Existence of global weak solutions for inviscid primitive equations

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19th of January 2017
Będlewo, Poland

[1] In collaboration with E. Chiodaroli. The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078

What are the primitive equations?

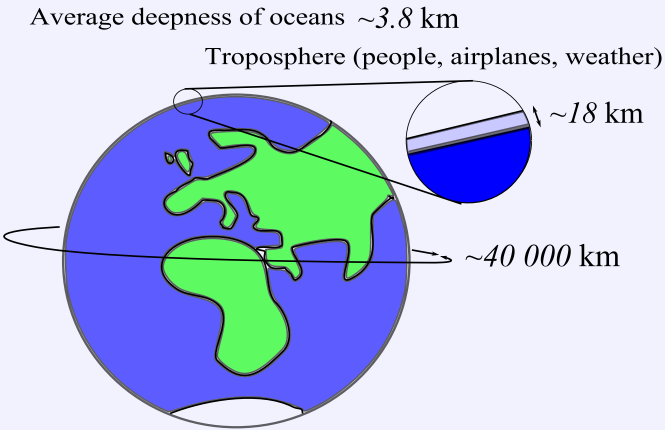

Figure: Scales in global oceanography/weather models

The Boussinesq approximation

The starting point in the large-scale oceanography is the following coupled system of evolutionary partial differential equations

$$\operatorname{div} \mathbf{u} = 0 \qquad \text{in } (0,T) \times \mathbb{T}^3,$$

$$\partial_t \mathbf{u} + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla p = -C_1 \theta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + C_2 \triangle \mathbf{u} \qquad \text{in } (0,T) \times \mathbb{T}^3,$$

$$\partial_t \theta + \mathbf{u} \cdot \nabla \theta = C_3 \triangle \theta \qquad \text{in } (0,T) \times \mathbb{T}^3$$

for unknown velocity field $\mathbf{u}$, pressure p and temperature θ.

What are the primitive equations?

We will investigate a geometrically simplified Cauchy problem[2]: to find $\mathbf{u} = (u, v, w)$, p, $\theta \colon [0, T) \times \mathbb{T}^3 \to \mathbb{R}$ satisfying

$$
\begin{aligned}
\operatorname{div} \mathbf{u} &= 0 && \text{in } (0,T) \times \mathbb{T}^3, \\
u_t + u u_x + v u_y + w u_z + p_x &= \mu_1 (u_{xx} + u_{yy}) + \mu_2 u_{zz} && \text{in } (0,T) \times \mathbb{T}^3, \\
v_t + u v_x + v v_y + w v_z + p_y &= \mu_1 (v_{xx} + v_{yy}) + \mu_2 v_{zz} && \text{in } (0,T) \times \mathbb{T}^3, \\
p_z &= -\theta && \text{in } (0,T) \times \mathbb{T}^3, \\
\theta_t + u \theta_x + v \theta_y + w \theta_z &= \lambda_1 (\theta_{xx} + \theta_{yy}) + \lambda_2 \theta_{zz} && \text{in } (0,T) \times \mathbb{T}^3
\end{aligned}
$$

with initial conditions $u(0) = u_0$, $v(0) = v_0$ and $\theta(0) = \theta_0$.

[2] The mathematical formulation was done by J. L. Lions, R. Temam and S. H. Wang: **New formulations of the primitive equations of atmosphere and applications.** In *Nonlinearity* (1992).

What are the primitive equations?

We will investigate a geometrically simplified Cauchy problem[3]: to find $\mathbf{u} = (u, v, w)$, p, $\theta \colon [0, T) \times \mathbb{T}^3 \to \mathbb{R}$ satisfying

$$\begin{aligned}
\operatorname{div} \mathbf{u} &= 0 && \text{in } (0,T) \times \mathbb{T}^3,\\
u_t + u u_x + v u_y + w u_z + p_x &= 0 && \text{in } (0,T) \times \mathbb{T}^3,\\
v_t + u v_x + v v_y + w v_z + p_y &= 0 && \text{in } (0,T) \times \mathbb{T}^3,\\
p_z &= -\theta && \text{in } (0,T) \times \mathbb{T}^3,\\
\theta_t + u\theta_x + v\theta_y + w\theta_z &= \lambda_1(\theta_{xx} + \theta_{yy}) + \lambda_2 \theta_{zz} && \text{in } (0,T) \times \mathbb{T}^3
\end{aligned}$$

with initial conditions $u(0) = u_0$, $v(0) = v_0$ and $\theta(0) = \theta_0$.
When $\mu_1 = \mu_2 = 0$ we use the term *inviscid primitive equations*.
A special feature: non-deterministic role of w.

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The definition of the weak solution

Definition

We call the quintet of functions (u, v, w, p, θ) a *weak solution of the inviscid primitive equations* if

- $\mathbf{u} = (u, v, w) \in L^2((0,T) \times \mathbb{T}^3; \mathbb{R}^3)$, $u,\ v \in \mathcal{C}([0,T]; L^2_w(\mathbb{T}^3))$, $p \in L^1((0,T) \times \mathbb{T}^3)$, $\partial_z p \in L^1((0,T) \times \mathbb{T}^3)$ and equations and the equalities

$$\int_0^T \int_{\mathbb{T}^3} u \partial_t \phi_1 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^3} u \mathbf{u} \cdot \nabla_{\mathbf{x}} \phi_1 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ - \int_{\mathbb{T}^3} u_0(\cdot) \phi_1(0, \cdot) \,\mathrm{d}\mathbf{x} + \int_0^T \int_{\mathbb{T}^3} p \partial_x \phi_1 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = 0,$$

$$\int_0^T \int_{\mathbb{T}^3} v \partial_t \phi_2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^3} v \mathbf{u} \cdot \nabla_{\mathbf{x}} \phi_2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ - \int_{\mathbb{T}^3} v_0(\cdot) \phi_2(0, \cdot) \,\mathrm{d}\mathbf{x} + \int_0^T \int_{\mathbb{T}^3} p \partial_y \phi_2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = 0$$

  hold for any $\phi_1,\ \phi_2 \in \mathcal{D}([0,T) \times \mathbb{T}^3)$,

The definition of the weak solution

- the incompressibility condition

$$\operatorname{div} \mathbf{u} = 0$$

is satisfied in the sense of distributions on $\mathbb{T}^3$,
- θ is a strong solution of

$$\theta_t + u\theta_x + v\theta_y + w\theta_z = \lambda_1(\theta_{xx} + \theta_{yy}) + \lambda_2\theta_{zz}$$

in $(0, T) \times \mathbb{T}^3$ and $\theta(0, \cdot) = \theta_0(\cdot)$,
- the equation

$$p_z = -\theta$$

holds for the weak derivative of p almost everywhere in $(0, T) \times \mathbb{T}^3$.

Why are they interesting? (for mathematicians)

The **viscid case, three spatial dimensions**:

- Existence of global weak solutions (Navier-Stokes-like theory) - J. L. Lions, R. Temam and S. H. Wang: **On the equations of the large-scale ocean.** In *Nonlinearity* (1992).
- Local in time existence of smooth solutions (the same paper),
- Global in time regularity of solutions for smooth initial conditions - C. Cao, E. S. Titi: **Global well-posedness of the 3D viscous primitive equations of large scale ocean and atmosphere dynamics.** In *Ann. Math* (2007).

Why are they interesting? (for mathematicians)

The **inviscid case**:

- The term *primitive* becomes a bit misleading.
- If we erase the diffusion in the heat equation, the system is not hyperbolic - J. Oliger and A. Sundström: **Theoretical and practical aspects of some initial boundary value problems in fluid dynamics.** In *SIAM J. Appl. Math.*, (1978).
- In 3D, there are (to the best knowledge of the author) *no* a priori estimates for velocities and temperature. In 2D and $\theta \equiv 0$, there exist local in time smooth solutions - Y. Brenier: **Homogeneous hydrostatic flows with convex velocity profiles.** In *Nonlinearity*, (1999).
- Finite time blow-up for some smooth initial data - C. Cao, S. Ibrahim, K. Nakanishi and E. S. Titi: **Finite-time blowup for the inviscid primitive equations of oceanic and atmospheric dynamics.** In *Comm. Math. Phys.*, (2015).

Global existence of weak solutions for the inviscid case[4]

Theorem

Assume that $T > 0$ (arbitrary), λ_1, $\lambda_2 > 0$. Let u_0, $v_0 \in \mathcal{C}(\overline{U})$, $\theta_0 \in \mathcal{C}^2(\overline{U})$ and suppose that there exists $w_0 \in \mathcal{C}(\overline{U})$ such that

$$\operatorname{div}((u_0, v_0, w_0)\chi_U) = 0 \quad \text{in the sense of distributions on } \mathbb{R}^3.$$

Then there are infinitely many weak solutions of the inviscid primitive equations emanating from the initial conditions u_0, v_0, θ_0.

- Canonically, there will be a jump of the kinetic energy at time $t = 0$. If we denote

$$E(t) = \int_U \frac{1}{2}|u(t,x)|^2 + |v(t,x)|^2 + |w(t,x)|^2 \, \mathrm{d}x$$

then

$$\liminf_{t \to 0^+} E(t) > E(0).$$

[4] E. Chiodaroli, M. M.- **Existence and non–uniqueness of global weak solutions to inviscid primitive and Boussinesq equations.** *To appear in Comm. Math. Phys.*

Infinitely many dissipative solutions

Definition

We call solutions dissipative if $E(t) \leq E(s)$ whenever $0 \leq s \leq t$.

Theorem

Assume that $T > 0$, λ_1, $\lambda_2 > 0$ *and* $\theta_0 \in \mathcal{C}^2(\overline{U})$. *Then there exist* u_0, $v_0 \in L^\infty(U)$ *for which we can find infinitely many weak dissipative solutions of the inviscid primitive equations emanating from the initial data* u_0, v_0, θ_0.

- **What is the main technique which can be used to proof the theorems?**
- *Convex integration.*

The curious case of De Lellis and Székelyhidi

Theorem (De Lellis and Székelyhidi, 2011)

Let $\bar{e} \in C((0,T) \times \mathbb{T}^3) \cap C([0,T]; L^1(\mathbb{T}^3))$ *be positive in* $(0,T) \times \mathbb{T}^3$. *Then there exist infinitely many weak solutions* $\mathbf{u}$ *of the Euler equations*

$$\begin{aligned} \operatorname{div} \mathbf{u} &= 0 \quad \textit{in the sense of distributions,} \\ \partial_t \mathbf{u} + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla p &= 0 \quad \textit{in the sense of distributions,} \end{aligned}$$

with pressure $p = -\frac{1}{3}|\mathbf{u}|^2$ *such that* $\mathbf{u} \in C([0,T]; L^2_{weak}(\mathbb{T}^3))$, $\mathbf{u}(0,x) = 0$ *for* $t = 0, T$ *a. e.* $x \in \mathbb{T}^3$,

$$\frac{1}{2}|\mathbf{u}(t,x)|^2 = \bar{e}(t,x) \quad \textit{for every } t \in (0,T) \textit{ a. e. } x \in \mathbb{T}^3.$$

A generalization for an abstract Euler system

Observation (E. Feireisl, 2015)

Let $\mathbb{H}\colon C([0,T]; L^2_{weak}(\mathbb{T}^3)) \to C([0,T]\times\mathbb{T}^3; \mathbb{R}^{3\times 3}_{0,sym})$,

$$\Pi \in C([0,T]; L^2_{weak}(\mathbb{T}^3)) \to C([0,T]\times\mathbb{T}^3)$$

be bounded and continuous operators satisfying some additional technical assumptions and assume that $\Pi[\mathbf{u}]$ is bounded independently on $\mathbf{u}$. Then there exist infinitely many weak solutions $\mathbf{u}$ of the following abstract version of the Euler system

$$\begin{aligned} \operatorname{div}\mathbf{u} &= 0 \quad \text{in the sense of distributions,} \\ \partial_t\mathbf{u} + \operatorname{div}(\mathbf{u}\otimes\mathbf{u} + \mathbb{H}[\mathbf{u}]) + \nabla\Pi[\mathbf{u}] &= 0 \quad \text{in the sense of distributions,} \end{aligned}$$

such that $\mathbf{u} \in C([0,T]; L^2_{weak}(\mathbb{T}^3))$, $\mathbf{u}(0,x) = 0$ for $t = 0, T$ a. e. $x \in \mathbb{T}^3$.

The primitive equations as a differential inclusion

- We would like to apply the machinery of convex integration. The first step is to recast the primitive equations into the form

$$\begin{aligned} \operatorname{div} \mathbf{u} &= 0, \\ \partial_t \mathbf{u} + \operatorname{div} (\mathbf{u} \otimes \mathbf{u} + \mathbb{H}[\mathbf{u}]) + \nabla \Pi[\mathbf{u}] &= 0. \end{aligned}$$

- The main problem - the equation for the third component of the velocity is degenerated.

Inviscid primitive equations

Let us take the primitive equations

$$
\begin{aligned}
u_x + v_y + w_z &= 0 \\
u_t + uu_x + vu_y + wu_z + p_x &= 0, \\
v_t + uv_x + vv_y + wv_z + p_y &= 0, \\
p_z &= \theta \\
\theta_t + u\theta_x + v\theta_y + w\theta_z &= \lambda_1(\theta_{xx} + \theta_{yy}) + \lambda_2\theta_{zz}
\end{aligned}
$$

Extended inviscid primitive equations

...and supplement it by an extra equation

$$
\begin{aligned}
u_x + v_y + w_z &= 0 \\
u_t + uu_x + vu_y + wu_z + p_x &= 0, \\
v_t + uv_x + vv_y + wv_z + p_y &= 0, \\
w_t + uw_x + vw_y + ww_z + p_z &= 0, \\
p_z &= \theta \\
\theta_t + u\theta_x + v\theta_y + w\theta_z &= \lambda_1(\theta_{xx} + \theta_{yy}) + \lambda_2\theta_{zz}
\end{aligned}
$$

Extended inviscid primitive equations

Let $\theta = \Theta[\mathbf{u}]$ be the solving operator for the convection diffusion equaiton, then:

$$
\begin{aligned}
\operatorname{div} \mathbf{u} &= 0 \\
\mathbf{u}_t + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla p &= 0 \\
p_z &= \Theta[\mathbf{u}] \\
\Theta[\mathbf{u}]_t + \mathbf{u} \cdot \nabla \Theta[\mathbf{u}] &= \lambda_1 (\Theta[\mathbf{u}]_{xx} + \Theta[\mathbf{u}]_{yy}) + \lambda_2 \Theta[\mathbf{u}]_{zz}
\end{aligned}
$$

Extended inviscid primitive equations

Then we can find a solving operator for the equation for p_z by taking

$$\Pi[u](t,x,y,z) \approx \int_{z_0}^{z} \Theta[\mathbf{u}](t,x,y,s)\,\mathrm{d}s.$$

Hence,

$$\begin{aligned} \operatorname{div}\mathbf{u} &= 0 \\ \mathbf{u}_t + \operatorname{div}(\mathbf{u}\otimes\mathbf{u}) + \nabla\Pi[\mathbf{u}] &= 0 \\ p_z &= \Theta[\mathbf{u}]. \end{aligned}$$