\{ \frac{\|\nabla^{m+1-\ell} F\|_{X_{n+m+1-\ell}}}{(|y|+1)^{n+m+1-\ell}} \right. \\
&\qquad \left. + \sum_{i=0}^{m+1-\ell} \frac{\|\nabla^i u\|_{X_{n-1+i}} \|\nabla^{m+1-\ell-i} u\|_{X_{n+m-\ell-i}}}{(|y|+1)^{2n-1+m-\ell}} \right\} dS_y \\
&\le C \sum_{\ell=1}^{m+1} (|x|+1)^{2-n-\ell} (|x|+1)^{-n-m-1+\ell} \int_{|x-y|=\frac{|x|+1}{2}} dS_y \\
&\le C(|x|+1)^{-n-m}.
\end{aligned} \tag{3.5}
$$

Here we have used $F \in X_{n+\delta} \subset X_n$. Hence

$$
|I_4| \le C(|x|+1)^{-n-m}. \tag{3.6}
$$

Since $\gamma = n+m+1$ if $n \ge m+3$ and $\gamma = 2n-2$ if $n \le m+3$, it follows from (3.3) and (3.6) that

$$
\begin{aligned}
|D^\alpha u(x)| &\le C\left\{ (|x|+1)^{1-\gamma} + (|x|+1)^{-n-m} \right\} \\
&\le \begin{cases} C(|x|+1)^{-n-m} & \text{if } n \ge m+3, \\ C(|x|+1)^{3-2n} & \text{if } n \le m+3. \end{cases}
\end{aligned}
$$

The multiindex α of order $m+1$ is arbitrary, and we thus obtain $\nabla^{m+1} u \in X_{n+m}$ for $n \ge m+3$ and $\nabla^{m+1} u \in X_{2n-3}$ for $n \le m+3$. Hence the induction on m is completed if $n \ge m+3$. For $n < m+3$, in view of the estimates above, we have already obtained the desired estimate (3.6) for I_4 and it suffices to estimate I_3 again using $\nabla^{m+1} u \in X_{2n-3}$, instead of $\nabla^{m+1} u \in X_{n-1}$, to get more rapid decay of $D^\alpha u$ (and thus $\nabla^{m+1} u$). Starting from $\nabla^{m+1} u \in X_{n-1}$, we repeat this procedure for ℓ times to deduce

$$
|D^\alpha u(x)| \le \begin{cases} C(|x|+1)^{-n-m} & \text{if } n \ge \dfrac{m+1}{\ell} + 2, \\[2mm] C(|x|+1)^{1-n+\ell(2-n)} & \text{if } n \le \dfrac{m+1}{\ell} + 2. \end{cases}
$$

We choose $\ell = m+1$ to obtain $|D^\alpha u(x)| \le C(|x|+1)^{-n-m}$ for all $n \ge 3$ and any multiindex α of order $m+1$. Therefore

$$
\nabla^{m+1} u \in X_{n+m} \quad \text{for all } n \ge 3.
$$

This completes the induction on m and we conclude that

$$
\nabla^j u \in X_{n-1+j} \quad \text{for all } j = 1, \ldots, m.
$$

Furthermore, from this conclusion and the property that $\nabla^2 E$ is the Calderón-Zygmund kernel, we can deduce that $\nabla^{m+1} u \in L^{1,\infty}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ for $1 < q < \infty$.

Next, we prove the decay property (2.4) of the associated pressure p. Let $0 \le k \le m-1$ and let β be any multiindex of order k. By the integration by parts, we see

$$D^\beta p(x) = \int_{\mathbb{R}^n} Q(x-y) \cdot D^\beta(\mathrm{div}\ F - u \cdot \nabla u)(y)\, dy \le I_5 + I_6 + I_7,$$

where

$$\begin{aligned}
I_5 &:= \int_{|x-y|<\frac{|x|+1}{2}} |Q(x-y)| \left(|\nabla^{k+1} F| + \sum_{\ell=0}^{k+1} |\nabla^\ell u||\nabla^{k+1-\ell} u| \right)(y)\, dy,\\
I_6 &:= \int_{|x-y|>\frac{|x|+1}{2}} |\nabla^{k+1} Q(x-y)||(F - u\otimes u)(y)|\, dy,\\
I_7 &:= \sum_{\ell=0}^{k} \int_{|x-y|=\frac{|x|+1}{2}} |\nabla^\ell Q(x-y)| \left(|\nabla^{k-\ell} F| + \sum_{i=0}^{k-\ell} |\nabla^i u||\nabla^{k-\ell-i} u| \right)(y)\, dS_y.
\end{aligned}$$

Recalling that $|\nabla^j Q(x-y)| \le C|x-y|^{1-n-j}$ and $\nabla^j u \in X_{n-1+j}$ $(j = 0, 1, \dots, m)$, we estimate I_5, I_6 and I_7 in the same way as (3.3), (3.4) and (3.5), respectively, to get

$$I_5 \le C(|x|+1)^{-n-k}, \quad I_6 \le C(|x|+1)^{-n-k}, \quad I_7 \le C(|x|+1)^{-n-k}.$$

Consequently, we obtain $|D^\beta p(x)| \le C(|x|+1)^{-n-k}$ for any multiindex β with $|\beta| = k$. Therefore

$$\nabla^k p \in X_{n+k} \quad \text{for all } k = 0, \dots, m-1.$$

Finally, we prove the uniqueness of the solution u obtained above in the class of solutions $v \in X_{n-1}$ with $\nabla v \in X_n$ to (2.1). We verify that u is a weak solution of (1.1) and can satisfy the smallness condition in [4, 10]. Indeed, the pair $\{u, p\}$ obtained above satisfies $(1.1)_{1,2}$ in the sense of distributions and the class of u implies $u \in \dot{H}^1_{0,\sigma}(\mathbb{R}^n)$. Thus, we see that

$$(\nabla u, \nabla \varphi) + (u \cdot \nabla u, \varphi) = -(F, \nabla \varphi) \quad \text{for all } \varphi \in C^\infty_{0,\sigma}(\mathbb{R}^n).$$

Here $(\cdot,\cdot)$ denotes the inner product in $L^2(\mathbb{R}^n)$. Hence u is a weak solution of (1.1) in the sense of [9, 3]. Furthermore, in view of the construction of the solution u, we can check that the estimate $\|u\|_{X_{n-1}} \le C\|F\|_{X_{n+\delta}}$ holds for some constant C depending only on n and δ. Since this implies

$$\|u\|_{X_1} \le C\|F\|_{X_{n+\delta}}$$

with the same constant C, the solution u can satisfy the smallness condition in [4, 10] provided that F is sufficiently small in $X_{n+\delta}$. Therefore, we can apply the result of [4, 10] to deduce, by restricting the size of F in $X_{n+\delta}$, that the solution u obtained above is unique in the class of weak solutions $w \in \dot{H}^1_{0,\sigma}(\mathbb{R}^n)$ of (1.1) with the energy inequality $\|\nabla w\|_2^2 \le -(F, \nabla w)$.

Let $v \in X_{n-1}$ with $\nabla v \in X_n$ be a solution of (2.1). As we just saw above, v is a weak solution of (1.1). It is known that every weak solution $w \in \dot{H}^1_{0,\sigma}(\mathbb{R}^n)$ with $w \in L^4(\mathbb{R}^n)$

fulfills the energy equality $\|\nabla w\|_2^2 = -(F, \nabla w)$ (see [4, IX, Theorem 2.1 and Remark 2.3]). Clearly, $v \in L^4(\mathbb{R}^n)$ and thus v satisfies the energy equality. Consequently, if F is sufficiently small in $X_{n+\delta}$, then the argument above yields $u = v$. The proof of Theorem 2.1 is complete. □

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