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Embedding into monothetic groups

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EMBEDDINGS INTO MONOTHETIC GROUPS

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Abstract. We provide a very short elementary proof that every bounded separable metric group embeds into a monothetic bounded metric group, in such a way that the result of Morris and Pestov that every separable abelian topological group embeds into a monothetic group is an immediate corollary. We show that the boundedness assumption is essential.

In [2] Morris and Pestov prove that every separable abelian topological group embeds into a monothetic group (following their previous generalization of the Higman-Neumann-Neumann theorem for topological groups from [1]). Since their proof is rather long and uses several non-elementary results we provide here a short elementary proof of their result which is actually a generalization: it is in the category of metric groups.

Theorem. *Let G be a separable abelian group with a bounded invariant metric d. Then d extends to a (bounded by the same constant) metric D on $G \oplus C$, where C is a cyclic group which is dense in $G \oplus C$. In particular, G embeds into a monothetic metric group.*

Proof. Instead of metric, we shall work with the corresponding norm, i.e. the distance of an element from the group zero, which we shall still denote d, or D. That is, $d(g)$ denotes $d(g, 0)$. Without loss of generality, suppose that d is bounded by 1. Let $H = \{h_n : n \in \mathbb{N}\}$ be a countable dense subgroup of G and let $\pi : \mathbb{N} \to \mathbb{N}^2$ be some bijection; we denote by $\pi(n)(1)$, resp. $\pi(n)(2)$, the respective coordinates of $\pi(n)$. Suppose the cyclic group C is generated by some c. By induction, we shall construct a partial norm D' on $H \oplus C$ which extends d, i.e. a partial function satisfying $D'(g) = 0$ iff $g = 0$, $D'(g) = D'(-g)$ and $D'(g_1 + \ldots + g_n) \leq D'(g_1) + \ldots + D'(g_n)$, whenever the corresponding elements are in the domain of D'. Moreover, we shall produce a strictly increasing sequence $(k_n)_n$ such that $D'(c^{k_n} - h_{\pi(n)(1)}) \leq 1/\pi(n)(2)$. At the end, we may define D by $D(x) = \min\{1, \inf\{\sum_{i=1}^m D'(x_i) : x = \sum_{i=1}^m x_i, (x_i)_i \subseteq \mathrm{dom}(D')\}\}$, for $x \in H \oplus C$. Since D' was a partial norm, D extends D', thus it extends d, and by the induction we will have guaranteed that C is dense in $H \oplus C$.

At the first step of the induction, we set D' to be equal to d on H, then we set $k_1 = 1$ and define $D'(c - h_{\pi(1)(1)}) = D'(h_{\pi(1)(1)} - c) = 1/\pi(1)(2)$. D' is clearly a partial norm. Suppose we have done the first $n-1$ steps, found

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$k_1, \dots, k_{n-1}$ and defined D' appropriately so that it is a partial norm. Let $\delta = \min\{1/\pi(i)(2) : i < n\}$ and let $k_n > k_{n-1}$ be arbitrary satisfying $k_n > k_{n-1}/\delta$. Then we set $D'(c^{k_n} - h_{\pi(n)(1)}) = D'(h_{\pi(n)(1)} - c^{k_n}) = 1/\pi(n)(2)$. We claim that D' is still a partial norm. Suppose on the contrary that the triangle inequality is broken, i.e. there are $x, x_1, \dots, x_n \in \mathrm{dom}(D')$ such that $x = x_1 + \dots + x_n$ and $D'(x) > D'(x_1) + \dots + D'(x_n)$. We shall suppose that $x \in H$, that is the most important case and the other case is treated analogously. For any $z \in H \oplus C$, denote by $k(z)$ the unique integer such that z can be written as $h \oplus c^{k(z)}$. Since $k(x) = 0$, we must have $\sum_{i=1}^n k(x_i) = 0$. For at least one $i \leq n$ we must have that $k(x_i)$ is k_n or $-k_n$, since before the extension at the n-th step, D' was a partial norm. Also, since H is abelian, we may suppose that for no $i, j \leq n$ we have $k(x_i) = -k(x_j) \neq 0$, since in that case $x_i + x_j = 0$ and we may remove x_i and x_j from the decomposition of x. Indeed, note that by the inductive construction for each $i \leq n$ there is a unique element u in the domain of D' such that $k(u) = k_i$, and a unique element v such that $k(v) = -k_n$; and obviously $v = -u$. Let $I \subseteq \{i \leq n : 0 \neq k(x_i) \neq k_n\}$. It follows that $|\sum_{i \in I} k(x_i)| \geq k_n$, so by definition of k_n we have $|I| > 1/\delta$, so $\sum_{i \in I} D'(x_i) > 1$, a contradiction. □

Notice that the numbers $(k_n)_n$ were chosen completely independently of the metric/norm, and the same sequence may be used for any metric/norm bounded by 1. Secondly, notice that there is no change in the proof if we replace metric, resp. norm by pseudometric, resp. pseudonorm.

Corollary (Morris, Pestov). *Every separable abelian topological group G embeds into a monothetic group.*

Proof. Let H be a countable dense subgroup of G and let $(\rho_\alpha)_{\alpha < \kappa}$ be a collection of pseudonorms bounded by 1 which give the topology of G, and H. By the previous proof we may extend each ρ_α on $H \oplus C$, where C is cyclic, using the same numbers $(k_n)_n$ and $\pi : \mathbb{N} \to \mathbb{N}^2$. By either completing $H \oplus C$ with respect to these pseudonorms or extending the pseudonorms on $G \oplus C$ by amalgamation, we obtain a monothetic group to which G embeds. □

One may wonder why we assume boundedness of the metric/norm. It turns out it is essential. Consider ℓ_1^2, the two-dimensional Banach space with ℓ_1 norm, and let H be its metric subgroup generated by the two basis elements e_1, e_2. Suppose there is a norm D on $H \oplus C$, with C cyclic, which extends the norm on H, and C is dense in $H \oplus C$. Then there exist $n, m \in \mathbb{Z}$ such that $D(c^n - e_1) < 1/2$ and $D(e_2 - c^m) < 1/2$. Then however $|m| + |n| = D(-m \cdot e_1 + n \cdot e_2) = D(c^{mn} - m \cdot e_1 + n \cdot e_2 - c^{nm}) \leq mD(c^n - e_1) + nD(e_2 - c^m) < 1/2(|m| + |n|)$, a contradiction.

Problem. Prove a metric version of the Higman-Neumann-Neumann theorem. Either for separable groups with general left-invariant metrics, or for groups with bi-invariant metric.

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References

[1] S. A. Morris and V. Pestov, *A topological generalization of the Higman–Neumann–Neumann theorem*, J. Group Theory 1 (1998), 181–187

[2] S. A. Morris, V. Pestov, *Subgroups of monothetic groups*, J. Group Theory 3 (2000), no. 4, 407–417

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