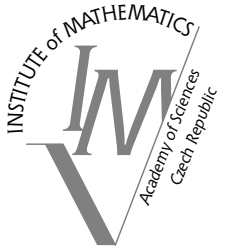

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WEIGHTED ITERATED HARDY-TYPE INEQUALITIES

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Abstract. In this paper a reduction and equivalence theorems for the boundedness of the composition of a quasilinear operator T with the Hardy and Copson operators in weighted Lebesgue spaces are proved. New equivalence theorems are obtained for the operator T to be bounded in weighted Lebesgue spaces restricted to the cones of monotone functions, which allow to change the cone of non-decreasing functions to the cone of non-increasing functions and vice versa not changing the operator T. New characterizations of the weighted Hardy-type inequalities on the cones of monotone functions are given. The validity of so-called weighted iterated Hardy-type inequalities are characterized.

1. Introduction

The well-known two-weight Hardy-type inequalities

$$\left(\int_0^\infty \left(\int_0^x f(\tau)\,d\tau\right)^q w(x)\,dx\right)^{1/q} \le c\left(\int_0^\infty f^p(x)v(x)\,dx\right)^{1/p} \tag{1.1}$$

and

$$\left(\int_0^\infty \left(\int_x^\infty f(\tau)\,d\tau\right)^q w(x)\,dx\right)^{1/q} \le c\left(\int_0^\infty f^p(x)v(x)\,dx\right)^{1/p} \tag{1.2}$$

for all non-negative measurable functions f on $(0,\infty)$, where $0 < p, q < \infty$ with c being a constant independent of f, have a broad variety of applications and represents now a basic tool in many parts of mathematical analysis, namely in the study of weighted function inequalities. For the results, history and applications of this problem, see [33, 34, 36].

Throughout the paper we assume that $I := (a, b) \subseteq (0,\infty)$. By $\mathfrak{M}(I)$ we denote the set of all measurable functions on I. The symbol $\mathfrak{M}^+(I)$ stands for the collection of all $f \in \mathfrak{M}(I)$ which are non-negative on I, while $\mathfrak{M}^+(I;\downarrow)$ and $\mathfrak{M}^+(I;\uparrow)$ are used to denote the subset of those functions which are non-increasing and non-decreasing on I, respectively. When $I = (0,\infty)$, we write simply $\mathfrak{M}^\downarrow$ and $\mathfrak{M}^\uparrow$ instead of $\mathfrak{M}^+(I;\downarrow)$ and $\mathfrak{M}^+(I;\uparrow)$, accordingly. The family of all weight functions (also called just weights) on I, that is, locally integrable non-negative functions on $(0,\infty)$, is given by $\mathcal{W}(I)$.

For $p \in (0,\infty]$ and $w \in \mathfrak{M}^+(I)$, we define the functional $\|\cdot\|_{p,w,I}$ on $\mathfrak{M}(I)$ by

$$\|f\|_{p,w,I} := \begin{cases} \left(\int_I |f(x)|^p w(x)\,dx\right)^{1/p} & \text{if} \quad p < \infty \\ \operatorname{ess\,sup}_I |f(x)| w(x) & \text{if} \quad p = \infty. \end{cases}$$

If, in addition, $w \in \mathcal{W}(I)$, then the weighted Lebesgue space $L^p(w, I)$ is given by

$$L^p(w, I) = \{f \in \mathfrak{M}(I) : \ \|f\|_{p,w,I} < \infty\},$$

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and it is equipped with the quasi-norm $\|\cdot\|_{p,w,I}$.

When $w \equiv 1$ on I, we write simply $L^p(I)$ and $\|\cdot\|_{p,I}$ instead of $L^p(w,I)$ and $\|\cdot\|_{p,w,I}$, respectively.

Suppose f be a measurable a.e. finite function on $\mathbb{R}^n$. Then its non-increasing rearrangement f^* is given by

$$f^*(t) = \inf\{\lambda > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| \le t\}, \quad t \in (0,\infty),$$

and let f^{**} denotes the Hardy-Littlewood maximal function of f, i.e.

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(\tau)\,d\tau, \quad t > 0.$$

Quite many familiar function spaces can be defined using the non-increasing rearrangement of a function. One of the most important classes of such spaces are the so-called classical Lorentz spaces.

Let $p \in (0,\infty)$ and $w \in \mathcal{W}$. Then the classical Lorentz spaces $\Lambda^p(w)$ and $\Gamma^p(w)$ consist of all functions $f \in \mathfrak{M}$ for which $\|f\|_{\Lambda^p(w)} < \infty$ and $\|f\|_{\Gamma^p(w)} < \infty$, respectively. Here it is

$$\|f\|_{\Lambda^p(w)} := \|f^*\|_{p,w,(0,\infty)} \qquad \text{and} \qquad \|f\|_{\Gamma^p(w)} := \|f^{**}\|_{p,w,(0,\infty)}.$$

For more information about the Lorentz Λ and Γ see e.g. [11] and the references therein.

There has been considerable progress in the circle of problems concerning characterization of boundedness of classical operators acting in weighted Lorentz spaces since the beginnig of the 1990s. The first results on the problem $\Lambda^p(v) \hookrightarrow \Gamma^p(v)$, $1 < p < \infty$, which is equivalent to inequality (1.1) restricted to the cones of non-increasing functions, were obtained by Boyd [5] and in an explicit form by Ariño and Muckenhoupt [3]. The problem with $w \neq v$ and $p \neq q$, $1 < p, q < \infty$ was first successfully solved by Sawyer [40]. Many articles on this topic followed, providing the results for a wider range of parameters. In particular, much attention was paid to inequalities (1.1) and (1.2) restricted to the cones of monotone functions; see for instance [3, 4, 10, 12, 15, 22–32, 35, 37, 40, 43, 45–47], survey [11], the monographs [33, 34], for the latest development of this subject see [27], and references given there. The restricted operator inequalities may often be handled by the so-called "reduction theorems". These, in general, reduce a restricted inequality into certain non-restricted inequalities. For example, the restriction to non-increasing or quasi-concave functions may be handled in this way, see e.g. [24–27, 42]. At the initial stage the main tool was the Sawyer duality principle [40], which allowed one to reduce an $L^p - L^q$ inequality for monotone functions with $1 < p, q < \infty$ to a more manageable inequality for arbitrary non-negative functions. This principle was extended by Stepanov in [46] to the case $0 < p < 1 < q < \infty$. In the same work Stepanov applied a different approach to this problem, so-called reduction theorems, which enabled to extend the range of parameters to $1 < p < \infty$, $0 < q < \infty$. The case $p \le q$, $0 < p \le 1$ was alternatively characterized in [8, 12, 35, 46, 47]. Later on some direct reduction theorems were found in [10, 23, 27] involving supremum operators which work for the case $0 < q < p \le 1$.

In this paper we consider operators $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ satisfying the following conditions:

(i) $T(\lambda f) = \lambda T f$ for all $\lambda \ge 0$ and $f \in \mathfrak{M}^+$;

(ii) $Tf(x) \le cTg(x)$ for almost all $x \in \mathbb{R}_+$ if $f(x) \le g(x)$ for almost all $x \in \mathbb{R}_+$, with constant $c > 0$ independent of f and g;

(iii) $T(f + \lambda\mathbf{1}) \le c(Tf + \lambda T\mathbf{1})$ for all $f \in \mathfrak{M}^+$ and $\lambda \ge 0$, with a constant $c > 0$ independent of f and λ.

Given a operator $T : \mathfrak{M}^+ \to \mathfrak{M}^+$, for $0 < p < \infty$ and $u \in \mathfrak{M}^+$, denote by

$$T_{p,u}(g) := (T(g^p u))^{1/p}, \qquad g \in \mathfrak{M}^+.$$

Hence $T_{1,\mathbf{1}} \equiv T$. When $p = 1$, we write T_u instead of $T_{1,u}$.

Denote by

$$Hg(t) := \int_0^t g(s)\,ds, \qquad g \in \mathfrak{M}^+,$$

and

$$H^*g(t) := \int_t^\infty g(s)\,ds, \qquad g \in \mathfrak{M}^+,$$

the Hardy operator and Copson operator, respectively.

In the paper we prove a reduction and equivalence theorems for the boundedness of the composition operators $T \circ H$ or $T \circ H^*$ of a quasiliear operator $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ with the operators H and H^* in weighted Lebesgue spaces. To be more precise, we consider inequalities

$$\left\| T\Big(\int_0^x h \Big) \right\|_{\beta,w,(0,\infty)} \le c\,\|h\|_{s,v,(0,\infty)}, \ h \in \mathfrak{M}^+, \tag{1.3}$$

and

$$\left\| T\Big(\int_x^\infty h \Big) \right\|_{\beta,w,(0,\infty)} \le c\,\|h\|_{s,v,(0,\infty)}, \ h \in \mathfrak{M}^+. \tag{1.4}$$

Using these equivalence theorems, in particular, we completely characterize the validity of the iterated Hardy-type inequalities

$$\left\| H_{p,u}\Big(\int_0^x h \Big) \right\|_{q,w,(0,\infty)} \le c\,\|h\|_{s,v,(0,\infty)}, \tag{1.5}$$

and

$$\left\| H_{p,u}\Big(\int_x^\infty h \Big) \right\|_{q,w,(0,\infty)} \le c\,\|h\|_{s,v,(0,\infty)}, \tag{1.6}$$

where $0 < p < \infty$, $0 < q \le \infty$, $1 \le s < \infty$, u, w and v are weight functions on $(0, \infty)$.

It is worth to mentoin that the characterizations of "dual" inequalities

$$\left\| H^*_{p,u}\Big(\int_x^\infty h \Big) \right\|_{q,w,(0,\infty)} \le c\,\|h\|_{s,v,(0,\infty)}, \tag{1.7}$$

and

$$\left\| H^*_{p,u}\Big(\int_0^x h \Big) \right\|_{q,w,(0,\infty)} \le c\,\|h\|_{s,v,(0,\infty)}, \tag{1.8}$$

can be easily obtained from the solutions of inequalities (1.5) - (1.6), respectively, by change of variables.

In the case when $p = 1$, using the Fubini Theorem, inequalities (1.5) and (1.6) can be reduced to the weighted $L^s - L^q$ boundedness problem of the Volterra operator

$$(Kh)(x) := \int_0^x k(x,y)h(y)\,dy, \quad x > 0,$$

with the kernel

$$k(x,y) := \int_y^x u(t)\,dt, \quad 0 < y \le x < \infty,$$

and the Stieltjes operator

$$(S\,h)(x) = \int_0^\infty \frac{h(t)\,dt}{U(x) + U(t)},$$

respectively, and consequently, can be easily solved. Indeed:

By the Fubini Theorem, we see that

$$\int_0^x \left(\int_0^t h(\tau)\, d\tau \right) u(t)\, dt = \int_0^x k(x,\tau) h(\tau)\, d\tau, \qquad h \in \mathfrak{M}^+(0,\infty).$$

On the other hand, it is easy to see that

$$\int_0^x \left(\int_t^\infty h(s)\, ds \right) u(t)\, dt \approx U(x) \cdot S(hU)(x), \qquad h \in \mathfrak{M}^+(0,\infty).$$

Note that the weighted $L^s - L^q$ boundedness of Volterra operators K, that is, inequality

$$\|Kh\|_{q,w,(0,\infty)} \le c \|h\|_{s,v,(0,\infty)}, \quad h \in \mathfrak{M}^+(0,\infty), \tag{1.9}$$

is completely characterized for $1 \le s \le \infty$, $0 < q \le \infty$ (see [27] and references given there).

The usual Stieltjes transform is obtained on putting $U(x) \equiv x$. In the case $U(x) \equiv x^\lambda$, $\lambda > 0$, the boundedness of the operator S between weighted L^s and L^q spaces, namely inequality

$$\|Sh\|_{q,w,(0,\infty)} \le c \|h\|_{s,v,(0,\infty)}, \quad h \in \mathfrak{M}^+(0,\infty), \tag{1.10}$$

was investigated in [2] (when $1 \le s \le q \le \infty$), in [41] (when $1 \le q < s \le \infty$), in [13] (see also [14]) (when $1 < s < \infty$, $0 < q \le \infty$), where the result is presented without proof. This problem also was considered in [16] and [20, 21], where completely different approach was used, based on the so called "gluing lemma" (see also [17]). It is proved in [19] (when $1 \le s \le \infty$, $0 < q \le \infty$) that inequality (1.10) holds if and only if

$$\left\| H_u \left(\int_x^\infty h \right) \right\|_{q,w,(0,\infty)} \le c \|hU\|_{s,v,(0,\infty)}, \quad h \in \mathfrak{M}^+(0,\infty), \tag{1.11}$$

holds, and the solution of (1.10) is obtained using characterization of inequality (1.11).

Note that inequality (1.6) has been completely characterized in [18] and [19] in the case $0 < p < \infty$, $0 < q \le \infty$, $1 \le s \le \infty$ by using difficult discretization and anti-discretization methods. Inequalities (1.5) - (1.6) and (1.7) - (1.8) were considered also in [38] and [39], but characterization obtained there is not complete and seems to us unsatisfactory from a practical point of view.

We pronounce that the characterizations of inequalities (1.5)-(1.6) and (1.7)-(1.8) are important because many inequalities for classical operators can be reduced to them (for illustrations of this important fact, see, for instance, [19]). These inequalities play an important role in the theory of Morrey-type spaces and other topics (see [6], [7] and [9]). It is worth to mention that using characterizations of weighted Hardy inequalities we can show that the characterization of the boundedness of bilinear Hardy inequalities, namely of the inequality

$$\|T_1 f \cdot T_2 g\|_{w,q,(0,\infty)} \le c \|f\|_{p_1,v_1,(0,\infty)} \|g\|_{p_2,v_2,(0,\infty)}, \tag{1.12}$$

for all $f \in L^{p_1}(v_1,(0,\infty))$ and $g \in L^{p_2}(v_2,(0,\infty))$ with constant c independent of f and g, where $T_i = H$ or H^*, $i = 1, 2$, are equivalent to inequalities (1.5)-(1.6) and (1.7)-(1.8) (see, for instance, [1]).

It is well-known that when T is a integral operator then by substitution of variables it is possible to change the cone of non-decreasing functions to the cone of non-increasing functions and vice versa, when considering inequalities

$$\|Tf\|_{\beta,w,(0,\infty)} \le c \|f\|_{s,v,(0,\infty)}, \qquad f \in \mathfrak{M}^{\downarrow}(0,\infty), \tag{1.13}$$

and

$$\|Tf\|_{\beta,w,(0,\infty)} \le c \|f\|_{s,v,(0,\infty)}, \qquad f \in \mathfrak{M}^{\uparrow}(0,\infty), \tag{1.14}$$

but this procedure changes T also as usually to the "dual" operator. Theorems proved in Section 4 allows to change the cones to each other not changing the operator T. This new observation enables to state that if we know solution of one inequality on any cone of monotone functions, then we could characterize the inequality on the other cone of monotone functions.

The paper is organized as follows. Section 2 contains some preliminaries along with the standard ingredients used in the proofs. In Section 3 we prove the reduction and equivalence theorems for the boundedness of the composition operators $T \circ H$ or $T \circ H^*$ in weighted Lebesgue spaces. In Section 4 the equivalence theorems which allow to change the cones of monotone functions to each other not changing the operator T are proved. In Section 5 we obtain a new characterizations of the weighted Hardy-type inequalities on the cones of monotone functions. In Section 6 we give complete characterization of inequalities (1.5) - (1.6) and (1.7) - (1.8).

2. Notations and Preliminaries

Throughout the paper, we always denote by c or C a positive constant, which is independent of main parameters but it may vary from line to line. However a constant with subscript or superscript such as c_1 does not change in different occurrences. By $a \lesssim b$, $(b \gtrsim a)$ we mean that $a \le \lambda b$, where $\lambda > 0$ depends on inessential parameters. If $a \lesssim b$ and $b \lesssim a$, we write $a \approx b$ and say that a and b are equivalent. We will denote by $\mathbf{1}$ the function $\mathbf{1}(x) = 1$, $x \in (0, \infty)$. Unless a special remark is made, the differential element dx is omitted when the integrals under consideration are the Lebesgue integrals. Everywhere in the paper, u, v and w are weights.

Convention 2.1. We adopt the following conventions:

(i) Throughout the paper we put $0 \cdot \infty = 0$, $\infty/\infty = 0$ and $0/0 = 0$.

(ii) If $p \in [1, +\infty]$, we define p' by $1/p + 1/p' = 1$.

(iii) If $0 < q < p < \infty$, we define r by $1/r = 1/q - 1/p$.

(iv) If $I = (a, b) \subseteq \mathbb{R}$ and g is monotone function on I, then by $g(a)$ and $g(b)$ we mean the limits $\lim_{x \to a+} g(x)$ and $\lim_{x \to b-} g(x)$, respectively.

To state the next statements we need the following notations:

$$U(t) := \int_0^t u, \quad U_*(t) := \int_t^\infty u,$$

$$V(t) := \int_0^t v, \quad V_*(t) := \int_t^\infty v,$$

$$W(t) := \int_0^t w, \quad W_*(t) := \int_t^\infty w.$$

Theorem 2.2 ([27], Theorem 3.1)**.** *Let $0 < \beta \le \infty$ and $1 \le s < \infty$, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ be a positive operator. Then the inequality*

$$\|Tf\|_{\beta,w,(0,\infty)} \le c\|f\|_{s,v,(0,\infty)}, \qquad f \in \mathfrak{M}^{\downarrow}(0,\infty) \tag{2.1}$$

implies the inequality

$$\left\| T\left(\int_x^\infty h \right) \right\|_{\beta,w,(0,\infty)} \le c\|h\|_{s,V^s v^{1-s},(0,\infty)}, \qquad h \in \mathfrak{M}^+(0,\infty). \tag{2.2}$$

If $V(\infty) = \infty$ and if T is an operator satisfying conditions (i)-(ii)*, then the condition* (2.2) *is sufficient for inequality* (2.1) *to hold on the cone $\mathfrak{M}^{\downarrow}$. Further, if $0 < V(\infty) < \infty$, then a sufficient condition for* (2.1) *to*

hold on $\mathfrak{M}^{\downarrow}$ is that both (2.2) *and*

$$\|T\mathbf{1}\|_{\beta,w,(0,\infty)} \le c\|\mathbf{1}\|_{s,v,(0,\infty)} \tag{2.3}$$

hold in the case when T satisfies the conditions (i)-(iii).

Theorem 2.3 ([27], Theorem 3.2)**.** *Let $0 < \beta \le \infty$ and $1 \le s < \infty$, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ satisfies conditions* (i) *and* (ii). *Then a sufficient condition for inequality* (2.1) *to hold is that*

$$\left\| T\Big(\frac{1}{V^2(x)}\int_0^x hV\Big)\right\|_{\beta,w,(0,\infty)} \le c\|h\|_{s,v^{1-s},(0,\infty)}, \qquad h \in \mathfrak{M}^+(0,\infty). \tag{2.4}$$

Moreover, (2.1) *is necessary for* (2.4) *to hold if conditions* (i)-(iii) *are all satisfied.*

Theorem 2.4 ([27], Theorem 3.3)**.** *Let $0 < \beta \le \infty$ and $1 \le s < \infty$, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ be a positive operator. Then the inequality*

$$\|Tf\|_{\beta,w,(0,\infty)} \le c\|f\|_{s,v,(0,\infty)}, \qquad f \in \mathfrak{M}^{\uparrow}(0,\infty) \tag{2.5}$$

implies the inequality

$$\left\| T\Big(\int_0^x h\Big)\right\|_{\beta,w,(0,\infty)} \le c\|h\|_{s,V_*^s v^{1-s},(0,\infty)}, \qquad h \in \mathfrak{M}^+(0,\infty). \tag{2.6}$$

If $V_*(0) = \infty$ and if T is an operator satisfying the conditions (i)-(ii)*, then the condition* (2.6) *is sufficient for inequality* (2.5) *to hold. If $0 < V_*(0) < \infty$ and T is an operator satisfying the conditions* (i)-(iii)*, then* (2.5) *follows from* (2.6) *and* (2.3).

Theorem 2.5 ([27], Theorem 3.4)**.** *Let $0 < \beta \le \infty$ and $1 \le s < \infty$, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ satisfies conditions* (i) *and* (ii). *Then a sufficient condition for inequality* (2.5) *to hold is that*

$$\left\| T\Big(\frac{1}{V_*^2(x)}\int_x^\infty hV_*\Big)\right\|_{\beta,w,(0,\infty)} \le c\|h\|_{s,v^{1-s},(0,\infty)}, \qquad h \in \mathfrak{M}^+(0,\infty). \tag{2.7}$$

Moreover, (2.5) *is necessary for* (2.7) *to hold if conditions* (i)-(iii) *are all satisfied.*

3. Reduction and equivalence theorems

In this section we prove some reduction and equivalence theorems for inequalities (1.3) and (1.4).

3.1. The case $1 < s < \infty$.

The following theorem allows to reduce the iterated inequality (1.3) to the inequality on the cone of non-increasing functions.

Theorem 3.1. *Let $0 < \beta \le \infty$, $1 < s < \infty$, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ satisfies conditions* (i)-(iii). *Assume that u, $w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(0,\infty)$ be such that*

$$\int_0^x v^{1-s'}(t)\,dt < \infty \qquad \text{for all} \qquad x > 0. \tag{3.1}$$

Then inequality (1.3) *holds iff*

$$\|T_{\Phi^2} f\|_{\beta,w,(0,\infty)} \le c\|f\|_{s,\phi,(0,\infty)}, \ f \in \mathfrak{M}^{\downarrow}, \tag{3.2}$$

holds, where

$$\phi(x) \equiv \phi[v;s](x) := \Big(\int_0^x v^{1-s'}(t)\,dt\Big)^{-\frac{s'}{s'+1}} v^{1-s'}(x)$$

and

$$\Phi(x) \equiv \Phi[v;s](x) = \int_0^x \phi(t)\,dt = \Big(\int_0^x v^{1-s'}(t)\,dt\Big)^{\frac{1}{s'+1}}.$$

Proof. Note that $\Phi^{-s}\phi^{1-s} \approx v$. Inequality (1.3) is equivalent to the inequality

$$\left\| T_{\Phi^2}\left(\frac{1}{\Phi^2(x)}\int_0^x h\right)\right\|_{\beta,w,(0,\infty)} \le c\|h\|_{s,\Phi^{-s}\phi^{1-s},(0,\infty)},\ h\in\mathfrak{M}^+. \tag{3.3}$$

Obviously, (3.3) is equivalent to

$$\left\| T_{\Phi^2}\left(\frac{1}{\Phi^2(x)}\int_0^x h\Phi\right)\right\|_{\beta,w,(0,\infty)} \le c\|h\|_{s,\phi^{1-s},(0,\infty)},\ h\in\mathfrak{M}^+. \tag{3.4}$$

By Theorem 2.3, inequality (3.4) is equivalent to

$$\|T_{\Phi^2}f\|_{\beta,w,(0,\infty)} \le c\|f\|_{s,\phi,(0,\infty)},\ f\in\mathfrak{M}^{\downarrow}.$$

This completes the proof. □

We immediately get the following equivalence statements.

Corollary 3.2. *Let $0<\beta\le\infty$, $1<s<\infty$, $0<\delta\le s$, and let $T:\mathfrak{M}^+\to\mathfrak{M}^+$ satisfies conditions* (i)-(iii). *Assume that u, $w\in\mathcal{W}(0,\infty)$ and $v\in\mathcal{W}(0,\infty)$ be such that* (3.1) *holds. Then inequality* (1.3) *holds iff both*

$$\left\| T_{\Phi^2}\left(\left\{\int_x^\infty h^\delta\right\}^{1/\delta}\right)\right\|_{\beta,w,(0,\infty)} \le c\|h\|_{s,\Phi^{s/\delta}\phi^{1-s/\delta},(0,\infty)},\ h\in\mathfrak{M}^+, \tag{3.5}$$

and

$$\|T_{\Phi^2}(\mathbf{1})\|_{\beta,w,(0,\infty)} \le c\|\mathbf{1}\|_{s,\phi,(0,\infty)}, \tag{3.6}$$

hold.

Proof. By Theorem 3.1, inequality (1.3) is equivalent to

$$\|T_{\Phi^2}f\|_{\beta,w,(0,\infty)} \le c\|f\|_{s,\phi,(0,\infty)},\ f\in\mathfrak{M}^{\downarrow}. \tag{3.7}$$

Since (3.7) is equivalent to

$$\|\widetilde{T}f\|_{\beta/\delta,w,(0,\infty)} \le c^\delta\|f\|_{s/\delta,\phi,(0,\infty)},\ f\in\mathfrak{M}^{\downarrow}, \tag{3.8}$$

with

$$\widetilde{T}(f) := \left\{T_{\Phi^2}(f^{1/\delta})\right\}^\delta,$$

it remains to apply Theorem 2.2. □

Corollary 3.3. *Let $0<\beta\le\infty$, $1<s<\infty$, $0<\delta\le s$, and let $T:\mathfrak{M}^+\to\mathfrak{M}^+$ satisfies conditions* (i)-(iii). *Assume that u, $w\in\mathcal{W}(0,\infty)$ and $v\in\mathcal{W}(0,\infty)$ be such that* (3.1) *holds. Then inequality* (1.3) *holds iff*

$$\left\| T_{\Phi^{2(1-1/\delta)}}\left(\left\{\int_0^x h^\delta\Phi\right\}^{1/\delta}\right)\right\|_{\beta,w,(0,\infty)} \le c\|h\|_{s,\phi^{1-s/\delta},(0,\infty)},\ h\in\mathfrak{M}^+ \tag{3.9}$$

holds.

Proof. By Theorem 3.1, inequality (1.3) is equivalent to

$$\|T_{\Phi^2}f\|_{\beta,w,(0,\infty)} \le c\|f\|_{s,\phi,(0,\infty)},\ f\in\mathfrak{M}^{\downarrow}. \tag{3.10}$$

We know that (3.10) is equivalent to

$$\|\widetilde{T}f\|_{\beta/\delta,w,(0,\infty)} \le c^\delta\|f\|_{s/\delta,\phi,(0,\infty)},\ f\in\mathfrak{M}^{\downarrow}, \tag{3.11}$$

with

$$\widetilde{T}(f) := \left\{T_{\Phi^2}(f^{1/\delta})\right\}^\delta,$$

By Theorem 2.3, we see that (3.11) is equivalent to

$$\left\|\widetilde{T}\Big(\frac{1}{\Phi^2(x)}\int_0^x h\Phi\Big)\right\|_{\beta/\delta,w,(0,\infty)} \le c^{\delta}\|h\|_{s/\delta,\phi^{1-s/\delta},(0,\infty)},\ h\in\mathfrak{M}^+(0,\infty). \tag{3.12}$$

To complete the proof it suffices to note that (3.12) is equivalent to (3.9). □

The following "dual" version of the reduction and equivalence statements also hold true and may be proved analogously.

Theorem 3.4. *Let $0<\beta\le\infty$, $1<s<\infty$, and let $T:\mathfrak{M}^+\to\mathfrak{M}^+$ satisfies conditions* (i)-(iii). *Assume that u, $w\in\mathcal{W}(0,\infty)$ and $v\in\mathcal{W}(0,\infty)$ be such that*

$$\int_x^\infty v^{1-s'}(t)\,dt<\infty \qquad \text{for all} \qquad x>0. \tag{3.13}$$

Then inequality (1.4) *holds iff*

$$\|T_{\Psi^2}f\|_{\beta,w,(0,\infty)}\le c\|f\|_{s,\psi,(0,\infty)},\ f\in\mathfrak{M}^{\uparrow} \tag{3.14}$$

holds, where

$$\psi(x)\equiv\psi[v;s](x):=\Big(\int_x^\infty v^{1-s'}(t)\,dt\Big)^{-\frac{s'}{s'+1}}v^{1-s'}(x)$$

and

$$\Psi(x)\equiv\Psi[v;s](x):=\int_x^\infty\psi(t)\,dt=\Big(\int_x^\infty v^{1-s'}(t)\,dt\Big)^{\frac{1}{s'+1}}$$

Corollary 3.5. *Let $0<\beta\le\infty$, $1<s<\infty$, $0<\delta\le s$, and let $T:\mathfrak{M}^+\to\mathfrak{M}^+$ satisfies conditions* (i)-(iii). *Assume that u, $w\in\mathcal{W}(0,\infty)$ and $v\in\mathcal{W}(0,\infty)$ be such that* (3.13) *holds. Then inequality* (1.4) *holds iff both*

$$\left\|T_{\Psi^2}\Big(\Big\{\int_0^x h^{\delta}\Big\}^{1/\delta}\Big)\right\|_{\beta,w,(0,\infty)}\le c\|h\|_{s,\Psi^{s/\delta}\psi^{1-s/\delta},(0,\infty)},\ h\in\mathfrak{M}^+, \tag{3.15}$$

and

$$\|T_{\Psi^2}\mathbf{1}\|_{\beta,w,(0,\infty)}\le c\|\mathbf{1}\|_{s,\psi,(0,\infty)}, \tag{3.16}$$

hold.

Corollary 3.6. *Let $0<\beta\le\infty$, $1<s<\infty$, $0<\delta\le s$, and let $T:\mathfrak{M}^+\to\mathfrak{M}^+$ satisfies conditions* (i)-(iii). *Assume that u, $w\in\mathcal{W}(0,\infty)$ and $v\in\mathcal{W}(0,\infty)$ be such that* (3.13) *holds. Then inequality* (1.4) *holds iff*

$$\left\|T_{\Psi^{2(1-1/\delta)}}\Big(\Big\{\int_x^\infty h^{\delta}\Psi\Big\}^{1/\delta}\Big)\right\|_{\beta,w,(0,\infty)}\le c\|h\|_{s,\psi^{1-s/\delta},(0,\infty)},\ h\in\mathfrak{M}^+ \tag{3.17}$$

holds.

The following theorem allows to reduce the iterated inequality (1.3) to the inequality on the cone of non-decreasing functions.

Theorem 3.7. *Let $0<\beta\le\infty$, $1<s<\infty$, and let $T:\mathfrak{M}^+\to\mathfrak{M}^+$ satisfies conditions* (i)-(iii). *Assume that u, $w\in\mathcal{W}(0,\infty)$ and $v\in\mathcal{W}(0,\infty)$ be such that* (3.1) *holds. Then inequality* (1.3) *holds iff both*

$$\left\|T_{\Phi^2[v;s]\cdot\Psi^{2/\delta}[\Phi[v;s]^{s/\delta}\phi[v;s]^{1-s/\delta};s/\delta]}f\right\|_{\beta,w,(0,\infty)}\le c\|f\|_{s,\psi[\Phi[v;s]^{s/\delta}\phi[v;s]^{1-s/\delta};s/\delta],(0,\infty)},\ f\in\mathfrak{M}^{\uparrow},$$

where $0<\delta<s$,

$$\psi[\Phi[v;s]^{s/\delta}\phi[v;s]^{1-s/\delta};s/\delta](x)$$

$$\approx \left\{\int_x^\infty \left(\int_0^t v^{1-s'}\right)^{-\frac{s'+(s/\delta)'}{1+s'}} v^{1-s'}(t)\,dt\right\}^{-\frac{(s/\delta)'}{1+(s/\delta)'}} \left(\int_0^x v^{1-s'}\right)^{-\frac{s'+(s/\delta)'}{1+s'}} v^{1-s'}(x),$$

$$\Psi[\Phi[v;s]^{s/\delta}\phi[v;s]^{1-s/\delta};s/\delta](x) \approx \left\{\int_x^\infty \left(\int_0^t v^{1-s'}\right)^{-\frac{s'+(s/\delta)'}{1+s'}} v^{1-s'}(t)\,dt\right\}^{\frac{1}{1+(s/\delta)'}},$$

and (3.6) *hold.*

Proof. By Corollary 3.2, (1.3) holds iff both (3.5) and (3.6) hold. It is easy to see that (3.5) is equivalent to

$$\left\|\left[T_{\Phi^2}\left(\left\{\int_x^\infty h\right\}^{1/\delta}\right)\right]^\delta\right\|_{\beta/\delta,w,(0,\infty)} \le c^\delta \|h\|_{s/\delta,\Phi^{s/\delta}\phi^{1-s/\delta},(0,\infty)},\ h\in\mathfrak{M}^+, \tag{3.18}$$

Since

$$\begin{aligned}
&\psi[\Phi[v;s]^{s/\delta}\phi[v;s]^{1-s/\delta};s/\delta](x)\\
&\quad= \left(\int_x^\infty \Phi[v;s]^{-(s/\delta)'}\phi[v;s]\right)^{-\frac{(s/\delta)'}{(s/\delta)'+1}} \Phi[v;s]^{-(s/\delta)'}(x)\phi[v;s](x)\\
&\quad\approx \left(\Phi[v;s]^{1-(s/\delta)'}(x) - \Phi[v;s]^{1-(s/\delta)'}(\infty)\right)^{-\frac{(s/\delta)'}{(s/\delta)'+1}} \Phi[v;s]^{-(s/\delta)'}(x)\phi[v;s](x)\\
&\quad\approx \left\{\left(\int_0^x v^{1-s'}\right)^{\frac{1-(s/\delta)'}{1+s'}} - \left(\int_0^\infty v^{1-s'}\right)^{\frac{1-(s/\delta)'}{1+s'}}\right\}^{-\frac{(s/\delta)'}{1+(s/\delta)'}} \left(\int_0^x v^{1-s'}\right)^{-\frac{s'+(s/\delta)'}{1+s'}} v^{1-s'}(x)\\
&\quad\approx \left\{\int_x^\infty \left(\int_0^t v^{1-s'}\right)^{-\frac{s'+(s/\delta)'}{1+s'}} v^{1-s'}(t)\,dt\right\}^{-\frac{(s/\delta)'}{1+(s/\delta)'}} \left(\int_0^x v^{1-s'}\right)^{-\frac{s'+(s/\delta)'}{1+s'}} v^{1-s'}(x),
\end{aligned}$$

and

$$\begin{aligned}
&\Psi[\Phi[v;s]^{s/\delta}\phi[v;s]^{1-s/\delta};s/\delta](x)\\
&\quad= \left(\int_x^\infty \Phi[v;s]^{-(s/\delta)'}\phi[v;s]\right)^{\frac{1}{(s/\delta)'+1}}\\
&\quad\approx \left(\Phi[v;s]^{1-(s/\delta)'}(x) - \Phi[v;s]^{1-(s/\delta)'}(\infty)\right)^{\frac{1}{(s/\delta)'+1}}\\
&\quad\approx \left\{\left(\int_0^x v^{1-s'}\right)^{\frac{1-(s/\delta)'}{1+s'}} - \left(\int_0^\infty v^{1-s'}\right)^{\frac{1-(s/\delta)'}{1+s'}}\right\}^{\frac{1}{1+(s/\delta)'}}\\
&\quad\approx \left\{\int_x^\infty \left(\int_0^t v^{1-s'}\right)^{-\frac{s'+(s/\delta)'}{1+s'}} v^{1-s'}(t)\,dt\right\}^{\frac{1}{1+(s/\delta)'}},
\end{aligned}$$

by Theorem 3.4, we complete the proof. □

Corollary 3.8. *Let* $0<\beta\le\infty$, $1<s<\infty$, *and let* $T:\mathfrak{M}^+\to\mathfrak{M}^+$ *satisfies conditions* (i)-(iii). *Assume that* u, $w\in\mathcal{W}(0,\infty)$ *and* $v\in\mathcal{W}(0,\infty)$ *be such that* (3.1) *holds. Then inequality* (1.3) *holds iff both*

$$\left\|T_{\Phi^2[v;s]\cdot\Psi^{4/s}[\Phi[v;s]^2\phi[v;s]^{-1};2]}f\right\|_{\beta,w,(0,\infty)} \le c\|f\|_{s,\psi[\Phi[v;s]^2\phi[v;s]^{-1};2],(0,\infty)},\ f\in\mathfrak{M}^{\uparrow},$$

where

$$\begin{aligned}
&\psi[\Phi[v;s]^2\phi[v;s]^{-1};2](x)\\
&\quad\approx \left\{\int_x^\infty \left(\int_0^t v^{1-s'}\right)^{-\frac{2+s'}{1+s'}} v^{1-s'}(t)\,dt\right\}^{-\frac{2}{3}} \left(\int_0^x v^{1-s'}\right)^{-\frac{2+s'}{1+s'}} v^{1-s'}(x),
\end{aligned}$$

$$\Psi[\Phi[v;s]^2\phi[v;s]^{-1};2](x)\approx\Big\{\int_x^\infty\Big(\int_0^t v^{1-s'}\Big)^{-\frac{2+s'}{1+s'}}v^{1-s'}(t)\,dt\Big\}^{\frac{1}{3}},$$

and (3.6) *hold.*

Proof. The statement follows by Theorem 3.7 with $\delta=s/2$. □

The following "dual" statement also holds true and may be proved analogously.

Theorem 3.9. *Let* $0<\beta\le\infty$, $1<s<\infty$, *and let* $T:\mathfrak{M}^+\to\mathfrak{M}^+$ *satisfies conditions* (i)-(iii). *Assume that* u, $w\in\mathcal{W}(0,\infty)$ *and* $v\in\mathcal{W}(0,\infty)$ *be such that* (3.13) *holds. Then inequality* (1.4) *holds iff both*

$$\big\|T_{\Psi^2[v;s]\cdot\Phi^{2/\delta}[\Psi[v;s]^{s/\delta}\psi[v;s]^{1-s/\delta};s/\delta]}f\big\|_{\beta,w,(0,\infty)}\le c\|f\|_{s,\phi[\Psi[v;s]^{s/\delta}\psi[v;s]^{1-s/\delta};s/\delta],(0,\infty)},\ f\in\mathfrak{M}^{\downarrow},$$

where $0<\delta<s$,

$$\begin{aligned}&\phi[\Psi[v;s]^{s/\delta}\psi[v;s]^{1-s/\delta};s/\delta](x)\\&\qquad\approx\Big\{\int_0^x\Big(\int_t^\infty v^{1-s'}\Big)^{-\frac{s'+(s/\delta)'}{1+s'}}v^{1-s'}(t)\,dt\Big\}^{-\frac{(s/\delta)'}{1+(s/\delta)'}}\Big(\int_x^\infty v^{1-s'}\Big)^{-\frac{s'+(s/\delta)'}{1+s'}}v^{1-s'}(x),\\&\Phi[\Psi[v;s]^{s/\delta}\psi[v;s]^{1-s/\delta};s/\delta](x)\approx\Big\{\int_0^x\Big(\int_t^\infty v^{1-s'}\Big)^{-\frac{s'+(s/\delta)'}{1+s'}}v^{1-s'}(t)\,dt\Big\}^{\frac{1}{1+(s/\delta)'}},\end{aligned}$$

and (3.6) *hold.*

Corollary 3.10. *Let* $0<\beta\le\infty$, $1<s<\infty$, *and let* $T:\mathfrak{M}^+\to\mathfrak{M}^+$ *satisfies conditions* (i)-(iii). *Assume that* u, $w\in\mathcal{W}(0,\infty)$ *and* $v\in\mathcal{W}(0,\infty)$ *be such that* (3.13) *holds. Then inequality* (1.4) *holds iff both*

$$\big\|T_{\Psi^2[v;s]\cdot\Phi^{4/s}[\Psi[v;s]^2\psi[v;s]^{-1};2]}f\big\|_{\beta,w,(0,\infty)}\le c\|f\|_{s,\phi[\Psi[v;s]^2\psi[v;s]^{-1};2],(0,\infty)},\ f\in\mathfrak{M}^{\downarrow},$$

where

$$\begin{aligned}&\phi[\Psi[v;s]^2\psi[v;s]^{-1};2](x)\\&\qquad\approx\Big\{\int_0^x\Big(\int_t^\infty v^{1-s'}\Big)^{-\frac{2+s'}{1+s'}}v^{1-s'}(t)\,dt\Big\}^{-\frac{2}{3}}\Big(\int_x^\infty v^{1-s'}\Big)^{-\frac{2+s'}{1+s'}}v^{1-s'}(x),\\&\Phi[\Psi[v;s]^2\psi[v;s]^{-1};2](x)\approx\Big\{\int_0^x\Big(\int_t^\infty v^{1-s'}\Big)^{-\frac{2+s'}{1+s'}}v^{1-s'}(t)\,dt\Big\}^{\frac{1}{3}},\end{aligned}$$

and (3.6) *hold.*

3.2. **The case** $s=1$**.** In this case we have the following results.

Theorem 3.11. *Let* $0<\beta\le\infty$, *and let* $T:\mathfrak{M}^+\to\mathfrak{M}^+$ *satisfies conditions* (i)-(iii). *Assume that* u, $w\in\mathcal{W}(0,\infty)$ *and* $v\in\mathcal{W}(0,\infty)$ *be such that* $V(x)<\infty$ *for all* $x>0$. *Then inequality*

$$\Big\|T\Big(\int_0^x h\Big)\Big\|_{\beta,w,(0,\infty)}\le c\,\|h\|_{1,V^{-1},(0,\infty)},\ h\in\mathfrak{M}^+,\tag{3.19}$$

holds iff

$$\|T_{V^2}f\|_{\beta,w,(0,\infty)}\le c\|f\|_{1,v,(0,\infty)},\ f\in\mathfrak{M}^{\downarrow}.\tag{3.20}$$

Proof. Inequality (3.19) is equivalent to the inequality

$$\Big\|T_{V^2}\Big(\frac{1}{V^2(x)}\int_0^x hV\Big)\Big\|_{\beta,w,(0,\infty)}\le c\,\|h\|_{1,(0,\infty)},\ h\in\mathfrak{M}^+.\tag{3.21}$$

By Theorem 2.3, inequality (3.21) is equivalent to (3.20). □

Corollary 3.12. *Let $0 < \beta \le \infty$, $0 < \delta \le 1$, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ satisfies conditions* (i)-(iii). *Assume that u, $w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(0,\infty)$ be such that $V(x) < \infty$ for all $x > 0$. Then inequality* (3.19) *holds iff both*

$$\left\| T_{V^2}\left(\left\{ \int_x^\infty h^\delta \right\}^{1/\delta} \right) \right\|_{\beta,w,(0,\infty)} \le c\|h\|_{1,V^{1/\delta}v^{1-1/\delta},(0,\infty)},\ h \in \mathfrak{M}^+, \tag{3.22}$$

and

$$\|T_{V^2}(\mathbf{1})\|_{\beta,w,(0,\infty)} \le c\|\mathbf{1}\|_{1,v,(0,\infty)}, \tag{3.23}$$

hold.

Proof. By Theorem 3.11, inequality (3.19) is equivalent to (3.20). Since (3.20) is equivalent to

$$\left\| \left\{ T_{V^2}(f^{1/\delta}) \right\}^\delta \right\|_{\beta/\delta,w,(0,\infty)} \le c^\delta \|f\|_{1/\delta,v,(0,\infty)},\ f \in \mathfrak{M}^\downarrow, \tag{3.24}$$

it remains to apply Theorem 2.2. □

Corollary 3.13. *Let $0 < \beta \le \infty$, $0 < \delta \le 1$, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ satisfies conditions* (i)-(iii). *Assume that u, $w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(0,\infty)$ be such that $V(x) < \infty$ for all $x > 0$. Then inequality* (3.19) *holds iff*

$$\left\| T_{V^{2(1-1/\delta)}}\left(\left\{ \int_0^x h^\delta V \right\}^{1/\delta} \right) \right\|_{\beta,w,(0,\infty)} \le c\|h\|_{1,v^{1-1/\delta},(0,\infty)},\ h \in \mathfrak{M}^+ \tag{3.25}$$

holds.

Proof. By Theorem 3.11, inequality (3.19) is equivalent to (3.24). By Theorem 2.3, we see that (3.24) is equivalent to

$$\left\| \left\{ T_{V^2}\left(\left[\frac{1}{V^2(x)} \int_0^x hV \right]^{1/\delta} \right) \right\}^\delta \right\|_{\beta/\delta,w,(0,\infty)} \le c^\delta \|h\|_{1/\delta,v^{1-1/\delta},(0,\infty)},\ h \in \mathfrak{M}^+(0,\infty). \tag{3.26}$$

To complete the proof it suffices to note that (3.26) is equivalent to (3.25). □

The following theorem allows to reduce the iterated inequality (3.19) to the inequality on the cone of non-decreasing functions.

Theorem 3.14. *Let $0 < \beta \le \infty$, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ satisfies conditions* (i)-(iii). *Assume that u, $w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(0,\infty)$ be such that $V(x) < \infty$ for all $x > 0$. Then inequality* (3.19) *holds iff both*

$$\left\| T_{V^2 \cdot \Psi^{2/\delta}[V^{1/\delta}v^{1-1/\delta};1/\delta]} f \right\|_{\beta,w,(0,\infty)} \le c\|f\|_{1,\psi[V^{1/\delta}v^{1-1/\delta};1/\delta],(0,\infty)},\ f \in \mathfrak{M}^\uparrow, \tag{3.27}$$

where $0 < \delta < 1$,

$$\psi[V^{1/\delta}v^{1-1/\delta};1/\delta](x) \approx \left(\int_x^\infty V^{-(1/\delta)'} v \right)^{-\frac{(1/\delta)'}{1+(1/\delta)'}} V^{-(1/\delta)'}(x)v(x),$$

$$\Psi[V^{1/\delta}v^{1-(1/\delta)};1/\delta](x) \approx \left(\int_x^\infty V^{-(1/\delta)'} v \right)^{\frac{1}{1+(1/\delta)'}},$$

and (3.23) *hold.*

Proof. By Corollary 3.12, inequality (3.19) holds iff both (3.22) and (3.23) hold. It is easy to see that (3.22) is equivalent to

$$\left\|\left[T_{V^2}\left(\left\{\int_x^\infty h\right\}^{1/\delta}\right)\right]^\delta\right\|_{\beta/\delta,w,(0,\infty)} \le c^\delta \|h\|_{1/\delta,V^{1/\delta}v^{1-1/\delta},(0,\infty)},\ h\in\mathfrak{M}^+. \tag{3.28}$$

By Theorem 3.4, inequality (3.28) is equivalent to

$$\left\|\left[T_{V^2\Psi^{2/\delta}\left[V^{1/\delta}v^{1-1/\delta};1/\delta\right]}(f^{1/\delta})\right]^\delta\right\|_{\beta/\delta,w,(0,\infty)} \le c^\delta \|f\|_{1/\delta,\psi[V^{1/\delta}v^{1-1/\delta};1/\delta],(0,\infty)},\ f\in\mathfrak{M}^\uparrow,$$

which is evidently equivalent to (3.27).

It remains to note that

$$\psi[V^{1/\delta}v^{1-1/\delta};1/\delta](x)\approx\left(\int_x^\infty V^{-(1/\delta)'}v\right)^{-\frac{(1/\delta)'}{1+(1/\delta)'}}V^{-(1/\delta)'}(x)v(x),$$
$$\Psi[V^{1/\delta}v^{1-(1/\delta)};1/\delta](x)\approx\left(\int_x^\infty V^{-(1/\delta)'}v\right)^{\frac{1}{1+(1/\delta)'}}.$$

□

Corollary 3.15. *Let $0<\beta\le\infty$, and let $T:\mathfrak{M}^+\to\mathfrak{M}^+$ satisfies conditions* (i)-(iii). *Assume that u, $w\in\mathcal{W}(0,\infty)$ and $v\in\mathcal{W}(0,\infty)$ be such that $V(x)<\infty$ for all $x>0$. Then inequality* (3.19) *holds iff both*

$$\left\|T_{V^2\Psi^4\left[V^2v^{-1};2\right]}(f)\right\|_{\beta,w,(0,\infty)} \le c\|f\|_{1,\psi[V^2v^{-1};2],(0,\infty)},\ f\in\mathfrak{M}^\uparrow, \tag{3.29}$$

where

$$\psi[V^2v^{-1};2](x)\approx\left(\int_x^\infty V^{-2}v\right)^{-2/3}V^{-2}(x)v(x)$$
$$\Psi[V^2v^{-1};2](x)\approx\left(\int_x^\infty V^{-2}v\right)^{1/3},$$

and (3.23) *hold.*

Proof. The statement follows by Theorem 3.14 with $\delta=1/2$. □

The following statement immediately follows from Theorem 3.11.

Corollary 3.16. *Let $0<\beta\le\infty$, and let $T:\mathfrak{M}^+\to\mathfrak{M}^+$ satisfies conditions* (i)-(iii). *Assume that u, $w\in\mathcal{W}(0,\infty)$ and $v\in\mathcal{W}(0,\infty)$ be such that $V_*(x)<\infty$ for all $x>0$ and $V_*(0)=\infty$. Then inequality*

$$\left\|T\left(\int_0^x h\right)\right\|_{\beta,w,(0,\infty)} \le c\,\|h\|_{1,V_*,(0,\infty)},\ h\in\mathfrak{M}^+, \tag{3.30}$$

holds iff

$$\|T_{V_*^{-2}}f\|_{\beta,w,(0,\infty)} \le c\|f\|_{1,v/V_*^2,(0,\infty)},\ f\in\mathfrak{M}^\downarrow \tag{3.31}$$

holds.

Proof. Since

$$V_*(x)=\left(\int_0^x\frac{v}{V_*^2}\right)^{-1},\ x>0,$$

it remains to apply Theorem 3.11. □

Corollary 3.17. *Let $0 < \beta \le \infty$, $0 < \delta \le 1$, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ satisfies conditions* (i)-(iii). *Assume that u, $w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(0,\infty)$ be such that $V_*(x) < \infty$ for all $x > 0$ and $V_*(0) = \infty$. Then inequality* (3.30) *holds iff both*

$$\left\| T_{V_*^{-2}}\left(\left\{ \int_x^\infty h^\delta \right\}^{1/\delta} \right) \right\|_{\beta,w,(0,\infty)} \le c \|h\|_{1,V_*^{1/\delta-2}v^{1-1/\delta},(0,\infty)}, \ h \in \mathfrak{M}^+, \tag{3.32}$$

holds.

Corollary 3.18. *Let $0 < \beta \le \infty$, $0 < \delta \le 1$, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ satisfies conditions* (i)-(iii). *Assume that u, $w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(0,\infty)$ be such that $V_*(x) < \infty$ for all $x > 0$ and $V_*(0) = \infty$. Then inequality* (3.30) *holds iff*

$$\left\| T_{V_*^{2(1/\delta-1)}}\left(\left\{ \int_0^x h^\delta \right\}^{1/\delta} \right) \right\|_{\beta,w,(0,\infty)} \le c \|h\|_{1,V_*^{3/\delta-2}v^{1-1/\delta},(0,\infty)}, \ h \in \mathfrak{M}^+ \tag{3.33}$$

holds.

The following "dual" statements also hold true and may be proved analogously.

Theorem 3.19. *Let $0 < \beta \le \infty$, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ satisfies conditions* (i)-(iii). *Assume that u, $w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(0,\infty)$ be such that $V_*(x) < \infty$ for all $x > 0$. Then inequality*

$$\left\| T\left(\int_x^\infty h \right) \right\|_{\beta,w,(0,\infty)} \le c \, \|h\|_{1,V_*^{-1},(0,\infty)}, \ h \in \mathfrak{M}^+, \tag{3.34}$$

holds iff

$$\|T_{V_*^2} f\|_{\beta,w,(0,\infty)} \le c \|f\|_{1,v,(0,\infty)}, \ f \in \mathfrak{M}^{\uparrow}. \tag{3.35}$$

Corollary 3.20. *Let $0 < \beta \le \infty$, $0 < \delta \le 1$, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ satisfies conditions* (i)-(iii). *Assume that u, $w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(0,\infty)$ be such that $V_*(x) < \infty$ for all $x > 0$. Then inequality* (3.34) *holds iff both*

$$\left\| T_{V_*^2}\left(\left\{ \int_0^x h^\delta \right\}^{1/\delta} \right) \right\|_{\beta,w,(0,\infty)} \le c \|h\|_{1,V_*^{1/\delta}v^{1-1/\delta},(0,\infty)}, \ h \in \mathfrak{M}^+, \tag{3.36}$$

and

$$\|T_{V_*^2}(\mathbf{1})\|_{\beta,w,(0,\infty)} \le c \|\mathbf{1}\|_{1,v,(0,\infty)}, \tag{3.37}$$

hold.

Corollary 3.21. *Let $0 < \beta \le \infty$, $0 < \delta \le 1$, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ satisfies conditions* (i)-(iii). *Assume that u, $w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(0,\infty)$ be such that $V_*(x) < \infty$ for all $x > 0$. Then inequality* (3.34) *holds iff*

$$\left\| T_{V_*^{2(1-1/\delta)}}\left(\left\{ \int_x^\infty h^\delta V_* \right\}^{1/\delta} \right) \right\|_{\beta,w,(0,\infty)} \le c \|h\|_{1,v^{1-1/\delta},(0,\infty)}, \ h \in \mathfrak{M}^+ \tag{3.38}$$

holds.

Theorem 3.22. *Let $0 < \beta \le \infty$, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ satisfies conditions* (i)-(iii). *Assume that u, $w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(0,\infty)$ be such that $V_*(x) < \infty$ for all $x > 0$. Then inequality* (3.34) *holds iff both*

$$\left\| T_{V^{2\cdot\Phi^{2/\delta}[V_*^{1/\delta}v^{1-1/\delta};1/\delta]}} f \right\|_{\beta,w,(0,\infty)} \le c \|f\|_{1,\phi[V_*^{1/\delta}v^{1-1/\delta};1/\delta],(0,\infty)}, \ f \in \mathfrak{M}^{\downarrow}, \tag{3.39}$$

where $0 < \delta < 1$,

$$\phi[V_*^{1/\delta}v^{1-1/\delta};1/\delta](x) \approx \Big(\int_0^x V_*^{-(1/\delta)'}v\Big)^{-\frac{(1/\delta)'}{1+(1/\delta)'}} V_*^{-(1/\delta)'}(x)v(x),$$
$$\Phi[V_*^{1/\delta}v^{1-(1/\delta)};1/\delta](x) \approx \Big(\int_0^x V_*^{-(1/\delta)'}v\Big)^{\frac{1}{1+(1/\delta)'}},$$

and (3.37) *hold.*

Corollary 3.23. *Let* $0 < \beta \le \infty$, *and let* $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ *satisfies conditions* (i)-(iii). *Assume that* u, $w \in \mathcal{W}(0,\infty)$ *and* $v \in \mathcal{W}(0,\infty)$ *be such that* $V_*(x) < \infty$ *for all* $x > 0$. *Then inequality* (3.34) *holds iff both*

$$\Big\|T_{V_*^2\Phi^4[V_*^2v^{-1};2]}(f)\Big\|_{\beta,w,(0,\infty)} \le c\|f\|_{1,\phi[V_*^2v^{-1};2],(0,\infty)},\ f \in \mathfrak{M}^{\downarrow},$$

where

$$\phi[V_*^2v^{-1};2](x) \approx \Big(\int_0^x V_*^{-2}v\Big)^{-2/3} V_*^{-2}(x)v(x)$$
$$\Phi[V_*^2v^{-1};2](x) \approx \Big(\int_0^x V_*^{-2}v\Big)^{1/3},$$

and (3.37) *hold.*

Corollary 3.24. *Let* $0 < \beta < \infty$, *and let* $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ *satisfies conditions* (i)-(iii). *Assume that* u, $w \in \mathcal{W}(0,\infty)$ *and* $v \in \mathcal{W}(0,\infty)$ *be such that* $V(x) < \infty$ *for all* $x > 0$ *and* $V(\infty) = \infty$. *Then inequality*

$$\Big\|T\Big(\int_x^\infty h\Big)\Big\|_{\beta,w,(0,\infty)} \le c\,\|h\|_{1,V,(0,\infty)},\ h \in \mathfrak{M}^+, \tag{3.40}$$

holds iff

$$\|T_{V^{-2}}f\|_{\beta,w,(0,\infty)} \le c\|f\|_{1,v/V^2,(0,\infty)},\ f \in \mathfrak{M}^{\downarrow}. \tag{3.41}$$

Corollary 3.25. *Let* $0 < \beta \le \infty$, $0 < \delta \le 1$, *and let* $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ *satisfies conditions* (i)-(iii). *Assume that* u, $w \in \mathcal{W}(0,\infty)$ *and* $v \in \mathcal{W}(0,\infty)$ *be such that* $V(x) < \infty$ *for all* $x > 0$ *and* $V(\infty) = \infty$. *Then inequality* (3.40) *holds iff both*

$$\Big\|T_{V^{-2}}\Big(\Big\{\int_0^x h^\delta\Big\}^{1/\delta}\Big)\Big\|_{\beta,w,(0,\infty)} \le c\|h\|_{1,V^{1/\delta-2}v^{1-1/\delta},(0,\infty)},\ h \in \mathfrak{M}^+, \tag{3.42}$$

holds.

Corollary 3.26. *Let* $0 < \beta \le \infty$, $0 < \delta \le 1$, *and let* $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ *satisfies conditions* (i)-(iii). *Assume that* u, $w \in \mathcal{W}(0,\infty)$ *and* $v \in \mathcal{W}(0,\infty)$ *be such that* $V(x) < \infty$ *for all* $x > 0$ *and* $V(\infty) = \infty$. *Then inequality* (3.40) *holds iff*

$$\Big\|T_{V^{2(1/\delta-1)}}\Big(\Big\{\int_x^\infty h^\delta\Big\}^{1/\delta}\Big)\Big\|_{\beta,w,(0,\infty)} \le c\|h\|_{1,V^{3/\delta-2}v^{1-1/\delta},(0,\infty)},\ h \in \mathfrak{M}^+ \tag{3.43}$$

holds.

4. Equivalence theorems for the weighted inequalities on the cones of monotone functions

As it is mentioned in the introduction, by substitution of variables it is possible to change the cone of non-decreasing functions to the cone of non-increasing functions and vice versa, when considering inequalities (2.1) and (2.5) for integral operators T. But this procedure changes T also as usually to the "dual" operator.

The following theorems allows to change the cones to each other not changing the operator T.

Theorem 4.1. *Let $0 < \beta \le \infty$, $0 < s < \infty$, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ satisfies conditions* (i)-(iii). *Assume that u, $w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(0,\infty)$ be such that $V(x) < \infty$ for all $x > 0$ holds. Then inequality* (2.1) *holds if and only if both*

$$\left\|T_{\{\Psi[V^{s/\delta}v^{1-s/\delta};s/\delta]\}^{2/\delta}}(f)\right\|_{\beta,w,(0,\infty)} \le c\|f\|_{s,\psi[V^{s/\delta}v^{1-s/\delta};s/\delta],(0,\infty)},\ f \in \mathfrak{M}^{\uparrow}, \tag{4.1}$$

where $0 < \delta < s$ and

$$\psi[V^{s/\delta}v^{1-s/\delta};s/\delta](x) \approx \left(\int_x^\infty V^{-(s/\delta)'}v\right)^{-\frac{(s/\delta)'}{(s/\delta)'+1}} V^{-(s/\delta)'}(x)v(x),\ (x>0),$$

$$\Psi[V^{s/\delta}v^{1-s/\delta};s/\delta](x) \approx \left(\int_x^\infty V^{-(s/\delta)'}v\right)^{\frac{1}{(s/\delta)'+1}},\ (x>0)$$

and (2.3) *hold.*

Proof. Inequality (2.1) is equivalent to

$$\left\|\left\{T(f^{1/\delta})\right\}^{\delta}\right\|_{\beta/\delta,w,(0,\infty)} \le c^{\delta}\|f\|_{s/\delta,v,(0,\infty)},\ f \in \mathfrak{M}^{\downarrow}. \tag{4.2}$$

By Theorems 2.2, (4.2) holds if and only if

$$\left\|\left\{T\left(\int_x^\infty h\right)^{1/\delta}\right\}^{\delta}\right\|_{\beta/\delta,w,(0,\infty)} \le c^{\delta}\|h\|_{s/\delta,V^{s/\delta}v^{1-s/\delta},(0,\infty)},\ h \in \mathfrak{M}^{+}, \tag{4.3}$$

and (2.3) hold. By Theorem 3.4, (4.3) is equivalent to

$$\left\|\left\{T_{\{\Psi[V^{s/\delta}v^{1-s/\delta};s/\delta]\}^2}(f^{1/\delta})\right\}^{\delta}\right\|_{\beta/\delta,w,(0,\infty)} \le c^{\delta}\|f\|_{s/\delta,\psi[V^{s/\delta}v^{1-s/\delta};s/\delta],(0,\infty)},\ f \in \mathfrak{M}^{\uparrow}, \tag{4.4}$$

with

$$\psi[V^{s/\delta}v^{1-s/\delta};s/\delta] \approx (V^{1-(s/\delta)'} - V^{1-(s/\delta)'}(\infty))^{-(s/\delta)'/((s/\delta)'+1)}V^{-(s/\delta)'}v$$

$$\Psi[V^{s/\delta}v^{1-s/\delta};s/\delta] \approx (V^{1-(s/\delta)'} - V^{1-(s/\delta)'}(\infty))^{1/((s/\delta)'+1)}.$$

Note that (4.4) is equivalent to (4.1), and this completes the proof. □

To state the next statements we need the following notations:

$$V_1(x) := \left(\int_x^\infty V^{-2}v\right)^{1/3}, \qquad (x>0).$$

The following statement holds true.

Corollary 4.2. *Let $0 < \beta \le \infty$, $0 < s < \infty$, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ satisfies conditions* (i)-(iii). *Assume that u, $w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(0,\infty)$ be such that $V(x) < \infty$ for all $x > 0$ holds. Then inequality* (2.1) *holds if and only if both*

$$\left\|T_{\{\Psi[V^2v^{-1};2]\}^{4/s}}(f)\right\|_{\beta,w,(0,\infty)} \le c\|f\|_{s,\psi[V^2v^{-1};2],(0,\infty)},\ f \in \mathfrak{M}^{\uparrow}, \tag{4.5}$$

where

$$\psi[V_*^2 v^{-1};2](x) \approx \{V_1 \cdot V\}^{-2}(x)v(x),\ (x>0),$$
$$\Psi[V_*^2 v^{-1};2](x) \approx V_1(x),\ (x>0),$$

and (2.3) *hold.*

Proof. The statement follows by Theorem 4.1 with $\delta = s/2$. □

The following "dual" statement also holds true and can be proved analogously.

Theorem 4.3. *Let* $0<\beta\le\infty$, $0<s<\infty$, *and let* $T:\mathfrak{M}^+\to\mathfrak{M}^+$ *satisfies conditions* (i)-(iii). *Assume that* u, $w\in\mathcal{W}(0,\infty)$ *and* $v\in\mathcal{W}(0,\infty)$ *be such that* $V_*(x)<\infty$ *for all* $x>0$ *holds. Then inequality* (2.5) *holds if and only if both*

$$\left\|T_{\{\Phi[V_*^{s/\delta}v^{1-s/\delta};s/\delta]\}^{2/\delta}}(f)\right\|_{\beta,w,(0,\infty)} \le c\|f\|_{s,\phi[V_*^{s/\delta}v^{1-s/\delta};s/\delta],(0,\infty)},\ f\in\mathfrak{M}^{\downarrow}, \tag{4.6}$$

where $0<\delta<s$ *and*

$$\phi[V_*^{s/\delta}v^{1-s/\delta};s/\delta](x) \approx \left(\int_0^x V_*^{-(s/\delta)'}v\right)^{-\frac{(s/\delta)'}{(s/\delta)'+1}} V_*^{-(s/\delta)'}(x)v(x),\ (x>0),$$
$$\Phi[V_*^{s/\delta}v^{1-s/\delta};s/\delta](x) \approx \left(\int_0^x V_*^{-(s/\delta)'}v\right)^{\frac{1}{(s/\delta)'+1}},\ (x>0),$$

and (2.3) *hold.*

To state the next statement we need the following notations:

$$V_1^*(x) := \left(\int_0^x V_*^{-2}v\right)^{1/3}, \qquad (x>0).$$

Corollary 4.4. *Let* $0<\beta\le\infty$, $0<s<\infty$, *and let* $T:\mathfrak{M}^+\to\mathfrak{M}^+$ *satisfies conditions* (i)-(iii). *Assume that* u, $w\in\mathcal{W}(0,\infty)$ *and* $v\in\mathcal{W}(0,\infty)$ *be such that* $V_*(x)<\infty$ *for all* $x>0$ *holds. Then inequality* (2.5) *holds if and only if both*

$$\left\|T_{\{\Phi[V_*^2v^{-1};2]\}^{4/p}}(f)\right\|_{\beta,w,(0,\infty)} \le c\|f\|_{s,\phi[V_*^2v^{-1};2],(0,\infty)},\ f\in\mathfrak{M}^{\downarrow}, \tag{4.7}$$

where

$$\phi[V_*^2v^{-1};2](x) \approx \{V_1^*\cdot V_*\}^{-2}(x)v(x),\ (x>0),$$
$$\Phi[V_*^2v^{-1};2](x) \approx V_1^*(x),\ (x>0),$$

and (2.3) *hold.*

5. The weighted Hardy-type inequalities on the cones of monotone functions

In this section we consider weighted Hardy inequalities on the cones of monotone functions.

Note that inequality

$$\|H_u(f)\|_{q,w,(0,\infty)} \le c\|f\|_{p,v,(0,\infty)},\ f\in\mathfrak{M}^{\downarrow} \tag{5.1}$$

was considered by many authors and there exist several characterizations of this inequality (see, survey paper [11], [4], [15], [10], and [27]).

Using change of variables $x=1/t$, we can easily obtain full characterization of the weighted inequality

$$\|H_u^*(f)\|_{q,w,(0,\infty)} \le c\|f\|_{p,v,(0.\infty)},\ f\in\mathfrak{M}^{\uparrow}. \tag{5.2}$$

Our aim in this section is to give the characterization of the inequalities

$$\|H_u(f)\|_{q,w,(0,\infty)} \le c\|f\|_{p,v,(0,\infty)},\ f \in \mathfrak{M}^{\uparrow} \tag{5.3}$$

and

$$\|H_u^*(f)\|_{q,w,(0,\infty)} \le c\|f\|_{p,v,(0,\infty)},\ f \in \mathfrak{M}^{\downarrow}. \tag{5.4}$$

Inequality (5.3) was considered in [31] in the case when $1 < p, q < \infty$, and recently, completely characterized in [29, 30] and [27] in the case $0 < p, q < \infty$. It is worth to mention that in the most difficult case when $0 < q < p \le 1$, the characterization obtained in [27, Theorem 3.12] involves additional function $\varphi(x) := W^{-1}(4W(x))$, where $W^{-1}(t) := \inf\{s \ge 0 :\ W(s) = t\}$ is the generalized inverse function of W. Theorem 5.3 give us a another characterization of (5.3) and its proof does not use the discretization technique.

Recall the following complete characterization of the weighted Hardy inequality on the cone of non-increasing functions.

Theorem 5.1 ([27], Theorems 2.5, 3.15, 3.16)**.** *Let $0 < q, p \le \infty$. Then inequality (5.1) with the best constant c holds if and only if:*

(i) *$1 < p \le q < \infty$, and in this case $c \approx A_0 + A_1$, where*

$$A_0 := \sup_{t>0}\Big(\int_0^t U^q(\tau)w(\tau)\,d\tau\Big)^{\frac{1}{q}} V^{-\frac{1}{p}}(t),$$

$$A_1 := \sup_{t>0} W_*^{\frac{1}{q}}(t)\Big(\int_0^t \Big(\frac{U(\tau)}{V(\tau)}\Big)^{p'} v(\tau)\,d\tau\Big)^{\frac{1}{p'}};$$

(ii) *$q < p < \infty$ and $1 < p < \infty$, and in this case $c \approx B_0 + B_1$, where*

$$B_0 := \Big(\int_0^\infty V^{-\frac{r}{p}}(t)\Big(\int_0^t U^q(\tau)w(\tau)\,d\tau\Big)^{\frac{r}{p}} U^q(t)w(t)\,dt\Big)^{\frac{1}{r}},$$

$$B_1 := \Big(\int_0^\infty W_*^{\frac{r}{p}}(t)\Big(\int_0^t \Big(\frac{U(\tau)}{V(\tau)}\Big)^{p'} v(\tau)\,d\tau\Big)^{\frac{r}{p'}} w(t)\,dt\Big)^{\frac{1}{r}};$$

(iii) *$q < p \le 1$, and in this case $c \approx B_0 + C_1$, where*

$$C_1 := \Big(\int_0^\infty \Big(\operatorname*{ess\,sup}_{\tau\in(0,t)} \frac{U^p(\tau)}{V(\tau)}\Big)^{\frac{r}{p}} W_*^{\frac{r}{p}}(t)w(t)\,dt\Big)^{\frac{1}{r}};$$

(iv) *$p \le q < \infty$ and $p \le 1$, and in this case $c = D_0$, where*

$$D_0 := \sup_{t>0} V^{-\frac{1}{p}}(t)\Big(\int_0^\infty U^q(\min\{\tau,t\})w(\tau)\,d\tau\Big)^{\frac{1}{q}};$$

(v) *$p \le 1$ and $q = \infty$, and in this case $c = E_0$, where*

$$E_0 := \operatorname*{ess\,sup}_{t>0} V^{-\frac{1}{p}}(t)\Big(\operatorname*{ess\,sup}_{\tau>0}\ U(\min\{\tau,t\})w(\tau)\Big);$$

(vi) *$1 < p < \infty$ and $q = \infty$, and in this case $c = F_0$, where*

$$F_0 := \operatorname*{ess\,sup}_{t>0} w(t)\Big(\int_0^t \Big(\int_\tau^t u(y)V^{-1}(y)\,dy\Big)^{p'} v(\tau)\,d\tau\Big)^{\frac{1}{p'}};$$

(vii) *$p = \infty$ and $0 < q < \infty$, and in this case $c = G_0$, where*

$$G_0 := \Big(\int_0^\infty \Big(\int_0^t \frac{u(y)\,dy}{\operatorname{ess\,sup}_{\tau\in(0,y)} v(\tau)}\Big)^q w(t)\,dt\Big)^{\frac{1}{q}};$$

(viii) $p = q = \infty$, *and in this case* $c = H_0$, *where*

$$H_0 := \operatorname*{ess\,sup}_{t>0}\Big(\int_0^t \frac{u(y)\,dy}{\operatorname{ess\,sup}_{\tau\in(0,y)} v(\tau)}\Big)w(t).$$

The following theorem holds true.

Theorem 5.2. *Let* $0 < q,\ p \le \infty$. *Then inequality* (5.2) *with the best constant* c *holds if and only if:*

(i) $1 < p \le q < \infty$, *and in this case* $c \approx A_0^* + A_1^*$, *where*

$$A_0^* := \sup_{t>0}\Big(\int_t^\infty U_*^q(\tau)w(\tau)\,d\tau\Big)^{\frac{1}{q}} V_*^{-\frac{1}{p}}(t),$$

$$A_1^* := \sup_{t>0} W^{\frac{1}{q}}(t)\Big(\int_t^\infty \Big(\frac{U_*(\tau)}{V_*(\tau)}\Big)^{p'} v(\tau)\,d\tau\Big)^{\frac{1}{p'}};$$

(ii) $q < p < \infty$ *and* $1 < p < \infty$, *and in this case* $c \approx B_0^* + B_1^*$, *where*

$$B_0^* := \Big(\int_0^\infty V_*^{-\frac{r}{p}}(t)\Big(\int_t^\infty U_*^q(\tau)w(\tau)\,d\tau\Big)^{\frac{r}{p}} U_*^q(t)w(t)\,dt\Big)^{\frac{1}{r}},$$

$$B_1^* := \Big(\int_0^\infty W^{\frac{r}{p}}(t)\Big(\int_t^\infty \Big(\frac{U_*(\tau)}{V_*(\tau)}\Big)^{p'} v(\tau)\,d\tau\Big)^{\frac{r}{p'}} w(t)\,dt\Big)^{\frac{1}{r}};$$

(iii) $q < p \le 1$, *and in this case* $c \approx B_0^* + C_1^*$, *where*

$$C_1^* := \Big(\int_0^\infty \Big(\operatorname*{ess\,sup}_{y\in(t,\infty)} \frac{U_*^p(y)}{V_*(y)}\Big)^{\frac{r}{p}} W^{\frac{r}{p}}(t)w(t)\,dt\Big)^{\frac{1}{r}};$$

(iv) $p \le q < \infty$ *and* $p \le 1$, *and in this case* $c = D_0^*$, *where*

$$D_0^* := \sup_{t>0} V_*^{-\frac{1}{p}}(t)\Big(\int_0^\infty U_*^q(\max\{\tau,t\})w(\tau)\,d\tau\Big)^{\frac{1}{q}}.$$

(v) $p \le 1$ *and* $q = \infty$, *and in this case* $c = E_0$, *where*

$$E_0^* := \operatorname*{ess\,sup}_{t>0} V_*^{-\frac{1}{p}}(t)\Big(\operatorname*{ess\,sup}_{\tau>0}\ U_*(\max\{\tau,t\})w(\tau)\Big);$$

(vi) $1 < p < \infty$ *and* $q = \infty$, *and in this case* $c = F_0^*$, *where*

$$F_0^* := \operatorname*{ess\,sup}_{t>0} w(t)\Big(\int_t^\infty \Big(\int_t^\tau u(y)V_*^{-1}(y)\,dy\Big)^{p'} v(\tau)\,d\tau\Big)^{\frac{1}{p'}};$$

(vii) $p = \infty$ *and* $0 < q < \infty$, *and in this case* $c = G_0^*$, *where*

$$G_0^* := \Big(\int_0^\infty \Big(\int_t^\infty \frac{u(y)\,dy}{\operatorname{ess\,sup}_{\tau\in(y,\infty)} v(\tau)}\Big)^q w(t)\,dt\Big)^{\frac{1}{q}};$$

(viii) $p = q = \infty$, *and in this case* $c = H_0^*$, *where*

$$H_0^* := \operatorname*{ess\,sup}_{t>0}\Big(\int_t^\infty \frac{u(y)\,dy}{\operatorname{ess\,sup}_{\tau\in(y,\infty)} v(\tau)}\Big)w(t).$$

Proof. By change of variables $x = 1/t$, it is easy to see that inequality (5.2) holds if and only if

$$\big\|H_{p,\tilde{u}}(f)\big\|_{q,\tilde{w},(0,\infty)} \le c\,\|f\|_{p,\tilde{v},(0,\infty)}, \quad f \in \mathfrak{M}^{\downarrow}$$

holds, where

$$\tilde{u}(t) = u\Big(\frac{1}{t}\Big)\frac{1}{t^2},\ \tilde{w}(t) = w\Big(\frac{1}{t}\Big)\frac{1}{t^2},\ \tilde{v}(t) = v\Big(\frac{1}{t}\Big)\Big(\frac{1}{t^2}\Big),\ t > 0,$$

when $0 < p < \infty$, $0 < q < \infty$, and

$$\tilde{u}(t) = u\Big(\frac{1}{t}\Big)\frac{1}{t^2},\ \tilde{w}(t) = w\Big(\frac{1}{t}\Big),\ \tilde{v}(t) = v\Big(\frac{1}{t}\Big)\Big(\frac{1}{t^2}\Big),\ t > 0,$$

when $0 < p < \infty$, $q = \infty$, and

$$\tilde{u}(t) = u\Big(\frac{1}{t}\Big)\frac{1}{t^2},\ \tilde{w}(t) = w\Big(\frac{1}{t}\Big)\Big(\frac{1}{t^2}\Big),\ \tilde{v}(t) = v\Big(\frac{1}{t}\Big),\ t > 0,$$

when $p = q = \infty$, and

$$\tilde{u}(t) = u\Big(\frac{1}{t}\Big)\frac{1}{t^2},\ \tilde{w}(t) = w\Big(\frac{1}{t}\Big),\ \tilde{v}(t) = v\Big(\frac{1}{t}\Big),\ t > 0.$$

Using Theorem 5.1, and then applying substitution of variables mentioned above three times, we get the statement. □

The following theorem is true.

Theorem 5.3. *Let $0 < q \le \infty$ and $0 < p < \infty$. Assume that u, $w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(0,\infty)$ be such that $V_*(x) < \infty$ for all $x > 0$ holds. Recall that*

$$V_1^*(x) := \Big(\int_0^x V_*^{-2}v\Big)^{1/3}, \qquad (x > 0).$$

Denote by

$$U_1^*(x) := \int_0^x u(t)[V_1^*]^{\frac{4}{p}}(t)\,dt, \qquad (x > 0).$$

Then inequality (5.3) with the best constant c holds if and only if:

(i) $1 < p \le q < \infty$, *and in this case*

$$c \approx \tilde{A}_0 + \tilde{A}_1 + \|H_u(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0,\infty)},$$

where

$$\tilde{A}_0 := \sup_{t>0}\Big(\int_0^t [U_1^*]^q(\tau)w(\tau)\,d\tau\Big)^{\frac{1}{q}}[V_1^*]^{-\frac{1}{p}}(t),$$

$$\tilde{A}_1 := \sup_{t>0} W_*^{\frac{1}{q}}(t)\Big(\int_0^t [U_1^*]^{p'}(\tau)[V_1^*]^{-(2+p')}(\tau)V_*^{-2}(\tau)v(\tau)\,d\tau\Big)^{\frac{1}{p'}};$$

(ii) $q < p < \infty$ *and* $1 < p < \infty$, *and in this case*

$$c \approx \tilde{B}_0 + \tilde{B}_1 + \|H_u(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0,\infty)},$$

where

$$\tilde{B}_0 := \Big(\int_0^\infty [V_1^*]^{-\frac{r}{p}}(t)\Big(\int_0^t [U_1^*]^q(\tau)w(\tau)\,d\tau\Big)^{\frac{r}{p}}[U_1^*]^q(t)w(t)\,dt\Big)^{\frac{1}{r}},$$

$$\tilde{B}_1 := \Big(\int_0^\infty W_*^{\frac{r}{p}}(t)\Big(\int_0^t [U_1^*]^{p'}(\tau)[V_1^*]^{-(2+p')}(\tau)V_*^{-2}(\tau)v(\tau)\,d\tau\Big)^{\frac{r}{p'}}w(t)\,dt\Big)^{\frac{1}{r}};$$

(iii) $q < p \le 1$, *and in this case*

$$c \approx \tilde{B}_0 + \tilde{C}_1 + \|H_u(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0,\infty)},$$

where

$$\tilde{C}_1 := \Big(\int_0^\infty \Big(\operatorname*{ess\,sup}_{\tau\in[0,t]}\frac{[U_1^*]^p(\tau)}{V_1^*(\tau)}\Big)^{\frac{r}{p}}W_*^{\frac{r}{p}}(t)w(t)\,dt\Big)^{\frac{1}{r}};$$

(iv) $p \le q < \infty$ *and* $0 < p \le 1$, *and in this case*

$$c = \tilde{D}_0 + \|H_u(\mathbf{1})\|_{q,w,(0,\infty)} / \|\mathbf{1}\|_{p,v,(0,\infty)},$$

where

$$\tilde{D}_0 := \sup_{t>0} [V_1^*]^{-\frac{1}{p}}(t) \Big(\int_0^\infty [U_1^*]^q (\min\{\tau, t\}) w(\tau)\, d\tau \Big)^{\frac{1}{q}};$$

(v) $p \le 1$ *and* $q = \infty$, *and in this case*

$$c = \tilde{E}_0 + \|H_u(\mathbf{1})\|_{q,w,(0,\infty)} / \|\mathbf{1}\|_{p,v,(0,\infty)},$$

where

$$\tilde{E}_0 := \operatorname*{ess\,sup}_{t>0} [V_1^*]^{-\frac{1}{p}}(t) \Big(\operatorname*{ess\,sup}_{\tau>0} [U_1^*](\min\{\tau, t\}) w(\tau) \Big);$$

(vi) $1 < p < \infty$ *and* $q = \infty$, *and in this case*

$$c = \widetilde{F}_0 + \|H_u(\mathbf{1})\|_{q,w,(0,\infty)} / \|\mathbf{1}\|_{p,v,(0,\infty)},$$

where

$$\tilde{F}_0 := \operatorname*{ess\,sup}_{t>0} w(t) \Big(\int_0^t \Big(\int_\tau^t u(y) [V_1^*]^{\frac{4-p}{p}}(y)\, dy \Big)^{p'} [V_1^*]^{-2}(\tau) V_*^{-2}(\tau) v(\tau)\, d\tau \Big)^{\frac{1}{p'}}.$$

Proof. By Corollary 4.4 applied with $\beta = q$, $s = p$ and $T = H_u$, inequality (5.3) holds if and only if both

$$\big\| H_{u[V_1^*]^{4/p}}(f) \big\|_{q,w,(0,\infty)} \le c \|f\|_{p,\{V_1^* \cdot V_*\}^{-2} v,(0,\infty)},\ f \in \mathfrak{M}^{\downarrow}, \tag{5.5}$$

and

$$\|H_u(\mathbf{1})\|_{q,w,(0,\infty)} \le c \|\mathbf{1}\|_{p,v,(0,\infty)} \tag{5.6}$$

hold.

Now the statement follows by applying Theorem 5.1. □

Theorem 5.4. *Let* $0 < q \le \infty$ *and* $0 < p < \infty$. *Recall that*

$$V_1(x) := \Big(\int_x^\infty V^{-2} v \Big)^{\frac{1}{3}}, \qquad (x > 0).$$

Denote by

$$U_1(x) := \int_x^\infty u(t) V_1^{\frac{4}{p}}(t)\, dt, \qquad (x > 0).$$

Then inequality (5.4) *with the best constant c holds if and only if:*

(i) $1 < p \le q < \infty$, *and in this case*

$$c \approx \tilde{A}_0^* + \tilde{A}_1^* + \|H_u^*(\mathbf{1})\|_{q,w,(0,\infty)} / \|\mathbf{1}\|_{p,v,(0,\infty)},$$

where

$$\tilde{A}_0^* := \sup_{t>0} \Big(\int_t^\infty U_1^q(\tau) w(\tau)\, d\tau \Big)^{\frac{1}{q}} V_1^{-\frac{1}{p}}(t),$$

$$\tilde{A}_1^* := \sup_{t>0} W^{\frac{1}{q}}(t) \Big(\int_t^\infty U_1^{p'}(\tau) V_1^{-(2+p')}(\tau) V^{-2}(\tau) v(\tau)\, d\tau \Big)^{\frac{1}{p'}};$$

(ii) $q < p < \infty$ *and* $1 < p < \infty$, *and in this case*

$$c \approx \tilde{B}_0^* + \tilde{B}_1^* + \|H_u^*(\mathbf{1})\|_{q,w,(0,\infty)} / \|\mathbf{1}\|_{p,v,(0,\infty)},$$

where

$$\tilde{B}_0^* := \Big(\int_0^\infty V_1^{-\frac{r}{p}}(t)\Big(\int_t^\infty U_1^q(\tau)w(\tau)\,d\tau\Big)^{\frac{r}{p}} U_1^q(t)w(t)\,dt\Big)^{\frac{1}{r}},$$

$$\tilde{B}_1^* := \Big(\int_0^\infty W^{\frac{r}{p}}(t)\Big(\int_t^\infty U_1^{p'}(\tau)V_1^{-(2+p')}(\tau)V^{-2}(\tau)v(\tau)\,d\tau\Big)^{\frac{r}{p'}} w(t)\,dt\Big)^{\frac{1}{r}};$$

(iii) $q < p \le 1$, *and in this case*

$$c \approx \tilde{B}_0^* + \tilde{C}_1^* + \|H_u^*(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0,\infty)},$$

where

$$\tilde{C}_1^* := \Big(\int_0^\infty \Big(\operatorname*{ess\,sup}_{\tau\in(t,\infty)} \frac{U_1^p(\tau)}{V_1(\tau)}\Big)^{\frac{r}{p}} W^{\frac{r}{p}}(t)w(t)\,dt\Big)^{\frac{1}{r}};$$

(iv) $p \le q < \infty$ *and* $p \le 1$, *and in this case*

$$c = \tilde{D}_0^* + \|H_u^*(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0,\infty)},$$

where

$$\tilde{D}_0^* := \sup_{t>0} V_1^{-\frac{1}{p}}(t)\Big(\int_0^\infty U_1^q(\max\{s,t\})w(s)\,ds\Big)^{\frac{1}{q}}.$$

(v) $p \le 1$ *and* $q = \infty$, *and in this case*

$$c = \tilde{E}_0 + \|H_u^*(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0,\infty)},$$

where

$$\tilde{E}_0^* := \operatorname*{ess\,sup}_{t>0} V_1^{-\frac{1}{p}}(t)\Big(\operatorname*{ess\,sup}_{\tau>0}\ U_1(\max\{\tau,t\})w(\tau)\Big);$$

(vi) $1 < p < \infty$ *and* $q = \infty$, *and in this case*

$$c = \tilde{F}_0^* + \|H_u^*(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0,\infty)},$$

where

$$\tilde{F}_0^* := \operatorname*{ess\,sup}_{t>0} w(t)\Big(\int_t^\infty \Big(\int_t^\tau u(y)V_1^{-1}(y)\,dy\Big)^{p'} V_1^{-2}(\tau)V^{-2}(\tau)v(\tau)\,d\tau\Big)^{\frac{1}{p'}}.$$

Proof. By change of variables $x = 1/t$, it is easy to see that inequality (5.4) holds if and only if

$$\big\|H_{p,\tilde{u}}(f)\big\|_{q,\tilde{w},(0,\infty)} \le c\,\|f\|_{p,\tilde{v},(0,\infty)}, \quad f \in \mathfrak{M}^{\uparrow}$$

holds, where

$$\tilde{u}(t) = u\Big(\frac{1}{t}\Big)\frac{1}{t^2},\ \tilde{w}(t) = w\Big(\frac{1}{t}\Big)\frac{1}{t^2},\ \tilde{v}(t) = v\Big(\frac{1}{t}\Big)\Big(\frac{1}{t^2}\Big),\ t > 0,$$

when $0 < p < \infty$, $0 < q < \infty$, and

$$\tilde{u}(t) = u\Big(\frac{1}{t}\Big)\frac{1}{t^2},\ \tilde{w}(t) = w\Big(\frac{1}{t}\Big),\ \tilde{v}(t) = v\Big(\frac{1}{t}\Big)\Big(\frac{1}{t^2}\Big),\ t > 0,$$

when $0 < p < \infty$, $q = \infty$, and

$$\tilde{u}(t) = u\Big(\frac{1}{t}\Big)\frac{1}{t^2},\ \tilde{w}(t) = w\Big(\frac{1}{t}\Big)\Big(\frac{1}{t^2}\Big),\ \tilde{v}(t) = v\Big(\frac{1}{t}\Big),\ t > 0,$$

when $p = q = \infty$, and

$$\tilde{u}(t) = u\Big(\frac{1}{t}\Big)\frac{1}{t^2},\ \tilde{w}(t) = w\Big(\frac{1}{t}\Big),\ \tilde{v}(t) = v\Big(\frac{1}{t}\Big),\ t > 0.$$

Using Theorem 5.3, and then applying substitution of variables mentioned above three times, we get the statement. □

6. The weighted norm inequalities for iterated Hardy-type operators

In this section we give complete characterization of inequalities (1.5) - (1.6) and (1.7) - (1.8).

Using results obtained in the previous section we can reduce the characterization of inequality (1.5) to the weighted Hardy inequality on the cones of non-increasing functions.

The following theorem is true.

Theorem 6.1. *Let $0 < p < \infty$, $0 < q \le \infty$ and $1 < s < \infty$. Assume that u, $w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(0,\infty)$ be such that (3.1) holds. Recall that*

$$\Phi[v;s](x) = \Big(\int_0^x v^{1-s'}(t)\,dt\Big)^{\frac{1}{s'+1}},\ x > 0.$$

Denote by

$$\Phi_1(\tau) := \int_0^\tau u(x)\Phi[v;s]^{2p}(x)\,dx = \int_0^\tau u(x)\Big(\int_0^x v^{1-s'}(t)\,dt\Big)^{\frac{2p}{s'+1}}dx,\ \tau > 0.$$

Then inequality (1.5) with the best constant c_1 holds if and only if:

(i) *$p < s \le q < \infty$, and in this case $c_1 \approx A_{1,1} + A_{1,2}$, where*

$$A_{1,1} := \sup_{t>0}\Big(\int_0^t [\Phi_1]^{\frac{q}{p}}(\tau)w(\tau)\,d\tau\Big)^{\frac{1}{q}}\Phi[v;s]^{-\frac{1}{s}}(t),$$

$$A_{1,2} := \sup_{t>0} W_*^{\frac{1}{q}}(t)\Big(\int_0^t \Big(\frac{\Phi_1(\tau)}{\Phi[v;s](\tau)}\Big)^{\frac{s}{s-p}}\phi[v;s](\tau)\,d\tau\Big)^{\frac{s-p}{ps}};$$

(ii) *$q < s < \infty$ and $p < s$, and in this case $c_1 \approx B_{1,1} + B_{1,2}$, where*

$$B_{1,1} := \Big(\int_0^\infty \Phi[v;s]^{\frac{q}{q-s}}(t)\Big(\int_0^t [\Phi_1]^{\frac{q}{p}}(\tau)w(\tau)\,d\tau\Big)^{\frac{q}{s-q}}[\Phi_1]^{\frac{q}{p}}(t)w(t)\,dt\Big)^{\frac{s-q}{qs}},$$

$$B_{1,2} := \Big(\int_0^\infty W_*^{\frac{q}{s-q}}(t)\Big(\int_0^t \Big(\frac{\Phi_1(\tau)}{\Phi[v;s](\tau)}\Big)^{\frac{s}{s-p}}\phi[v;s](\tau)\,d\tau\Big)^{\frac{q(s-p)}{p(s-q)}}w(t)\,dt\Big)^{\frac{s-q}{qs}};$$

(iii) *$q < s \le p$, and in this case $c_1 \approx B_{1,1} + C_1$, where*

$$C_1 := \Big(\int_0^\infty \Big(\operatorname*{ess\,sup}_{\tau\in(0,t)}\frac{[\Phi_1]^{\frac{s}{p}}(\tau)}{\Phi[v;s](\tau)}\Big)^{\frac{q}{s-q}}W_*^{\frac{q}{s-q}}(t)w(t)\,dt\Big)^{\frac{s-q}{sq}};$$

(iv) *$s \le q < \infty$ and $s \le p$, and in this case $c_1 = D_1$, where*

$$D_1 := \sup_{t>0}\Phi[v;s]^{-\frac{1}{s}}(t)\Big(\int_0^\infty [\Phi_1]^{\frac{q}{p}}(\min\{\tau,t\})w(\tau)\,d\tau\Big)^{\frac{1}{q}};$$

(v) *$s \le p$ and $q = \infty$, and in this case $c_1 = E_1$, where*

$$E_1 := \operatorname*{ess\,sup}_{t>0}\Phi[v;s]^{-\frac{1}{s}}(t)\Big(\operatorname*{ess\,sup}_{\tau>0}\Phi_1(\min\{\tau,t\})w(\tau)\Big)^{\frac{1}{p}};$$

(vi) *$p < s$ and $q = \infty$, and in this case $c_1 = F_1$, where*

$$F_1 := \operatorname*{ess\,sup}_{t>0} w(t)\Big(\int_0^t\Big(\int_\tau^t u(y)\Phi[v;s]^{-1}(y)\,dy\Big)^{\frac{s}{s-p}}\phi[v;s](\tau)\,d\tau\Big)^{\frac{s-p}{sp}}.$$

Proof. By Theorem 3.1 (with the operator $T = H_{p,u}$), ineqality (1.5) holds if and only if

$$\Big\|\int_0^x fu\Phi[v;s]^{2p}\Big\|_{q/p,w,(0,\infty)} \le C_1^p\,\|f\|_{s/p,\phi[v;s],(0,\infty)},\ f \in \mathfrak{M}^{\downarrow} \tag{6.1}$$

holds. Moreover, $c_1 \approx C_1$. It remains to apply Theorem 5.1. □

We have the following statement when $s = 1$.

Theorem 6.2. *Let $0 < p < \infty$ and $0 < q \le \infty$. Assume that u, $w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(0,\infty)$ be such that $V(x) < \infty$ for all $x > 0$. Denote by*

$$V_2(\tau) := \int_0^\tau u(x)V^{2p}(x)\,dx,\ \tau > 0.$$

Then inequality

$$\left\|H_{p,u}\left(\int_0^x h\right)\right\|_{q,w,(0,\infty)} \le c_1^1\,\|h\|_{1,V^{-1},(0,\infty)},\ h \in \mathfrak{M}^+ \tag{6.2}$$

with the best constant c_1^1 holds if and only if:

(i) *$p < 1 \le q < \infty$, and in this case $c_1^1 \approx A_{1,1}^1 + A_{1,2}^1$, where*

$$A_{1,1}^1 := \sup_{t>0}\left(\int_0^t [V_2]^{\frac{q}{p}}(\tau)w(\tau)\,d\tau\right)^{\frac{1}{q}} V^{-1}(t),$$

$$A_{1,2}^1 := \sup_{t>0} W_*^{\frac{1}{q}}(t)\left(\int_0^t \left(\frac{V_2(\tau)}{V(\tau)}\right)^{\frac{1}{1-p}} v(\tau)\,d\tau\right)^{\frac{1-p}{p}};$$

(ii) *$q < 1$ and $p < 1$, and in this case $c_1^1 \approx B_{1,1}^1 + B_{1,2}^1$, where*

$$B_{1,1}^1 := \left(\int_0^\infty V^{\frac{q}{q-1}}(t)\left(\int_0^t [V_2]^{\frac{q}{p}}(\tau)w(\tau)\,d\tau\right)^{\frac{q}{1-q}} [V_2]^{\frac{q}{p}}(t)w(t)\,dt\right)^{\frac{1-q}{q}},$$

$$B_{1,2}^1 := \left(\int_0^\infty W_*^{\frac{q}{1-q}}(t)\left(\int_0^t \left(\frac{V_2(\tau)}{V(\tau)}\right)^{\frac{1}{1-p}} v(\tau)\,d\tau\right)^{\frac{q(1-p)}{p(1-q)}} w(t)\,dt\right)^{\frac{1-q}{q}};$$

(iii) *$q < 1 \le p$, and in this case $c_1^1 \approx B_{1,1}^1 + C_1^1$, where*

$$C_1^1 := \left(\int_0^\infty \left(\operatorname*{ess\,sup}_{\tau\in(0,t)} \frac{[V_2]^{\frac{1}{p}}(\tau)}{V(\tau)}\right)^{\frac{q}{1-q}} W_*^{\frac{q}{1-q}}(t)w(t)\,dt\right)^{\frac{1-q}{q}};$$

(iv) *$1 \le q < \infty$ and $1 \le p$, and in this case $c_1^1 = D_1^1$, where*

$$D_1^1 := \sup_{t>0} V^{-1}(t)\left(\int_0^\infty [V_2]^{\frac{q}{p}}(\min\{\tau,t\})w(\tau)\,d\tau\right)^{\frac{1}{q}};$$

(v) *$1 \le p$ and $q = \infty$, and in this case $c_1^1 = E_1^1$, where*

$$E_1^1 := \operatorname*{ess\,sup}_{t>0} V^{-1}(t)\left(\operatorname*{ess\,sup}_{\tau>0}\ V_2(\min\{\tau,t\})w(\tau)\right)^{\frac{1}{p}};$$

(vi) *$p < 1$ and $q = \infty$, and in this case $c_1^1 = F_1^1$, where*

$$F_1^1 := \operatorname*{ess\,sup}_{t>0} w(t)^{\frac{1}{p}}\left(\int_0^t \left(\int_\tau^t u(y)V^{2p-1}(y\,dy)\right)^{\frac{1}{1-p}} v(\tau)\,d\tau\right)^{\frac{1-p}{p}}.$$

Proof. By Theorem 3.11 applied to the operator $H_{p,u}$, inequality (6.2) with the best constant c_1 holds if and only if inequality

$$\left\|\int_0^x fV^{2p}u\right\|_{q/p,w,(0,\infty)} \le C_1^p\,\|f\|_{1/p,v,(0,\infty)},\ f \in \mathfrak{M}^{\downarrow} \tag{6.3}$$

holds. Moreover, $c_1 \approx C_1$. In order to complete the proof, it remains to apply Theorem 5.1. □

The following theorems give us another more simpler and natural method for characterization of inequality (1.6), which is different from that one worked out in [18] and [19].

Theorem 6.3. *Let $0 < p < \infty$, $0 < q \le \infty$ and $1 < s < \infty$. Assume that u, $w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(0,\infty)$ be such that (3.13) holds. Denote by*

$$\Phi_2(\tau) := \int_0^\tau u(x)\Big(\Psi[v;s]\cdot\Phi[\Psi[v;s]^s\psi[v;s]^{1-s};s]\Big)^{2p}(x)\,dx,\ \tau>0.$$

Recall that

$$\Psi[v;s](x) = \Big(\int_x^\infty v^{1-s'}(t)\,dt\Big)^{\frac{1}{s'+1}},\ x>0,$$

$$\phi[\Psi[v;s]^s\psi[v;s]^{1-s};s](x)$$
$$\approx \Big\{\int_0^x\Big(\int_t^\infty v^{1-s'}\Big)^{-\frac{2s'}{1+s'}}v^{1-s'}(t)\,dt\Big\}^{-\frac{s'}{1+s'}}\Big(\int_x^\infty v^{1-s'}\Big)^{-\frac{2s'}{1+s'}}v^{1-s'}(x),\ x>0,$$

$$\Phi[\Psi[v;s]^s\psi[v;s]^{1-s};s](x) \approx \Big\{\int_0^x\Big(\int_t^\infty v^{1-s'}\Big)^{-\frac{2s'}{1+s'}}v^{1-s'}(t)\,dt\Big\}^{\frac{1}{1+s'}},\ x>0.$$

Then inequality (1.6) with the best constant c_2 holds if and only if:

(i) *$p < s \le q < \infty$, and in this case*

$$c_2 \approx A_{2,1} + A_{2,2} + \|\|\mathbf{1}\|_{p,\Psi[v;s]^{2p}u,(0,t)}\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{s,\psi[v;s],(0,\infty)},$$

where

$$A_{2,1} := \sup_{t>0}\Big(\int_0^t[\Phi_2]^{\frac{q}{p}}(\tau)w(\tau)\,d\tau\Big)^{\frac{1}{q}}\Phi[\Psi^s\psi^{1-s};s]^{-\frac{1}{s}}(t),$$

$$A_{2,2} := \sup_{t>0}W_*^{\frac{1}{q}}(t)\Big(\int_0^t\Big(\frac{\Phi_2(\tau)}{\Phi[\Psi^s\psi^{1-s};s](\tau)}\Big)^{\frac{s}{s-p}}\phi[\Psi^s\psi^{1-s};s](\tau)\,d\tau\Big)^{\frac{s-p}{ps}};$$

(ii) *$q < s < \infty$ and $p < s$, and in this case*

$$c_2 \approx B_{2,1} + B_{2,2} + \|\|\mathbf{1}\|_{p,\Psi[v;s]^{2p}u,(0,t)}\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{s,\psi[v;s],(0,\infty)},$$

where

$$B_{2,1} := \Big(\int_0^\infty\Phi[\Psi^s\psi^{1-s};s]^{\frac{q}{q-s}}(t)\Big(\int_0^t[\Phi_2]^{\frac{q}{p}}(\tau)w(\tau)\,d\tau\Big)^{\frac{q}{s-q}}[\Phi_2]^{\frac{q}{p}}(t)w(t)\,dt\Big)^{\frac{s-q}{qs}},$$

$$B_{2,2} := \Big(\int_0^\infty W_*^{\frac{q}{s-q}}(t)\Big(\int_0^t\Big(\frac{\Phi_2(\tau)}{\Phi[\Psi^s\psi^{1-s};s](\tau)}\Big)^{\frac{s}{s-p}}\phi[\Psi^s\psi^{1-s};s](\tau)\,d\tau\Big)^{\frac{q(s-p)}{p(s-q)}}w(t)\,dt\Big)^{\frac{s-q}{qs}};$$

(iii) *$q < s \le p$, and in this case*

$$c_2 \approx B_{2,1} + C_2 + \|\|\mathbf{1}\|_{p,\Psi[v;s]^{2p}u,(0,t)}\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{s,\psi[v;s],(0,\infty)},$$

where

$$C_2 := \Big(\int_0^\infty\Big(\operatorname*{ess\,sup}_{\tau\in(0,t)}\frac{[\Phi_2]^{\frac{s}{p}}(\tau)}{\Phi[\Psi^s\psi^{1-s};s](\tau)}\Big)^{\frac{q}{s-q}}W_*^{\frac{q}{s-q}}(t)w(t)\,dt\Big)^{\frac{s-q}{sq}};$$

(iv) *$s \le q < \infty$ and $s \le p$, and in this case*

$$c_2 = D_2 + \|\|\mathbf{1}\|_{p,\Psi[v;s]^{2p}u,(0,t)}\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{s,\psi[v;s],(0,\infty)},$$

where

$$D_2 := \sup_{t>0}\Phi[\Psi^s\psi^{1-s};s]^{-\frac{1}{s}}(t)\Big(\int_0^\infty[\Phi_2]^{\frac{q}{p}}(\min\{\tau,t\})w(\tau)\,d\tau\Big)^{\frac{1}{q}};$$

(v) *$s \le p$ and $q = \infty$, and in this case*

$$c_2 = E_2 + \|\|\mathbf{1}\|_{p,\Psi[v;s]^{2p}u,(0,t)}\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{s,\psi[v;s],(0,\infty)},$$

where

$$E_2 := \operatorname*{ess\,sup}_{t>0} \Phi[\Psi^s\psi^{1-s};s]^{-\frac{1}{s}}(t)\Big(\operatorname*{ess\,sup}_{\tau>0}\ \Phi_2(\min\{\tau,t\})w(\tau)\Big)^{\frac{1}{p}};$$

(vi) $p<s$ *and* $q=\infty$, *and in this case*

$$c_2 = F_2 + \|\|\mathbf{1}\|_{p,\Psi[v;s]^{2p}u,(0,t)}\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{s,\psi[v;s],(0,\infty)},$$

where

$$F_2 := \operatorname*{ess\,sup}_{t>0} w(t)\Big(\int_0^t\Big(\int_\tau^t u(y)\Phi[\Psi^s\psi^{1-s};s]^{-1}(y)\,dy\Big)^{\frac{s}{s-p}}\phi[\Psi^s\psi^{1-s};s](\tau)\,d\tau\Big)^{\frac{s-p}{sp}}.$$

Proof. By Corollary 3.5 (applied to $H_{p,u}$ with $\delta=1$), inequality (1.6) with the best constant c_2 holds if and only if both

$$\Big\|H_{p,\Psi[v;s]^{2p}u}\Big(\int_0^x h\Big)\Big\|_{q,w,(0,\infty)} \le c_{2,1}\,\|h\|_{s,\Psi[v;s]^s\psi[v;s]^{1-s},(0,\infty)},\ h\in\mathfrak{M}^+, \tag{6.4}$$

and

$$\|\|\mathbf{1}\|_{p,\Psi[v;s]^{2p}u,(0,t)}\|_{q,w,(0,\infty)} \le c_{2,2}\|\mathbf{1}\|_{s,\psi[v;s],(0,\infty)}, \tag{6.5}$$

hold.

Moreover, $c_2 \approx c_{2,1} + \|\|\mathbf{1}\|_{p,\Psi[v;s]^{2p}u,(0,t)}\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{s,\psi[v;s],(0,\infty)}$.

Now the statement follows by Theorem 6.1. □

We have the following statement when $s=1$.

Theorem 6.4. *Let* $0<p<\infty$ *and* $0<q\le\infty$. *Assume that* $u,\ w\in\mathcal{W}(0,\infty)$ *and* $v\in\mathcal{W}(0,\infty)$ *be such that* $V_*(x)<\infty$ *for all* $x>0$. *Denote by*

$$V_3^*(\tau) := \int_0^\tau u(x)\{V_*\cdot[V_1^*]^2\}^{2p}(x)\,dx,\ \tau>0.$$

Recall that

$$V_1^*(x) := \Big(\int_0^x V_*^{-2}(t)v(t)\,dt\Big)^{1/3}, \qquad (x>0).$$

Then inequality

$$\Big\|H_{p,u}\Big(\int_x^\infty h\Big)\Big\|_{q,w,(0,\infty)} \le c_2^1\,\|h\|_{1,V_*^{-1},(0,\infty)},\ h\in\mathfrak{M}^+ \tag{6.6}$$

with the best constant c_2^1 *holds if and only if:*

(i) $p<1\le q<\infty$, *and in this case*

$$c_2^1 \approx A_{2,1}^1 + A_{2,2}^1 + \Big\|\|\mathbf{1}\|_{p,V_*^{2p}u,(0,t)}\Big\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{1,v,(0,\infty)},$$

where

$$A_{2,1}^1 := \sup_{t>0}\Big(\int_0^t [V_3^*]^{q/p}(\tau)w(\tau)\,d\tau\Big)^{1/q}[V_1^*]^{-1}(t),$$

$$A_{2,2}^1 := \sup_{t>0} W_*^{\frac{1}{q}}(t)\Big(\int_0^t\Big(\frac{V_3^*(\tau)}{V_1^*(\tau)}\Big)^{\frac{1}{1-p}}\{V_*\cdot[V_1^*]\}^{-2}(\tau)v(\tau)\,d\tau\Big)^{\frac{1-p}{p}};$$

(ii) $q<1$ *and* $p<1$, *and in this case*

$$c_2^1 \approx B_{2,1}^1 + B_{2,2}^1 + \Big\|\|\mathbf{1}\|_{p,V_*^{2p}u,(0,t)}\Big\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{1,v,(0,\infty)},$$

where

$$B^1_{2,1} := \Big(\int_0^\infty [V_1^*]^{\frac{q}{q-1}}(t)\Big(\int_0^t [V_3^*]^{\frac{q}{p}}(\tau)w(\tau)\,d\tau\Big)^{\frac{q}{1-q}}[V_3^*]^{\frac{q}{p}}(t)w(t)\,dt\Big)^{\frac{1-q}{q}},$$

$$B^1_{2,2} := \Big(\int_0^\infty W_*^{\frac{q}{1-q}}(t)\Big(\int_0^t \Big(\frac{V_3^*(\tau)}{V_1^*(\tau)}\Big)^{\frac{1}{1-p}}\{V_*\cdot[V_1^*]\}^{-2}(\tau)v(\tau)\,d\tau\Big)^{\frac{q(1-p)}{p(1-q)}}w(t)\,dt\Big)^{\frac{1-q}{q}};$$

(iii) $q<1\le p$, *and in this case*

$$c_2^1 \approx B^1_{2,1}+C^1_2+\Big\|\|\mathbf{1}\|_{p,V_*^{2p}u,(0,t)}\Big\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{1,v,(0,\infty)},$$

where

$$C^1_2 := \Big(\int_0^\infty \Big(\operatorname*{ess\,sup}_{\tau\in(0,t)}\frac{[V_3^*]^{\frac{1}{p}}(\tau)}{V_1^*(\tau)}\Big)^{\frac{q}{1-q}}W_*^{\frac{q}{1-q}}(t)w(t)\,dt\Big)^{\frac{1-q}{q}};$$

(iv) $1\le q<\infty$ *and* $1\le p$, *and in this case*

$$c_2^1 = D^1_2+\Big\|\|\mathbf{1}\|_{p,V_*^{2p}u,(0,t)}\Big\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{1,v,(0,\infty)},$$

where

$$D^1_2 := \sup_{t>0}[V_1^*]^{-1}(t)\Big(\int_0^\infty [V_3^*]^{\frac{q}{p}}(\min\{\tau,t\})w(\tau)\,d\tau\Big)^{\frac{1}{q}};$$

(v) $1\le p$ *and* $q=\infty$, *and in this case*

$$c_2^1 = E^1_2+\Big\|\|\mathbf{1}\|_{p,V_*^{2p}u,(0,t)}\Big\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{1,v,(0,\infty)},$$

where

$$E^1_2 := \operatorname*{ess\,sup}_{t>0}[V_1^*]^{-1}(t)\Big(\operatorname*{ess\,sup}_{\tau>0}[V_3^*](\min\{\tau,t\})w(\tau)\Big)^{\frac{1}{p}};$$

(vi) $p<1$ *and* $q=\infty$, *and in this case*

$$c_2^1 = F^1_2+\Big\|\|\mathbf{1}\|_{p,V_*^{2p}u,(0,t)}\Big\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{1,v,(0,\infty)},$$

where

$$F^1_2 := \operatorname*{ess\,sup}_{t>0}w(t)^{\frac{1}{p}}\Big(\int_0^t\Big(\int_\tau^t u(y)[V_1^*]^{2p-1}(y)\,dy\Big)^{\frac{1}{1-p}}\{V_*\cdot[V_1^*]\}^{-2}(\tau)v(\tau)\,d\tau\Big)^{\frac{1-p}{p}}.$$

Proof. By Corollary 3.23 applied to the operator $H_{p,u}$, inequality (6.6) with the best constant c_2^1 holds if and only if both

$$\Big\|\int_0^x\{V_*\cdot[V_1^*]^2\}^{2p}uf\Big\|_{q/p,w,(0,\infty)}\le c_{2,1}^p\,\|f\|_{1/p,\{V_*\cdot[V_1^*]\}^{-2}v,(0,\infty)},\ f\in\mathfrak{M}^{\downarrow}, \tag{6.7}$$

and

$$\Big\|\|\mathbf{1}\|_{p,V_*^{2p}u,(0,t)}\Big\|_{q,w,(0,\infty)}\le c_{2,2}\|\mathbf{1}\|_{1,v,(0,\infty)}, \tag{6.8}$$

hold. Moreover, $c_2^1\approx c_{2,1}+\Big\|\|\mathbf{1}\|_{p,V_*^{2p}u,(0,t)}\Big\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{1,v,(0,\infty)}$. Applying Theorem 5.1 we obtain the statement. □

For the sake of completeness we give the characterizations of inequalities of (1.7) and (1.8) here.

Theorem 6.5. *Let $0 < p < \infty$, $0 < q \le \infty$ and $1 < s < \infty$. Assume that u, $w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(0,\infty)$ be such that (3.13) holds. Recall that*

$$\Psi[v;s](x) = \Big(\int_x^\infty v^{1-s'}(t)\,dt\Big)^{\frac{1}{s'+1}},\ x > 0.$$

Denote by

$$\Psi_1(\tau) := \int_\tau^\infty u(x)\Psi[v;s]^{2p}(x)\,dx = \int_\tau^\infty u(x)\Big(\int_x^\infty v^{1-s'}(t)\,dt\Big)^{\frac{2p}{s'+1}}\,dx,\ \tau > 0.$$

Then inequality (1.7) with the best constant c_3 holds if and only if:

(i) *$p < s \le q < \infty$, and in this case $c_3 \approx A_{3,1} + A_{3,2}$, where*

$$A_{3,1} := \sup_{t>0}\Big(\int_t^\infty [\Psi_1]^{\frac{q}{p}}(\tau)w(\tau)\,d\tau\Big)^{\frac{1}{q}}\Psi[v;s]^{-\frac{1}{s}}(t),$$

$$A_{3,2} := \sup_{t>0} W^{\frac{1}{q}}(t)\Big(\int_t^\infty \Big(\frac{\Psi_1(\tau)}{\Psi[v;s](\tau)}\Big)^{\frac{s}{s-p}}\psi[v;s](\tau)\,d\tau\Big)^{\frac{s-p}{ps}};$$

(ii) *$q < s < \infty$ and $p < s$, and in this case $c_3 \approx B_{3,1} + B_{3,2}$, where*

$$B_{3,1} := \Big(\int_0^\infty \Psi[v;s]^{\frac{q}{q-s}}(t)\Big(\int_t^\infty [\Psi_1]^{\frac{q}{p}}(\tau)w(\tau)\,d\tau\Big)^{\frac{q}{s-q}}[\Psi_1]^{\frac{q}{p}}(t)w(t)\,dt\Big)^{\frac{s-q}{qs}},$$

$$B_{3,2} := \Big(\int_0^\infty W^{\frac{q}{s-q}}(t)\Big(\int_t^\infty \Big(\frac{\Psi_1(\tau)}{\Psi[v;s](\tau)}\Big)^{\frac{s}{s-p}}\psi[v;s](\tau)\,d\tau\Big)^{\frac{q(s-p)}{p(s-q)}}w(t)\,dt\Big)^{\frac{s-q}{qs}};$$

(iii) *$q < s \le p$, and in this case $c_3 \approx B_{3,1} + C_3$, where*

$$C_3 := \Big(\int_0^\infty \Big(\operatorname*{ess\,sup}_{\tau\in(t,\infty)}\frac{[\Psi_1]^{\frac{s}{p}}(\tau)}{\Psi[v;s](\tau)}\Big)^{\frac{q}{s-q}}W^{\frac{q}{s-q}}(t)w(t)\,dt\Big)^{\frac{s-q}{sq}};$$

(iv) *$s \le q < \infty$ and $s \le p$, and in this case $c_3 = D_3$, where*

$$D_3 := \sup_{t>0}\Psi[v;s]^{-\frac{1}{s}}(t)\Big(\int_0^\infty [\Psi_1]^{\frac{q}{p}}(\max\{\tau,t\})w(\tau)\,d\tau\Big)^{\frac{1}{q}};$$

(v) *$s \le p$ and $q = \infty$, and in this case $c_3 = E_3$, where*

$$E_3 := \operatorname*{ess\,sup}_{t>0}\Psi[v;s]^{-\frac{1}{s}}(t)\Big(\operatorname*{ess\,sup}_{\tau>0}\Psi_1(\max\{\tau,t\})w(\tau)\Big)^{\frac{1}{p}};$$

(vi) *$p < s$ and $q = \infty$, and in this case $c_3 = F_3$, where*

$$F_3 := \operatorname*{ess\,sup}_{t>0} w(t)\Big(\int_t^\infty \Big(\int_t^\tau u(y)\Psi[v;s]^{-1}(y)\,dy\Big)^{\frac{s}{s-p}}\psi[v;s](\tau)\,d\tau\Big)^{\frac{s-p}{sp}}.$$

Proof. By change of variables $x = 1/t$, it is easy to see that inequality (1.7) holds if and only if

$$\Big\|H_{p,\tilde{u}}\Big(\int_0^x h\Big)\Big\|_{q,\tilde{w},(0,\infty)} \le c\,\|h\|_{s,\tilde{v},(0,\infty)}$$

holds, where

$$\tilde{u}(t) = u\Big(\frac{1}{t}\Big)\frac{1}{t^2},\ \tilde{w}(t) = w\Big(\frac{1}{t}\Big)\frac{1}{t^2},\ \tilde{v}(t) = v\Big(\frac{1}{t}\Big)\Big(\frac{1}{t^2}\Big)^{1-s},\ t > 0,$$

when $0 < q < \infty$, and

$$\tilde{u}(t) = u\Big(\frac{1}{t}\Big)\frac{1}{t^2},\ \tilde{w}(t) = w\Big(\frac{1}{t}\Big),\ \tilde{v}(t) = v\Big(\frac{1}{t}\Big)\Big(\frac{1}{t^2}\Big)^{1-s},\ t > 0,$$

when $q = \infty$.

Using Theorem 6.1, and then applying substitution of variables mentioned above three times, we get the statement. □

Theorem 6.6. *Let $0 < p < \infty$ and $0 < q \le \infty$. Assume that u, $w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(0,\infty)$ be such that $V_*(x) < \infty$ for all $x > 0$. Denote by*

$$V_2^*(\tau) := \int_\tau^\infty u(x)V_*^{2p}(x)\,dx,\ \tau > 0.$$

Then inequality

$$\left\|H_{p,u}^*\left(\int_x^\infty h\right)\right\|_{q,w,(0,\infty)} \le c_3^1\,\|h\|_{1,V_*^{-1},(0,\infty)},\ h \in \mathfrak{M}^+ \tag{6.9}$$

with the best constant c_3^1 holds if and only if:

(i) *$p < 1 \le q < \infty$, and in this case $c_3^1 \approx A_{3,1}^1 + A_{3,2}^1$, where*

$$A_{3,1}^1 := \sup_{t>0}\left(\int_t^\infty [V_2^*]^{\frac{q}{p}}(\tau)w(\tau)\,d\tau\right)^{\frac{1}{q}} V_*^{-1}(t),$$

$$A_{3,2}^1 := \sup_{t>0} W^{\frac{1}{q}}(t)\left(\int_t^\infty \left(\frac{V_2^*(\tau)}{V_*(\tau)}\right)^{\frac{1}{1-p}} v(\tau)\,d\tau\right)^{\frac{1-p}{p}};$$

(ii) *$q < 1$ and $p < 1$, and in this case $c_3^1 \approx B_{3,1}^1 + B_{3,2}^1$, where*

$$B_{3,1}^1 := \left(\int_0^\infty V_*^{\frac{q}{q-1}}(t)\left(\int_t^\infty [V_2^*]^{\frac{q}{p}}(\tau)w(\tau)\,d\tau\right)^{\frac{q}{1-q}} [V_2^*]^{\frac{q}{p}}(t)w(t)\,dt\right)^{\frac{1-q}{q}},$$

$$B_{3,2}^1 := \left(\int_0^\infty W^{\frac{q}{1-q}}(t)\left(\int_t^\infty \left(\frac{V_2^*(\tau)}{V_*(\tau)}\right)^{\frac{1}{1-p}} v(\tau)\,d\tau\right)^{\frac{q(1-p)}{p(1-q)}} w(t)\,dt\right)^{\frac{1-q}{q}};$$

(iii) *$q < 1 \le p$, and in this case $c_3^1 \approx B_{3,1}^1 + C_3^1$, where*

$$C_3^1 := \left(\int_0^\infty \left(\operatorname*{ess\,sup}_{\tau\in(t,\infty)} \frac{[V_2^*]^{\frac{1}{p}}(\tau)}{V_*(\tau)}\right)^{\frac{q}{1-q}} W^{\frac{q}{1-q}}(t)w(t)\,dt\right)^{\frac{1-q}{q}};$$

(iv) *$1 \le q < \infty$ and $1 \le p$, and in this case $c_3^1 = D_3^1$, where*

$$D_3^1 := \sup_{t>0} V_*^{-1}(t)\left(\int_0^\infty [V_2^*]^{\frac{q}{p}}(\max\{\tau,t\})w(\tau)\,d\tau\right)^{\frac{1}{q}};$$

(v) *$1 \le p$ and $q = \infty$, and in this case $c_3^1 = E_3^1$, where*

$$E_3^1 := \operatorname*{ess\,sup}_{t>0} V_*^{-1}(t)\left(\operatorname*{ess\,sup}_{\tau>0}\ V_2^*(\max\{\tau,t\})w(\tau)\right)^{\frac{1}{p}};$$

(vi) *$p < 1$ and $q = \infty$, and in this case $c_3^1 = F_3^1$, where*

$$F_3^1 := \operatorname*{ess\,sup}_{t>0} w(t)^{\frac{1}{p}}\left(\int_t^\infty \left(\int_t^\tau u(y)V_*^{2p-1}(y)\,dy\right)^{\frac{1}{1-p}} v(\tau)\,d\tau\right)^{\frac{1-p}{p}}.$$

Proof. By change of variables $x = 1/t$, it is easy to see that inequality (6.9) holds if and only if

$$\left\|H_{p,\tilde{u}}\left(\int_0^x h\right)\right\|_{q,\tilde{w},(0,\infty)} \le c\,\|h\|_{1,\tilde{V}^{-1},(0,\infty)},\ h \in \mathfrak{M}^+$$

holds, where

$$\tilde{u}(t) = u\left(\frac{1}{t}\right)\frac{1}{t^2},\ \tilde{w}(t) = w\left(\frac{1}{t}\right)\frac{1}{t^2},\ \tilde{V}(t) = \int_0^t v\left(\frac{1}{y}\right)\frac{1}{y^2}\,dy,\ t > 0,$$

when $0 < q < \infty$, and

$$\tilde{u}(t) = u\Big(\frac{1}{t}\Big)\frac{1}{t^2},\ \tilde{w}(t) = w\Big(\frac{1}{t}\Big),\ \tilde{V}(t) = \int_0^t v\Big(\frac{1}{y}\Big)\frac{1}{y^2}\,dy,\ t > 0,$$

when $q = \infty$.

Applying Theorem 6.2, and then using substitution of variables mentioned above three times, we get the statement. □

Theorem 6.7. *Let $0 < p < \infty$, $0 < q \le \infty$ and $1 < s < \infty$. Assume that u, $w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(0,\infty)$ be such that (3.1) holds. Denote by*

$$\Psi_2(\tau) := \int_\tau^\infty u(x)\Big(\Phi[v;s]\cdot\Psi[\Phi[v;s]^s\phi[v;s]^{1-s};s]\Big)^{2p}(x)\,dx,\ \tau > 0.$$

Recall that

$$\Phi[v;s](x) = \Big(\int_0^x v^{1-s'}(t)\,dt\Big)^{\frac{1}{s'+1}},\ x > 0,$$

$$\psi[\Phi[v;s]^s\phi[v;s]^{1-s};s](x)$$

$$\approx \Big\{\int_x^\infty\Big(\int_0^t v^{1-s'}\Big)^{-\frac{2s'}{1+s'}}v^{1-s'}(t)\,dt\Big\}^{-\frac{s'}{1+s'}}\Big(\int_0^x v^{1-s'}\Big)^{-\frac{2s'}{1+s'}}v^{1-s'}(x),$$

$$\Psi[\Phi[v;s]^s\phi[v;s]^{1-s};s](x) \approx \Big\{\int_x^\infty\Big(\int_0^t v^{1-s'}\Big)^{-\frac{2s'}{1+s'}}v^{1-s'}(t)\,dt\Big\}^{\frac{1}{1+s'}},$$

Then inequality (1.8) with the best constant c_4 holds if and only if:

(i) *$p < s \le q < \infty$, and in this case*

$$c_4 \approx A_{4,1} + A_{4,2} + \|\|\mathbf{1}\|_{p,\Phi[v;s]^{2p}u,(t,\infty)}\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{s,\phi[v;s],(0,\infty)},$$

where

$$A_{4,1} := \sup_{t>0}\Big(\int_t^\infty[\Psi_2]^{\frac{q}{p}}(\tau)w(\tau)\,d\tau\Big)^{\frac{1}{q}}\Psi[\Phi^s\phi^{1-s};s]^{-\frac{1}{s}}(t),$$

$$A_{4,2} := \sup_{t>0}W^{\frac{1}{q}}(t)\Big(\int_t^\infty\Big(\frac{\Psi_2(\tau)}{\Psi[\Phi^s\phi^{1-s};s](\tau)}\Big)^{\frac{s}{s-p}}\phi[\Phi^s\phi^{1-s};s](\tau)\,d\tau\Big)^{\frac{s-p}{ps}};$$

(ii) *$q < s < \infty$ and $p < s$, and in this case*

$$c_4 \approx B_{4,1} + B_{4,2} + \|\|\mathbf{1}\|_{p,\Phi[v;s]^{2p}u,(t,\infty)}\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{s,\phi[v;s],(0,\infty)},$$

where

$$B_{4,1} := \Big(\int_0^\infty\Psi[\Phi^s\phi^{1-s};s]^{\frac{q}{q-s}}(t)\Big(\int_t^\infty[\Psi_2]^{\frac{q}{p}}(\tau)w(\tau)\,d\tau\Big)^{\frac{q}{s-q}}[\Psi_2]^{\frac{q}{p}}(t)w(t)\,dt\Big)^{\frac{s-q}{qs}},$$

$$B_{4,2} := \Big(\int_0^\infty W^{\frac{q}{s-q}}(t)\Big(\int_t^\infty\Big(\frac{\Psi_2(\tau)}{\Psi[\Phi^s\phi^{1-s};s](\tau)}\Big)^{\frac{s}{s-p}}\psi[\Phi^s\phi^{1-s};s](\tau)\,d\tau\Big)^{\frac{q(s-p)}{p(s-q)}}w(t)\,dt\Big)^{\frac{s-q}{qs}};$$

(iii) *$q < s \le p$, and in this case*

$$c_4 \approx B_{4,1} + C_4 + \|\|\mathbf{1}\|_{p,\Phi[v;s]^{2p}u,(t,\infty)}\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{s,\phi[v;s],(0,\infty)},$$

where

$$C_4 := \Big(\int_0^\infty\Big(\operatorname*{ess\,sup}_{\tau\in(t,\infty)}\frac{[\Psi_2]^{\frac{s}{p}}(\tau)}{\Psi[\Phi^s\phi^{1-s};s](\tau)}\Big)^{\frac{q}{s-q}}W^{\frac{q}{s-q}}(t)w(t)\,dt\Big)^{\frac{s-q}{sq}};$$

(iv) *$s \le q < \infty$ and $s \le p$, and in this case*

$$c_4 = D_4 + \|\|\mathbf{1}\|_{p,\Phi[v;s]^{2p}u,(t,\infty)}\|_{q,w,(0,\infty)} / \|\mathbf{1}\|_{s,\phi[v;s],(0,\infty)},$$

where

$$D_4 := \sup_{t>0} \Psi[\Phi^s\phi^{1-s};s]^{-\frac{1}{s}}(t)\Big(\int_0^\infty [\Psi_2]^{\frac{q}{p}}(\max\{\tau,t\})w(\tau)\,d\tau\Big)^{\frac{1}{q}};$$

(v) *$s \le p$ and $q = \infty$, and in this case*

$$c_4 = E_4 + \|\|\mathbf{1}\|_{p,\Phi[v;s]^{2p}u,(t,\infty)}\|_{q,w,(0,\infty)} / \|\mathbf{1}\|_{s,\phi[v;s],(0,\infty)},$$

where

$$E_4 := \operatorname*{ess\,sup}_{t>0} \Psi[\Phi^s\phi^{1-s};s]^{-\frac{1}{s}}(t)\Big(\operatorname*{ess\,sup}_{\tau>0}\ \Psi_2(\max\{\tau,t\})w(\tau)\Big)^{\frac{1}{p}};$$

(vi) *$p < s$ and $q = \infty$, and in this case*

$$c_4 = F_4 + \|\|\mathbf{1}\|_{p,\Phi[v;s]^{2p}u,(t,\infty)}\|_{q,w,(0,\infty)} / \|\mathbf{1}\|_{s,\phi[v;s],(0,\infty)},$$

where

$$F_4 := \operatorname*{ess\,sup}_{t>0} w(t)\Big(\int_t^\infty \Big(\int_t^\tau u(y)\Psi[\Phi^s\phi^{1-s};s]^{-1}(y)\,dy\Big)^{\frac{s}{s-p}}\psi[\Phi^s\phi^{1-s};s](\tau)\,d\tau\Big)^{\frac{s-p}{sp}}.$$

Proof. Obviously, inequality (1.8) holds if and only if

$$\Big\|H_{p,\tilde{u}}\Big(\int_x^\infty h\Big)\Big\|_{q,\tilde{w},(0,\infty)} \le c\,\|h\|_{s,\tilde{v},(0,\infty)}$$

holds, where

$$\tilde{u}(t) = u\Big(\frac{1}{t}\Big)\frac{1}{t^2},\ \tilde{w}(t) = w\Big(\frac{1}{t}\Big)\frac{1}{t^2},\ \tilde{v}(t) = v\Big(\frac{1}{t}\Big)\Big(\frac{1}{t^2}\Big)^{1-s},\ t>0,$$

when $0 < q < \infty$, and

$$\tilde{u}(t) = u\Big(\frac{1}{t}\Big)\frac{1}{t^2},\ \tilde{w}(t) = w\Big(\frac{1}{t}\Big),\ \tilde{v}(t) = v\Big(\frac{1}{t}\Big)\Big(\frac{1}{t^2}\Big)^{1-s},\ t>0,$$

when $q = \infty$.

Using Theorem 6.3, and then applying substitution of variables mentioned above three times, we get the statement. □

Theorem 6.8. *Let $0 < p < \infty$ and $0 < q \le \infty$. Assume that u, $w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(0,\infty)$ be such that $V(x) < \infty$ for all $x > 0$. Recall that*

$$V_1(x) := \Big(\int_x^\infty V^{-2}v\Big)^{\frac{1}{3}}, \qquad (x>0).$$

Denote by

$$V_3(\tau) := \int_\tau^\infty u(x)\{V\cdot V_1^2\}^{2p}(x)\,dx,\ \tau>0.$$

Then inequality

$$\Big\|H^*_{p,u}\Big(\int_0^x h\Big)\Big\|_{q,w,(0,\infty)} \le c_4^1\,\|h\|_{1,V^{-1},(0,\infty)}, \tag{6.10}$$

with the best constant c_4^1 holds if and only if:

(i) *$p < 1 \le q < \infty$, and in this case*

$$c_4^1 \approx A_{4,1}^1 + A_{4,2}^1 + \|\|\mathbf{1}\|_{p,V^{2p}u,(t,\infty)}\|_{q,w,(0,\infty)} / \|\mathbf{1}\|_{1,v,(0,\infty)},$$

where

$$A^1_{4,1} := \sup_{t>0}\Big(\int_t^\infty [V_3]^{\frac{q}{p}}(\tau)w(\tau)\,d\tau\Big)^{\frac{1}{q}}[V_1]^{-1}(t),$$

$$A^1_{4,2} := \sup_{t>0} W^{\frac{1}{q}}(t)\Big(\int_t^\infty \Big(\frac{V_3(\tau)}{V_1(\tau)}\Big)^{\frac{1}{1-p}}\{V\cdot[V_1]\}^{-2}(\tau)v(\tau)\,d\tau\Big)^{\frac{1-p}{p}};$$

(ii) $q<1$ *and* $p<1$, *and in this case*

$$c^1_4 \approx B^1_{4,1}+B^1_{4,2}+\|\|\mathbf{1}\|_{p,V^{2p}u,(t,\infty)}\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{1,v,(0,\infty)},$$

where

$$B^1_{4,1} := \Big(\int_0^\infty V_1^{\frac{q}{q-1}}(t)\Big(\int_t^\infty [V_3]^{\frac{q}{p}}(\tau)w(\tau)\,d\tau\Big)^{\frac{q}{1-q}}V_3^{\frac{q}{p}}(t)w(t)\,dt\Big)^{\frac{1-q}{q}},$$

$$B^1_{4,2} := \Big(\int_0^\infty W^{\frac{q}{1-q}}(t)\Big(\int_t^\infty \Big(\frac{V_3(\tau)}{V_1(\tau)}\Big)^{\frac{1}{1-p}}\{V\cdot[V_1]\}^{-2}(\tau)v(\tau)\,d\tau\Big)^{\frac{q(1-p)}{p(1-q)}}w(t)\,dt\Big)^{\frac{1-q}{q}};$$

(iii) $q<1\le p$, *and in this case*

$$c^1_4 \approx B^1_{4,1}+C^1_4+\|\|\mathbf{1}\|_{p,V^{2p}u,(t,\infty)}\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{1,v,(0,\infty)},$$

where

$$C^1_4 := \Big(\int_0^\infty \Big(\operatorname*{ess\,sup}_{\tau\in(t,\infty)}\frac{[V_3]^{\frac{1}{p}}(\tau)}{V_1(\tau)}\Big)^{\frac{q}{1-q}}W^{\frac{q}{1-q}}(t)w(t)\,dt\Big)^{\frac{1-q}{q}};$$

(iv) $1\le q<\infty$ *and* $1\le p$, *and in this case*

$$c^1_4 = D^1_4+\|\|\mathbf{1}\|_{p,V^{2p}u,(t,\infty)}\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{1,v,(0,\infty)},$$

where

$$D^1_4 := \sup_{t>0}V_1^{-1}(t)\Big(\int_0^\infty [V_3]^{\frac{q}{p}}(\max\{\tau,t\})w(\tau)\,d\tau\Big)^{\frac{1}{q}};$$

(v) $1\le p$ *and* $q=\infty$, *and in this case*

$$c^1_4 = E^1_4+\|\|\mathbf{1}\|_{p,V^{2p}u,(t,\infty)}\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{1,v,(0,\infty)},$$

where

$$E^1_4 := \operatorname*{ess\,sup}_{t>0}V_1^{-1}(t)\Big(\operatorname*{ess\,sup}_{\tau>0}[V_3](\max\{\tau,t\})w(\tau)\Big)^{\frac{1}{p}};$$

(vi) $p<1$ *and* $q=\infty$, *and in this case*

$$c^1_4 = F^1_4+\|\|\mathbf{1}\|_{p,V^{2p}u,(t,\infty)}\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{1,v,(0,\infty)},$$

where

$$F^1_4 := \operatorname*{ess\,sup}_{t>0}w(t)^{\frac{1}{p}}\Big(\int_t^\infty\Big(\int_t^\tau u(y)V_1^{2p-1}(y)\,dy\Big)^{\frac{1}{1-p}}\{V\cdot V_1\}^{-2}(\tau)v(\tau)\,d\tau\Big)^{\frac{1-p}{p}}.$$

Proof. Obviously, inequality (6.10) holds if and only if

$$\Big\|H_{p,\tilde u}\Big(\int_x^\infty h\Big)\Big\|_{q,\tilde w,(0,\infty)} \le c\,\|h\|_{1,\tilde V_*^{-1},(0,\infty)},\ h\in\mathfrak{M}^+$$

holds, where

$$\tilde u(t)=u\Big(\frac{1}{t}\Big)\frac{1}{t^2},\ \tilde w(t)=w\Big(\frac{1}{t}\Big)\frac{1}{t^2},\ \tilde V_*(t)=\int_t^\infty v\Big(\frac{1}{y}\Big)\frac{1}{y^2}\,dy,\ t>0,$$

when $0 < q < \infty$, and

$$\tilde{u}(t) = u\Big(\frac{1}{t}\Big)\frac{1}{t^2},\ \tilde{w}(t) = w\Big(\frac{1}{t}\Big),\ \tilde{V}_*(t) = \int_t^\infty v\Big(\frac{1}{y}\Big)\frac{1}{y^2}\,dy,\ t > 0,$$

when $q = \infty$.

Applying Theorem 6.4, and then using substitution of variables mentioned above three times, we get the statement. □

Remark 6.9. It is worth to mention that Theorem 6.3 - 6.8 can be proved by reducing corresponding iterated inequality to the cone of monotone functions. For instance: inequality (1.7) with the best constant c_3 holds if and only if inequality

$$\Big\| \int_0^x f u \Psi[v;s]^{2p} \Big\|_{q/p,w,(0,\infty)} \le c_3^p \, \|f\|_{s/p,\psi[v;s],(0,\infty)},\ f \in \mathfrak{M}^\uparrow$$

holds, and the statement of Theorem 6.5 immediately follows by Theorem 5.2.

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