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ON SOME PROBLEMS IN DISCRETE DYNAMICS (CYCLES, CHAOS, TOPOLOGICAL ENTROPY, MINIMALITY)

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Contents

1 INTRODUCTION

The dissertation falls within the theory of dynamical system. Recall that this theory is currently one of the important areas of mathematics, born with H. Poincaré's *"Les méthodes nouvelles de la mécanique céleste"* at the end of XIX century. Since 2000 it is given a separate heading in the Mathematical Reviews. Owing to its universal character, the theory uses methods from various branches of mathematical science (analysis, topology, algebra, geometry, ...). It has arisen from an attempt at an adequate description of phenomena in the surrounding world. Therefore it traditionally plays the role of the theoretical basis for various models in physics, biology, economics, etc. Nevertheless, at present also conversely, problems posed in the theory of dynamical systems penetrate other mathematical theories, giving them a fresh impulse, serving as a tool for solving complex problems within these theories and also opening completely new problems.

1.1 Structure of the dissertation

The dissertation consists of the following six papers and a commentary showing what is their role in the development of the theory of dynamical systems:

[1] Ľ. Snoha, *Characterization of potentially minimal periodic orbits of continuous mappings of an interval*, Acta Math. Univ. Comenianae **LII-LIII** (1987), 111-124.

[2] V. Jiménez López, Ľ. Snoha, *There are no piecewise linear maps of type* 2^∞, Trans. Amer. Math. Soc. **49** (1997), 1377-1387.

[3] V. Jiménez López, Ľ. Snoha, *All maps of type* 2^∞ *are boundary maps*, Proc. Amer. Math. Soc. **125** (1997), 1667-1673.

[4] Ľ. Snoha, *Generic chaos*, Comment. Math. Univ. Carolinae **31** (1990), 793 – 810.

[5] S. Kolyada, Ľ. Snoha, *Topological entropy of nonautonomous dynamical systems*, Random Comput. Dynamics **4(2 & 3)** (1996), 205–233.

[6] S. Kolyada, Ľ. Snoha, S. Trofimchuk, *Noninvertible minimal maps*, Fund. Math. **168** (2001), 141–163.

The criteria for the choice of the papers were twofold. On one hand, the papers have been chosen to cover the main present or past topics of interest of the author. On the other hand, the papers have been chosen either because of its influence on further development of the theory (papers [5], [6] and in an indirect way also [1]) or because, according to the author's opinion, they belong to his best papers ([1], [2],

[4], [6]). The paper [3] is an exception — it does not have many citations but has been chosen because it is very closely related to [2] and presents the solution of one of very few open problems formulated in the influential monograph *L. S. Block, W. A. Coppel, Dynamics in one dimension, LNM 1513, Springer-Verlag 1992.*

The four main sections of the dissertation (with those of the above papers which are covered by them) are: 'Cycles' ([1], [2], [3]), 'Chaos' ([4]), 'Topological entropy' ([5]) and 'Minimality' ([6]).

The number of papers is restricted to six in order not to exceed a reasonable length of the dissertation.

1.2 The aim of the dissertation and methods used in it

The dissertation covers some topics from combinatorial dynamics and some topics from topological dynamics. Its aim is twofold:

1. To solve some *concrete problems* which naturally appeared (or were even explicitely formulated by other authors) in the theory of dynamical systems, for instance to characterize all objects with a property explicitly or implicitly introduced by other authors or to answer questions posed by other authors. Of this type are all main problems solved in sections 'Cycles' and 'Chaos'.

2. To enrich the theory of dynamical systems by developing a *new part of the theory* (see the theory of topological entropy of nonautonomous dynamical systems in section 'Topological entropy') or by studying *new properties of old objects* (in section 'Minimality' we study for instance the topological properties of noninvertible minimal maps — the topic completely ignored before but quite popular and useful afterwards).

The methods we use are therefore combinatorial and topological, very often they are ad hoc methods. The dissertation is a mixture of theoretical results and constructions of examples and counterexamples, what is in fact quite typical for dynamical systems theory.

2 CYCLES

2.1 Starting point of the research

The notion of a cycle (periodic orbit) is basic in combinatorial dynamics. This part of the theory of dynamical systems has its roots in the Sharkovskiĭ Theorem describing possible sets of periods of periodic orbits of continuous selfmaps of an interval.

Consider the Sharkovskiĭ ordering of the set $\mathbb{N} \cup \{2^\infty\}$:

$$3 \succ 5 \succ 7 \succ \cdots \succ 2 \cdot 3 \succ 2 \cdot 5 \succ 2 \cdot 7 \succ \cdots \succ 4 \cdot 3 \succ 4 \cdot 5 \succ 4 \cdot 7 \succ \cdots \succ \cdots$$

$$\succ 2^n \cdot 3 \succ 2^n \cdot 5 \succ 2^n \cdot 7 \succ \cdots \succ \cdots \succ 2^\infty \succ \cdots \succ 2^n \succ \cdots \succ 4 \succ 2 \succ 1.$$

We will also use the symbol $\succeq$ in the natural way. For $t \in \mathbb{N} \cup \{2^\infty\}$ we denote by $S(t)$ the set $\{k \in \mathbb{N} : t \succeq k\}$ ($S(2^\infty)$ stands for the set $\{1, 2, 4, \ldots, 2^k, \ldots\}$). Let $C(I)$ be the set of all continuous selfmaps of a real compact interval I. Let $\mathrm{Per}(f)$ be the set of periods of all periodic points of a map f.

Theorem 1 (Sharkovskiĭ Theorem [81], [82]). *For every $f \in C(I)$ there exists a $t \in \mathbb{N} \cup \{2^\infty\}$ with $\mathrm{Per}(f) = S(t)$. On the other hand, for every $t \in \mathbb{N} \cup \{2^\infty\}$ there exists an $f \in C(I)$ with $\mathrm{Per}(f) = S(t)$.*

If $\mathrm{Per}(f) = S(t)$, then f is called to be *of type t*. When speaking of types we consider them to be ordered by the Sharkovskiĭ ordering. So if a map f is of type 2^∞ or greater than 2^∞ or less than 2^∞ then, respectively, $\mathrm{Per}(f) = \{1, 2, \ldots, 2^k, \ldots\}$ or f has a periodic point with period not a power of 2 or $\mathrm{Per}(f) = \{1, 2, \ldots, 2^n\}$ for some $n \in \mathbb{N} \cup \{0\}$.

Thus, if f has a cycle P of period n then it has also cycles of all periods $k \in \mathbb{N}$ with $n \succ k$ and it may or may not have a cycle with period $m \succ n$. Usually f does have also a cycle with such a period m. In fact, if $n > 3$ then most of the cyclic permutations describing how f may work on P are such that if f on P works in accordance with such a permutation then f necessarily has also a cycle with some period $m \succ n$.

In what follows we use the terminology from [87].

Definition 2. A periodic orbit P of f of period n is said to be a potentially minimal periodic orbit (shortly PMPO), if there exists a function $F \in C(I)$ such that $f|_P = F|_P$ and F has no periodic orbit of period $m \succ n$.

In 1980's there was an open problem of characterizing potentially minimal periodic orbits. Of course, all periodic orbits of periods 1, 2 and 3 are potentially minimal. Štefan [94] characterized PMPO of odd periods:

Theorem 3 (Štefan [94]). *Let P be a periodic orbit of $f \in C(I)$ of period $2p+1$, $p \in \mathbb{N}$. Then P is potentially minimal if and only if there is a point $b_1 \in P$ such that*

$$b_{2p+1} < b_{2p-1} < \cdots < b_3 < b_1 < b_2 < \cdots < b_{2p-2} < b_{2p}$$

or

$$b_{2p} < b_{2p-2} < \cdots < b_2 < b_1 < b_3 < \cdots < b_{2p-1} < b_{2p+1} \ ,$$

where, for every $1 \le i \le 2p$*,* $f(b_i) = b_{i+1}$ *and* $f(b_{2p+1}) = b_1$*.*

Another result on PMPO was implicitly contained in [20]: If P is a PMPO of $f \in C(I)$ whose period is a power of two then P is simple. For further use in this dissertation, let us give the definition of a simple orbit generalized to any even period.

Definition 4 (for $p = 0$ see [20]). Let P be a periodic orbit of $f \in C(I)$ of period $2^k \cdot (2p+1)$, $k \in \mathbb{N}$, $p \in \{0\} \cup \mathbb{N}$. Then P is said to be simple if for every positive integers n, r with $2^k \cdot (2p+1) = n \cdot r$ and $r = 2^s$ for some $s \in \{0, 1, 2, \dots, k-1\}$, and for every periodic orbit $\{q_1 < q_2 < \cdots < q_n\} \subset P$ of f^r it holds

$$f^r(\{q_1, \dots, q_{n/2}\}) = \{q_{n/2+1}, \dots, q_n\} \ .$$

In this situation, the problem of characterizing PMPO of even periods called for a solution.

The importance of cycles is not restricted to combinatorial dynamics. They play an important role also in topological dynamics. For instance, on the interval it turns out that the Sharkovskiĭ theorem allows us to organize a classification of the maps from $C(I)$ in terms of their dynamical complexity. Maps of type 2^n, $n = 0, 1, 2, \dots$, are simple: all their points are asymptotically periodic. On the other hand, maps of type greater than 2^∞ have a very complicated dynamics. For instance, they have positive topological entropy and are chaotic in the sense of Li and Yorke. Maps of type 2^∞ are located somewhere between these two groups. They always have zero topological entropy but there are examples of maps F, G of type 2^∞ respectively having only asymptotically periodic points [33] and chaotic in the sense of Li and Yorke [84], [72].

Let $\mathbb{R}$ be the real line and $J \subset \mathbb{R}$ a (not necessarily compact) interval. We say that a continuous map $f : J \to \mathbb{R}$ is *piecewise monotone* (resp. *piecewise linear*) if there are points $\inf I = a_0 < a_1 < \cdots < a_n = \sup I$ such that for every $k \in \{1, 2, \dots, n\}$, the restriction of f to the interval (a_{k-1}, a_k) is (not necessarily strictly) monotone (resp. linear and non-constant). Note that, according to the definitions, piecewise monotone maps can have constant pieces, while piecewise linear maps cannot. In the case of piecewise linear maps, notice also that the consecutive linear pieces need not be alternatively increasing and decreasing.

One could wonder whether the converse of the Sharkovskiĭ theorem holds for piecewise monotone (or even polynomial) maps from $C(I)$. It is well known that the answer is affirmative. For instance, the classical logistic family $\{F_\lambda\}_{\lambda \in [0,4]}$ defined by $F_\lambda(x) = \lambda x(1-x)$ contains examples of maps of all types in the Sharkovskiĭ ordering (see [44]). On the other hand, compare this with the family of "tent" maps $\{G_\mu\}_{\mu \in [0,1]}$ defined by $G_\mu(x) = \mu(1 - |2x-1|)$. The map G_μ is of type greater than 2^∞ if $\mu > 1/2$ but G_μ is of type 1 for any $\mu \leqslant 1/2$ (see, e.g., [33]). So, a question remains: does the converse of the Sharkovskiĭ theorem hold for piecewise linear maps? Examples of piecewise linear maps of all types except of type 2^∞

are well known (see e.g. Corollary 2.2.9 in [7]) and the map F from [33] of type 2^∞ mentioned earlier consists of an infinite number of non-constant linear pieces. Further, the map $H \in C([0,1])$ defined by $H(x) = \min\{\kappa, G_1(x)\}$, $\kappa \approx 0.8249...$ is of type 2^∞ (see [72]) but it has a constant piece and so it is not piecewise linear. No examples of piecewise linear maps of type 2^∞ were known.

So, in the described situation it was a challenging question whether piecewise linear maps of type 2^∞ exist at all. (The question is closely related to the problem of the existence of so called solenoids for piecewise linear maps, see below.)

Very often, maps of type 2^∞ are most difficult to handle with. In many aspects, they have a special position in $C(I)$. Recall that it is very easy to see that any neighborhood of any map f contains maps of types greater than 2^∞ (even maps of type 3). Contrary to the maps of types greater than 2^∞, the maps of types less than 2^∞ do not form a dense set in the space $C(I)$ endowed with the supremum metric. In fact, this set is nowhere dense in $C(I)$. To see this use the following

Theorem 5 (Block's Theorem on stability [21]). *Let $f \in C(I)$ and let $n \in \mathrm{Per}(f)$. Then there exists a neighborhood $U(f,n)$ of f such that for all $g \in U(f,n)$ we have $\mathrm{Per}(g) \supset S(n) \setminus \{n\}$.*

So, if f is of type greater than 2^∞, there is a neighbourhood of f containing no map of type less than (or equal to) 2^∞. L. S. Block and W. A. Coppel (see [22], the end of chapter II.4) posed a question whether any neighbourhood of any map of type 2^∞ contains a map of type less than 2^∞.

2.2 Main results of the dissertation ([87], [53], [54])

In [87], we have solved the problem of characterizing potentially minimal periodic orbits (PMPO) of even periods in the following three theorems.

Theorem 6 ([87]). *Let P be a periodic orbit of $f \in C(I)$ of period 2^k, $k \in \mathbb{N}$. Then P is potentially minimal if and only if P is simple.*

Theorem 7 ([87]). *Let P be a periodic orbit of $f \in C(I)$ of period $3 \cdot 2^k$, $k \in \mathbb{N}$. Then P is potentially minimal if and only if P is simple.*

Let $P = \{a_1 < a_2 < \cdots < a_m\}$ be a periodic orbit of f of period m. Let n divides m. For $k = 1, 2, \ldots, \frac{m}{n}$ write

$$P(n,k) = \{a_i : \ i = (k-1)n + 1, (k-1)n + 2, \ldots, kn\} \ .$$

Now let a, b be real numbers. Instead of $f(a) = b$ we will also use the notation $a \xrightarrow{f} b$. Let $E = \{e_1, e_2, \ldots, e_r\} \subset \mathbb{R}$. We will write $f \updownarrow E$ if there is a permutation $(\alpha(1), \alpha(2), \ldots, \alpha(r))$ of the set $\{1, 2, \ldots, r\}$ such that

$$e_{\alpha(1)} \xrightarrow{f} e_{\alpha(2)} \xrightarrow{f} \ldots \xrightarrow{f} e_{\alpha(r)} \ .$$

Theorem 8 ([87]). *Let P be a periodic orbit of $f \in C(I)$ of period $2^k \cdot (2p+1)$, where $k \in \mathbb{N}$, $p \in \mathbb{N}$ and $p \geq 2$. Let*

$$E = \bigcup_{j=1}^{2^k} \{\min P(2p+1, j), \max P(2p+1, j)\} \ .$$

Consider the following four conditions:

(C1) *P is simple,*

(C2) *the sets $P(2p+1, j)$, $j = 1, 2, 3, \ldots, 2^k$ are potentially minimal periodic orbits of the function f^{2^k} (see Theorem 3),*

(C3-a) *$f \updownarrow E$,*

(C3-b) *f is monotone on each of the sets $P(2p+1, j)$, $j = 1, 2, 3, \ldots, 2^k$, except of one of them.*

Then the following three conditions are equivalent:

(i) *P is potentially minimal,*

(ii) *(C1) and (C2) and (C3-a),*

(iii) *(C-1) and (C-2) and (C3-b).*

In the condition (C3-b) it is not important whether we understand it in the sense that f is monotone on exactly $2^k - 1$ of the mentioned sets or in the sense that it is monotone on at least $2^k - 1$ of them. Due to the fact that P is a cycle, the set E cannot be f-invariant and so both the formulations are equivalent.

The problem of (non)existence of piecewise linear maps of type 2^∞ was solved in [53].

Given a map $f \in C(I)$, we say that a sequence $(I_i)_{i=0}^{k-1}$ of closed subintervals of I is periodic of period k for f if the intervals I_i have pairwise disjoint interiors, $f(I_i) \subset I_{i+1}$ for any $i = 0, 1, \ldots, k-2$ and $f(I_{k-1}) \subset I_0$. We say that $A \subset I$ is a *solenoid* of f if there exist a strictly increasing sequence $(k_n)_{n=1}^\infty$ of positive integers and periodic sequences $\mathcal{C}^n = (I_i^n)_{i=0}^{k_n-1}$ of period k_n of closed intervals such that $\bigcup_{i=0}^{k_n-1} I_i^n \supset \bigcup_{i=0}^{k_{n+1}-1} I_i^{n+1}$ for any n and $A = \bigcap_{n=1}^\infty \bigcup_{i=0}^{k_n-1} I_i^n$. The sequence $(\mathcal{C}^n)_{n=1}^\infty$ will be called a *covering* of A of *type* $(k_n)_{n=1}^\infty$.

It is not difficult to show that for any $J \in \mathcal{C}^{n+1}$ there is $K \in \mathcal{C}^n$ such that $J \subset K$. Moreover, k_{n+1} divides k_n for any n and each interval from $\mathcal{C}^n$ contains exactly $\frac{k_{n+1}}{k_n}$ intervals from $\mathcal{C}^{n+1}$. If a solenoid A admits a covering of type $(k_n)_{n=1}^\infty$ with $\frac{k_{n+1}}{k_n} = 2$ for any n large enough then we call A a *doubling period* solenoid.

The following proposition is a part of folklore knowledge. Its short elementary proof can be found in [53].

Proposition 9. *If $f \in C(I)$ is a piecewise monotone map of type 2^∞ then it has a doubling period solenoid.*

The following theorem is the key result which enabled to prove the nonexistence of piecewise linear maps of type 2^∞.

Theorem 10 ([53]). *If $f \in C(I)$ is piecewise linear then it has no doubling period solenoids.*

From this theorem and the previous proposition we immediately get the desired result:

Corollary 11 ([53]). *There are no piecewise linear maps of type 2^∞ in $C(I)$.*

The Block's and Coppel's problem mentioned in Subsection 2.1 was answered in the affirmative in [54]:

Theorem 12 ([54]). *Let $f \in C(I)$ be of type 2^∞. Then every neighbourhood of the map f contains a piecewise monotone map of type less than 2^∞.*

2.3 Remarks and possibilities for further research

The author presented the characterization of potentially minimal periodic orbits of continuous mappings of an interval (see Theorems 6,7,8) at the 1st Czechoslovak Summer School on Dynamical Systems held at Račková Dolina, Czechoslovakia, in June 1984 (partial results even sooner, at the International Conference on Real Functions held in Bydgoszcz, Poland, in August 1983 (see [88])). A. N. Sharkovskiĭ, one of the participants, confirmed that the results were new. Later it turned out that some other mathematicians also were working on the problem. The problem was independently solved in [46] and [8] and a similar problem was solved in [34]. Some six years later, the characterization was refound by R. G. Rakhmankulov (unpublished (?)). I guess that today, rather paradoxically (?), the paper [8] has dozens of citations, while the other papers where the problem was solved have hardly any of them.

Let us also remark that the conditions (C3-a) and (ii) from Theorem 8 were overlooked in [8].

These results belong to the core of combinatorial dynamics. In fact, in 1993 the first and in 2000 the second edition of the book [7] appeared. One of the *central* topics treated in the book is the characterization of so called primary cycles and one can easily show that for cycles whose period is not of the form 3×2^m for some $m > 0$, to be primary is the same as to be potentially minimal (see also the small print after Corollary 2.11.2 in [7]).

In [89], it is studied the problem of finding possible types of maps having a cycle of a given period that is not potentially minimal.

Since 1980's, the combinatorial dynamics has developed very much. Many papers appeared where the authors study the forcing relation between cycles, the entropy of cycles etc. (and not only on the interval). In Czech Republic J. Bobok has achieved deep results in this area. We refer the reader to [7, 2nd. edition] as a general reference.

There are attempts to extend results from the interval to graphs. Results for the circle can be found for instance in [7]. The case of general graphs is very difficult.

On one hand, Sharkovskiĭ-type theorem strongly depends on how the graph looks like and, on the other hand, it is difficult to carry over even the notion of a pattern from the interval to a graph. For more details, see [7, 2nd. edition] and [2].

Solenoids are important in one-dimensional discrete dynamical systems because they play a key role in the description of the asymptotic behaviour of a large class of continuous selfmaps of a real compact interval. Namely, if $f \in C(I)$ is piecewise strictly monotone and smooth enough then there is a residual set R such that for any $x \in R$ the set of limit points of the sequence $(f^n(x))_{n=0}^{\infty}$ is either a periodic orbit, a finite union of closed intervals or a solenoid. Regarding to this, the reader may for example wish to see [77], [70].

Thus, it is important to know whether a considered class of maps admits solenoids at all. In the particular case of piecewise linear maps the nonexistence of period doubling solenoids was proved in Theorem 10. Later this result was strengthened to all solenoids [69], [3]. Finally it was proved in [3] that piecewise smooth maps (with a finite number of pieces of monotonicity) whose derivative is Lipschitz continuous and nowhere vanishing may have only solenoids which are doubling period. On the other hand, there exist examples of such maps having doubling period solenoids and, moreover, with the corresponding smooth pieces in the class C^∞ (see [53]). Such a map may be of type 2^∞.

One would expect that our result on the nonexistence of maps of type 2^∞ in the class of piecewise linear maps could be extended to larger classes of maps. However, in view of the examples from the previous paragraph, it is difficult to find such a class of maps. In particular, it is not clear whether the existence of piecewise analytic maps (with non-vanishing derivatives) of type 2^∞ should be reasonably expected or not.

In connection with solenoids, examples by Bobok and Kuchta [25] of an expanding countably piecewise linear map with a doubling period solenoid and by Misiurewicz [71] of a countably piecewise linear map with a doubling period solenoid of positive Lebesgue measure are also worth noticing.

Kolyada in [57] proved that if $f \in C(I)$ is such that for some $c \in I$, $f(x) = \frac{a_1x+b_1}{c_1x+d_1}$ for any $x \leq c$ and $f(x) = \frac{a_2x+b_2}{c_2x+d_2}$ for any $x \geq c$ with $a_id_i - b_ic_i \neq 0$ for $i = 1, 2$, then it has no doubling period solenoids. Denote the set of all such maps by $K_2(I)$ (the index 2 indicates that these maps consist of two pieces). Since $K_2(I)$ is included in the class of piecewise smooth maps with nowhere vanishing Lipschitz continuous derivative, the maps in $K_2(I)$ have no solenoids. We conjecture that this result can be extended to similar maps with an arbitrary (finite) number of pieces of monotonicity. This is an open problem.

Notice that the fact that piecewise smooth maps with nowhere vanishing Lipschitz continuous derivative have no solenoids implies that for any positive integer l the family of l-modal maps from the mentioned class cannot be full. In [41] it is shown that somewhat different but related families of piecewise smooth maps cannot be full either.

Concerning other papers of the author of the disseration, let us mention that [52] is closely related to [53] and [54]. Also [43] has a close connection with [54].

Nevertheless, at present the author is not active in combinatorial dynamics since he prefers the study of chaos, entropy and minimality.

3 CHAOS

3.1 Starting point of the research

The notion of *chaos* in connection with a map was first used by Li and Yorke [66] in 1975 although without giving any formal definition. Nevertheless, the following definition was implicitly contained in their paper: A map $f \in C(I)$ is *chaotic in the sense of Li and Yorke* if there exists an uncountable set S such that for every $x, y \in S$, $x \neq y$, and every periodic point p of f we have

(i) $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$,

(ii) $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$,

(iii) $\limsup_{n\to\infty} |f^n(x) - f^n(p)| > 0$.

Since the set S cannot contain more than one asymptotically periodic point, i.e., a point for which (iii) is not satisfied, the condition (iii) is redundant in the definition. Any set satisfying (i) and (ii) is called a *scrambled set*. Thus, f is chaotic in the sense of Li and Yorke if it has an uncountable scrambled set. A pair of points satisfying (i) and (ii) is called a Li-Yorke pair.

Several equivalent conditions with the chaos in the sense of Li and Yorke on the interval were found in [49]. The notion of Li-Yorke chaos fits well with Sharkovskiĭ's ordering. All maps of type greater than 2^∞ are chaotic, all maps of type smaller than 2^∞ are non-chaotic and in the family of maps of type 2^∞ there are chaotic as well as non-chaotic maps (see [86] and [72]).

The notion of a Li-Yorke chaos, implicitly contained in [66] in the setting of interval dynamical systems, may seem to be strange and in fact it was an object of a serious criticism. Two problems are here. The first objection is that chaos in this sense may not be 'physically' observable. Still, the Li and Yorke's idea of defining chaos has a good sense at least on the interval because it turns out to be the minimal requirement for a continuous selfmap of an interval to be 'complex'. This was shown by Smítal who proved in [86] that any interval map satisfies one of the following two mutually exclusive properties:

(i) f is Li-Yorke chaotic;

(ii) all trajectories of f are *approximable by cycles*, that is, for any x and any $\varepsilon > 0$ there is a periodic point p such that $\limsup_{n\to\infty} |f^n(x) - f^n(p)| < \varepsilon$.

The second, more formal, objection against the definition of Li-Yorke chaos is: 'why uncountable?' and not, say, 'infinite' or 'topologically large' or something else? Later Kuchta and Smítal [65] proved that if an *interval* map has a scrambled set with two points then it has also an uncountable, in fact Cantor, scrambled set. Though this result is not true for *general* systems, it is a good reason for studying the size of scrambled sets (see [84], [85] and [29]) and for modifications of the definition of chaos with the notion of Li-Yorke pairs kept in mind.

One of such definitions, much stronger but rather similar to that of Li and Yorke was proposed by A. Lasota (see [75]). Again, similarly as the Li-Yorke chaos, originally it was introduced for interval maps. A map $f \in C(I)$ is *generically chaotic* if the set of Li-Yorke pairs for f is residual in I^2 (i.e., its complement is a first category set in I^2). J. Piórek [75] in 1985 found examples of generically chaotic interval maps, so it became clear that maps satisfying such a strong definition of chaoticity exist.

Of course, it was natural to ask whether it is possible to find a reasonable characterization of generically chaotic maps on the interval.

3.2 Main results of the dissertation ([90])

In this subsection, a function will always be a function belonging to the space $C(I)$ of all continuous maps of a real compact interval I into itself, endowed with the topology of uniform convergence. For a function f and $\varepsilon > 0$ define the following planar sets:

$$C_1(f) = \Big\{[x,y] \in I^2 : \liminf_{n\to\infty} |f^n x - f^n y| = 0\Big\},$$

$$C_2(f) = \Big\{[x,y] \in I^2 : \limsup_{n\to\infty} |f^n x - f^n y| > 0\Big\},$$

$$C_2(f,\varepsilon) = \Big\{[x,y] \in I^2 : \limsup_{n\to\infty} |f^n x - f^n y| > \varepsilon\Big\},$$

$$C(f) = C_1(f) \cap C_2(f),$$

$$C(f,\varepsilon) = C_1(f) \cap C_2(f,\varepsilon) \ .$$

We say that f is *generically* or *densely chaotic* if the set $C(f)$ is residual or dense in I^2, respectively. Similarly, f is *generically* or *densely ε-chaotic* if the set $C(f,\varepsilon)$ is residual or dense in I^2, respectively. (The definition of generic chaos, due to A. Lasota, appeared for the first time in [75], the others in our paper [90]).

A function $f \in C(I)$ is *topologically transitive* if for any nonempty open sets $U, V \subset I$ there is a nonnegative integer n with $f^n(U) \cap V \neq \emptyset$. In the rest of this subsection, an interval will always be a nondegenerate interval lying in I. It will not necessarily be compact. If J is an interval then $\operatorname{diam} J$ is its length. If $A, B \subset I$ then $\operatorname{dist}(A,B) = \inf\{|x-y| : x \in A, y \in B\}$. We write $\operatorname{dist}(A,b)$ instead of $\operatorname{dist}(A,\{b\})$. A compact interval J will be called an *invariant transitive interval* of f if it is f-invariant (i.e., $f(J) \subset J$) and the restriction of f to the interval J is topologically transitive. For any set $A \subset I$, $\operatorname{int} A$ is the interior of A and $\operatorname{Orb}(f,A) = \bigcup_{n=0}^{\infty} f^n(A)$.

In [90] we have characterized generically chaotic maps on the interval in terms of behaviour of subintervals of I under iterates of f and also in terms of topological transitivity:

Theorem 13 ([90]). *Let $f \in C(I)$. The following conditions are equivalent:*

(a) f is generically chaotic,

(b) *for some* $\varepsilon > 0$, f *is generically* ε*-chaotic,*

(c) *for some* $\varepsilon > 0$, f *is densely* ε*-chaotic,*

(d) $C_1(f)$ *is dense in* I^2 *and* $C_2(f)$ *is a second Baire category set in any interval* $J^2 \subset I^2$,

(e) $C_1(f)$ *is dense in* I^2 *and for some* $\varepsilon > 0$, $C_2(f,\varepsilon)$ *is dense in* I^2,

(f) *the following two conditions are fulfilled simultaneously:*

(f-1) *for every two intervals* J_1, J_2, $\liminf_{n\to\infty} \operatorname{dist}(f^n(J_1), f^n(J_2)) = 0$,

(f-2) *there exists an* $a > 0$ *such that for every interval* J, $\limsup_{n\to\infty} \operatorname{diam} f^n(J) > a$,

(g) *the following two conditions are fulfilled simultaneously:*

(g-1) *there exists a fixed point* x_0 *of* f *such that for every interval* J, $\lim_{n\to\infty} \operatorname{dist}(f^n(J), x_0) = 0$,

(g-2) *there exists a* $b > 0$ *such that for every interval* J, $\liminf_{n\to\infty} \operatorname{diam} f^n(J) > b$,

(h) *the following two conditions are fulfilled simultaneously:*

(h-1) f *is not constant in any subinterval of* I [1], *and has a unique invariant transitive interval or two invariant transitive intervals having one point in common,*

(h-2) *for every interval* J *there is an invariant transitive interval* T *of* f *such that* $\operatorname{Orb}(f, J) \cap \operatorname{int} T \neq \emptyset$.

Moreover, the equivalences (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (e) $\Leftrightarrow$ (f) *hold with the same* ε *and with* $a = \varepsilon$ *in (f-2).*

Using this characterization, it is easy to prove the following three theorems:

Theorem 14 ([90]). *In the space* $C(I)$, *for every* $0 < \varepsilon < \operatorname{diam} I$ *the number* $(1/2)\log 2$ *is the minimum of the topologicaly entropies of all generically (or, equivalently, densely)* ε*-chaotic functions (and hence also of all generically chaotic functions).*

Theorem 15 ([90]). *Let* $f \in C(I)$ *be generically chaotic. Then* f *has a periodic orbit of period* 2×3 *and may or may not have periodic orbits of odd periods greater than* 1.

Theorem 16 ([90]). *The set of all generically chaotic functions is dense in itself but is nowhere dense in* $C(I)$. *The same is true for densely chaotic functions and also for generically (or, equivalently, densely)* ε*-chaotic functions.*

[1] there is a misprint in [90] — these words are missing there, see also the footnote in [74].

3.3 Remarks and possibilities for further research

In [91], we have found also a full characterization of densely chaotic interval maps and proved that in the class of piecewise monotone maps with finite number of pieces of monotonicity (in fact, in a bit larger class) the notion of generic chaos and that of dense chaos coincide. Recently, in [78], S. Ruette showed that the same is true if we replace piecewise monotonicity by continuous differentiability. Moreover, also some open problems posed in [90] and [91] were solved in [78] and [79].

In [92] we have generalized some results from [90] to what we call two-parameter chaos (still in the interval case).

The definitions of generic and dense chaos (ε-chaos) can be carried over in an obvious way to metric spaces. A possibility to extend some results from [90] to metric spaces was indicated in a concluding remark in [91]. Recently E. Murinová [74] showed that the characterization of generically ε-chaotic maps given in Theorem 13 for interval maps can be carried over to continuous selfmaps of a large class of metric spaces. While on the interval generic chaos implies generic ε-chaos for some $\varepsilon > 0$, in [74] an example of a convex continuum in the plane is given on which generic chaos does not imply generic ε-chaos for any $\varepsilon > 0$. Still, it would be interesting to try to extend the result 'generic chaos implies generic ε-chaos for some $\varepsilon > 0$' to as large class of spaces as possible (E. Murinová works on this problem).

Though the paper [90] (together with a personal discussion) was an inspiration for [74], [51] and partially also for [50] and [78], we dare to say that the paper is perhaps less known than it deserves in our opinion. For instance, in spite of a close connection with [47], it is not quoted there. Further, in [18] the authors give, on a disconnected space (symbolic dynamics is used), an example of a topologically transitive, generically chaotic system which is not weakly mixing. In our paper [90] they could find much simpler example and even on the interval (a piecewise linear map f on $[0,1]$ with three pieces of linearity such that $f(0) = f(1/2) = 1/2$, $f(1/4) = 1$, $f(1) = 0$, f linear in between; see Example 3.7 in [90]).

Finally, recall that today there are various definitions of what it means for a map to be chaotic, some of them working reasonably only in special phase spaces. Though one could say that 'as many authors, as many definitions of chaos' (but also: 'chaos is what everybody speaks about but nobody knows what it is'), behind these definitions there is very often the same idea of unpredictability of behaviour of all trajectories or 'many' trajectories or at least one trajectory when the position of the point whose trajectory is considered is given with an error (instability of points or sensitive dependence to initial conditions are terms usually used to describe this phenomenon). Generic chaos we discussed here is just one of these definitions. We are not going to list these definitions here. Let us only say that most of them are surveyed in [62], [80], [59], [78]. In Czech Republic, J. Smítal achieved deep results on scrambled sets and Li-Yorke chaos and is a coauthor of the theory of distributional chaos.

Further research in order to understand better various forms of 'chaotic' behaviour is very desirable. In the literature there are formulated also many open problems.

The author of the dissertation continues his research of chaos and, more generally, the complexity in dynamical systems. From the recent papers let us mention at least [55] and [17]. At present he investigates for instance the possible size of scrambled sets, see [19].

4 TOPOLOGICAL ENTROPY

4.1 Starting point of the research

The notion of *topological entropy* was introduced by Adler, Konheim and McAndrew [1] as an invariant of topological conjugacy and as an analogue of measure theoretic entropy. Topological entropy provides a numerical measure for the complexity of an endomorphism of a compact topological space. Later Bowen [26] and Dinaburg [37] gave a new, but equivalent, definition in the case when the space under consideration is metrizable. This definition led to proofs of the results connecting topological and measure-theoretic entropies. Bowen [28] also defined the entropy of a uniformly continuous map of a (not necessarily compact) metric space.

In the theory of dynamical systems on compact spaces in dimensions ≥ 2, usually homeomorphisms (diffeomorphisms) satisfying certain conditions are considered (see, e.g., [56]). The non-invertible dynamics in higher dimensions is much less understood. In order to understand it better it is natural to start with investigation of systems of such a special form which would enable us to apply the knowledge of the well understood non-invertible one-dimensional dynamics (see [7], [22], [56], [70], [76], [77], [83]). Good candidates for such systems are skew products of one-dimensional dynamical systems, i.e., dynamical systems given by so called *triangular maps* on cartesian products of several one-dimensional spaces. Triangularity means that the i-th coordinate of the image of a point depends only on the first i coordinates of the point. So, triangular maps of the cube I^n, $n > 1$ are continuous maps of the form:

$$F : (x_1, x_2, ..., x_n) \mapsto (F_1(x_1), F_2(x_1, x_2), ..., F_n(x_1, x_2, ..., x_n)).$$

In a more general setting, if (X, ρ_X) and (Y, ρ_Y) are compact metric spaces and the space $X \times Y$ is endowed with the usual metric given by $\rho([x_1, y_1], [x_2, y_2]) = \max\{\rho_X(x_1, x_2), \rho_Y(y_1, y_2)\}$ then the space $C_\triangle(X \times Y)$ of triangular maps from $X \times Y$ into itself consists of all continuous maps of the form

$$F(x, y) = (f(x), g_x(y))$$

where $f : X \to X$ is continuous and g_x, $x \in X$ is a family of continuous maps $Y \to Y$ depending continuously on x. Notice that the system $(X \times Y; F)$ is an extension of $(X; f)$.

The topological entropy of triangular maps was studied in [58], [4], [5], [39]. In general it is difficult to compute the entropy of a triangular map $F = (f, g_x) \in C_\triangle(X \times Y)$. To estimate it, one can use the Bowen's formula (see [26] or [70]). Let $h_f(F) := \sup_{x \in X} h(F; Y_x)$, where $h(F; Y_x)$ is the entropy of F on the fibre

$Y_x := \{x\} \times Y$. Then, from the mentioned formula one gets

$$\max\{h(f), h_f(F)\} \leq h(F) \leq h(f) + h_f(F)$$

where $h(f)$ or $h(F)$ is the topological entropy of f or F, respectively. So, without any doubt, when computing $h(F)$ it is useful to compute $h(F; Y_x)$ for $x \in X$. The problem is that in general Y_x is not F-invariant, so $h(F; Y_x)$ is not the 'usual' entropy of an *autonomous* dynamical system. To compute $h(F; Y_x)$, it is not sufficient to know the fibre map g_x. In fact, we need to know all the maps from the sequence $g_x, g_{f(x)}, g_{f^2(x)}, \dots$. This sequence of maps defines a *nonautonomous* dynamical system on Y.

Thus, to understand better the entropy of triangular maps, it was desirable to develop a theory of the entropy for nonautonomous dynamical systems. Besides this main motivation, one can also notice that the notion of sequence topological entropy (with respect to an increasing sequence $n_1, n_2, n_3, \dots$ of positive integers) of an autonomous dynamical system $(X; f)$ which was introduced in [42] and studied by several authors (see, e.g., [40] and [39]) is nothing else than the topological entropy of the nonautonomous dynamical system given on X by a sequence of maps $f^{n_1}, f^{n_2-n_1}, f^{n_3-n_2}, \dots$. So the investigation of the topological entropy of nonautonomous dynamical systems could be potentially useful also for further development of the theory of sequence topological entropy.

4.2 Main results of the dissertation ([61])

If $f_{1,\infty} = \{f_n\}_{n=1}^{\infty}$ is a sequence of continuous selfmaps of a compact topological space X, we will denote by $(X; f_{1,\infty})$ or $(X; f_1, f_2, ...)$ or $(X; \{f_n\}_{n=1}^{\infty})$ the *nonautonomous discrete dynamical system* in which the *trajectory of a point* $x \in X$ is defined to be the sequence

$$x, f_1(x), f_2(f_1(x)), \; ... \;, f_n(f_{n-1} \; ... \; (f_2(f_1(x)))...), \; ... \; .$$

In [61] we define the *topological entropy* $h(f_{1,\infty})$ of such a system both using open covers of X and, if X is metrizable, using separated and spanning sets. The definitions are equivalent, similarly as in the autonomous case. (Of course, for $f_1 = f_2 = ... = f$ we get the classical definitions.) We also extend the definition of $h(f_{1,\infty})$ by defining the entropy $h(f_{1,\infty}; Y)$ with respect to any (not necessarily compact and not necessarily invariant) subset Y of the space X (the extended definition is used in [61, Theorem H]).

We prove some basic properties of the entropy and we also introduce the notion of the *asymptotical topological entropy* $h^*(f_{1,\infty})$ of the considered system as the limit $\lim_{n\to\infty} h(f_{n,\infty})$ where $f_{n,\infty}$ is the tail $f_n, f_{n+1}, \dots$ of the sequence $f_{1,\infty}$ (the limit exists by [61, Lemma 4.5]). Further, in the case when $f_{1,\infty}$ is a sequence of equicontinuous selfmaps of X and $Y \subset X$ we define a new quantity $H(f_{1,\infty}; Y)$. We call it *topological* sup-*entropy* of the sequence $f_{1,\infty}$ on the set Y. It is used in [61, Theorem C].

Rather surprisingly, in [61, Theorem A], as a by-product of our investigation of nonautonomous dynamical systems we obtain a new property of the topological entropy of *autonomous* dynamical systems, the commutativity of the entropy. More precisely, the entropy of the composition of two continuous selfmaps of a compact topological space does not depend on the order in which we compose, i.e., $h(f \circ g) = h(g \circ f)$.

Then we study the topological conjugacy of nonautonomous dynamical systems. We introduce the notions of equisemiconjugacy and equiconjugacy, the separate conjugacy between f_i and g_i, $i = 1, 2, \dots$, being not sufficient to give $h(f_{1,\infty}) = h(g_{1,\infty})$. A corollary of [61, Theorem B] shows that the topological entropy of nonautonomous dynamical systems is an invariant of topological equiconjugacy (consequently, the same is true for the asymptotical topological entropy). [61, Theorem C] is an analogue of the Bowen's theorem for the entropy of an extension of a system.

In [61, Theorem D] we prove that any sequence of monotone continuous interval or circle maps has zero entropy.

Then we answer the question whether there is a relation between the entropy of a sequence of maps and the entropy of its uniform limit. In [61, Theorem E] it is shown that either they are the same or the entropy of the limit map is larger, both cases being possible. Then we study the lower semi-continuity of the topological entropy on the space of all sequences of continuous selfmaps of a compact metric space (X, ρ) with the metric $\mathcal{D}(f_{1,\infty}, g_{1,\infty}) = \sup_{i \geq 1} \max_{x \in X} \rho(f_i(x), g_i(x))$. The result is that at some points it is lower semi-continuous but in general it is not. Again we get consequences also for autonomous dynamical systems (see [61, Corollary 5.7, Corollary 5.8]).

Finally, we generalize to the case of nonautonomous dynamical systems the classical Bowen's result [27] saying that the topological entropy is concentrated on the set of nonwandering points (see [61, Theorem H]).

4.3 Remarks and possibilities for further research

Our theory of topological entropy of nonautonomous systems proved to be useful in [60] where we construct a large class of smooth triangular maps of the square of type 2^∞ and positive topological entropy. In fact, to show that the entropy is positive, we use results from [61], some of them slightly generalized to the case when not only maps but also spaces are different, i.e., the nonautonomous system is given by a sequence of compact metric spaces $(X_i)_{i=1}^\infty$ and a sequence of continuous maps $(f_i)_{i=1}^\infty$, $f_i : X_i \to X_{i+1}$. Let us also remark that in [60] we also further developed the theory of topological entropy of nonautonomous systems by proving that if all the spaces are compact real intervals and all the maps are piecewise monotone then, under some additional assumptions, a formula for the entropy of the system exists in terms of the number of pieces of monotonicity of $f_n \circ \cdots \circ f_2 \circ f_1$ (a generalization of Misiurewicz-Szlenk theorem [73]).

The commutativity of the entropy, the amazing by-product obtained in [61], turned out to be very surprising even for several experts in the theory of entropy. The first reaction usually was: "Is it really true?" Of course, people usually soon realized that in the special case when f and g are surjective, the result follows from the fact that $f \circ g$ is semiconjugate to $g \circ f$ and $g \circ f$ is semiconjugate to $f \circ g$ and so they started to believe that it was true also in general case. It seems that from the psychological point of view it is difficult to admit that such a nice property of the entropy could be overlooked in the past *if it were true*. But it really seems that in the mathematical literature this result did not appear before. Nevertheless, some 5 years later J. Canovas found that the commutativity of the entropy appeared in [35].

The paper [61], namely the commutativity of the entropy, was an inspiration for the papers [14], [15] where the commutativity was proved or disproved for other kinds of entropies.

The entropy of an interval map f is positive if and only if some iterate of f has a horseshoe. For a nonautonomous system given by a sequence of selfmaps of an interval it is unclear how to define a 'horseshoe'. There is no problem with the implication 'horseshoe implies entropy' but there are nonautonomous systems of positive entropy without having any reasonable kind of a horseshoe. We find the problem 'if a nonautonomous system on an interval has positive entropy then what ?' to be very challenging but very difficult.

The author of the dissertation continues his research of topological entropy, see e.g. the recent papers [6], [17], [11] and [93]. To compute the entropy of a map in [93], our theory from [61] and [60] has successfully been used.

5 MINIMALITY

5.1 Starting point of the research

The most fundamental dynamical systems are the minimal ones, see [9], [95]. These are systems which have no nontrivial subsystems. An equivalent condition is that the orbit of every point is dense.

In many important examples of minimal maps, these are homeomorphisms. In the sixties J. Auslander [10, p. 514] formulated the problem whether a continuous map of a compact metric space onto itself which is not one-to-one can be minimal. The answer, owing also to J. Auslander himself, is known — a class of examples of noninvertible minimal maps on some compact metric spaces can be found in [12, p. 186].

Interesting examples of noninvertible minimal maps are known in interval dynamics when a suitable interval map is restricted to an invariant Cantor set. In fact, it was proved in [30] that unimodal Fibonacci maps have a wild attractor (which is a Cantor set) provided that the order of the critical point is sufficiently high. By [23], the restriction of such a map to this Cantor set is minimal and by [68] the preimage of any point from this Cantor set is a singleton except of the critical point of the map whose preimage consists of two points. More generally, there are unimodal maps whose restriction to a Cantor set (the ω-limit set $\omega(c)$ of the critical point c) is minimal and fails to be invertible only at k points, each of them lying in the backward orbit of c (one of them is c itself) and having two preimages in $\omega(c)$ (all other points in $\omega(c)$ have only one preimage in $\omega(c)$), see [31].

Symbolic dynamics provides many examples of minimal noninvertible maps. Consider $A^{\mathbb{N}}$ endowed with the shift. One can show by a compactness argument that any infinite subshift (i.e., closed shift-invariant subset of $A^{\mathbb{N}}$) contains two different points with the same image. Thus, also any minimal subshift different from a periodic orbit is noninvertible. For instance, one-sided Sturmian and Toeplitz systems are minimal noninvertible subshifts.

None of the above mentioned examples of noninvertible minimal maps is on a manifold. On the interval there is no minimal map at all and it is well known that the circle admits a minimal homeomorphism but does not admit any noninvertible minimal map. Thus, a natural question is, whether for instance the two-dimensional torus admits a minimal noninvertible map. Further, we saw that a noninvertibility of a minimal map may be very small — there is a minimal map which is invertible except of one pair of points with the same image. So, another interesting question is, how 'large' the noninvertibility of a minimal map can be.

5.2 Main results of the dissertation ([63])

In [63] we studied, for a discrete dynamical system given by a compact Hausdorff space X and a continuous selfmap f of X (we write $f \in C(X)$), the connection between minimality, invertibility and openness of f.

Recall that a map is called *open* if it sends open sets to open sets and is called *feebly open* if it sends open sets to sets with nonempty interior.

Theorem 17 ([63]). *Let X be a compact Hausdorff space and $f \in C(X)$.*

(1) If f is minimal then it is feebly open.

(2) If f is minimal and open then it is a homeomorphism.

If f is minimal and $A \subseteq X$ then both $f(A)$ and $f^{-1}(A)$ share some topological properties with the set A — namely the ones which describe how large a set is. For completeness, the next theorem contains also some known results.

Theorem 18 ([63]). *Let X be a compact Hausdorff space and let $f \in C(X)$ be a minimal map. Let $A \subseteq X$.*

(1) If A is dense then both $f(A)$ and $f^{-1}(A)$ are dense.

(2) If A is nowhere dense then both $f(A)$ and $f^{-1}(A)$ are nowhere dense.

(3) If A is a 1st category set then both $f(A)$ and $f^{-1}(A)$ are 1st category sets.

(4) If A is a 2nd category set then both $f(A)$ and $f^{-1}(A)$ are 2nd category sets.

(5) If A has the Baire property then both $f(A)$ and $f^{-1}(A)$ have the Baire property.

(6) If A is residual then both $f(A)$ and $f^{-1}(A)$ are residual.

(7) If A has nonempty interior then both $f(A)$ and $f^{-1}(A)$ have nonempty interiors.

(8) If A is open then there is a positive integer r with the property $\bigcup_{k=0}^{r} f^{-k}(A) = \bigcup_{k=0}^{r} f^{k}(A) = X$.

(9) If A is open then there is an open set $B \subseteq X$ such that $B \subseteq f(A) \subseteq \overline{B}$ (here B may not be unique; the largest of such sets is always $B = \operatorname{int} f(A)$).

A map is called almost one-to-one if generically the preimage of a point is a singleton. Using the above results one can prove that any minimal map in a compact metric space is almost one-to-one.

Theorem 19 ([63]). *Let (X, ϱ) be a compact metric space and $f \in C(X)$ be minimal. Then the set $A = \{x \in X : \ \operatorname{card} f^{-1}(x) = 1\}$ is a G_δ-dense set in X. Hence, f is almost one-to-one.*

This theorem enables to show that a 'substantial' part of a minimal map is a minimal homeomorphism.

Theorem 20 ([63]). *Let (X, ϱ) be a compact metric space and $f \in C(X)$ be minimal. Then there exists a residual set $Y \subseteq X$ such that $f(Y) = Y$ and $f|_Y$ is a minimal homeomorphism. Moreover, $(f|_Y)^{-1}$ is also a minimal homeomorphism and while $f|_Y$ is uniformly continuous, $(f|_Y)^{-1}$ is uniformly continuous only in the case when f is a homeomorphism (then one can take $Y = X$).*

Finally, we solve in [63] our main problem: we show that the **torus admits minimal noninvertible maps**. In fact, two kinds of examples of noninvertible minimal maps on the torus are given — these are obtained either as a factor or as an extension of an appropriate minimal homeomorphism of the torus.

5.3 Remarks and possibilities for further research

A map $f : X \to Y$ is called *irreducible* if the only closed set $A \subseteq X$ for which $f(A) = Y$ is $A = X$. In [63] we show that if f is minimal then it is irreducible. Therefore one can shorten some of the proofs in [63] by using some properties of irreducible maps from [96]. Still, there is a place for further study of properties of minimal maps. For instance we conjectured that minimal maps on two-dimensional (and perhaps all) manifolds are monotone. Recently we learned about the paper [24] which was inspired by our papers [63] and [32] and where this (and even a stronger result) is proved for compact 2-manifolds.

While we have proved that minimal maps in compact spaces are almost one to one, i.e., the set $B_0 := \{x \in X : \#f^{-1}(x) > 1\}$ of points with more than one preimage is of first category, this set may not be negligible from a measure-theoretic point of view. In fact, in [38] we have showed that $\mu(B_0)$ may be positive for an f-invariant Borel probability measure μ. We have found a Toeplitz flow with such a property. We give also an example of a minimal selfmap of a continuum such that the set B_0 has positive measure for every invariant measure. Such a system is constructed as a semicocycle minimal almost 1-1 extension of an irrational circle rotation. Finally we show that there are even systems with $\mu(B_0) = 1$ for every invariant measure, and there are systems with $\mu(B_0) = 1$ for some ergodic measures and $\mu(B_0) = 0$ for some other ergodic measures.

There are compact spaces that do not admit any minimal map (say, the spaces with fixed point property), there are spaces that admit minimal homeomorphisms but do not admit any minimal noninvertible map (circle) and there are spaces that admit both minimal homeomorphisms and minimal noninvertible maps (Cantor set, torus). In [32] we show that there are also spaces that admit minimal noninvertible maps but do not admit any minimal homeomorphism.

One of the open questions is whether the circle is the only (infinite) continuum that admits a minimal homeomorphism but no minimal noninvertible map. The pseudo-circle is possibly a candidate for a counterexample. Further, we conjectured that none of the two-dimensional (and perhaps all) manifolds with boundary admits a minimal map (see also [32] for related problems). In the already mentioned recent preprint [24] it is proved that in fact the only 2-manifolds (compact or not, with or without boundary) which admit minimal maps are finite unions of tori and finite unions of Klein bottles (the fact that both admit minimal maps was known; they proved that no other 2-manifold admits a minimal map).

The basic fact discovered by G. D. Birkhoff is that any compact system $(X; f)$ has minimal subsystems $(M, f|_M)$. Such sets M are called minimal sets of $(X; f)$. The fundamental question is to describe (or even characterize) topological structure of minimal sets in a given space. This is an easy task in the case of the interval – every minimal set of a continuous selfmap of the interval is either finite or a Cantor set and, conversely, for any finite set as well as for any Cantor set in the interval there is a continuous selfmap of that interval such that the considered set is minimal for that map. While it is not difficult to generalize the result to the graphs (also the circles appear on the scene as minimal sets, see [13]), in the case of dendrites the problem is highly nontrivial and only very recently in [16] we have completely solved it (in fact we obtained even some more general results)

In higher dimensions there are few results about the topological structure of minimal sets of continuous maps. For some special classes of maps we have shown that all their minimal sets are nowhere dense (see [64] and [67]). We have used a nonhomogeneous minimal system of Floyd-Auslander type to construct a counterexample in triangular dynamics in [48]. Then in [93] we have found a new example of a nonhomogeneous minimal system with positive entropy which has successfully been used to solve so called Kolyada's problem.

On the other hand, still it is not known whether continuous maps on the torus (more generally, on two-dimensional and perhaps on all connected manifolds) have only minimal sets which are nowhere dense or coincide with the whole space. We conjecture that the answer is positive and we have been working on the problem.

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7 SUMMARY

The dissertation deals with discrete dynamical systems given by a continuous self-map of a compact metric space. Three problems on periodic points on the interval are solved: Minimal periodic orbits are characterized, the nonexistence of piecewise linear maps of type 2^∞ is proved and the existence of maps of type less than 2^∞ in any neighbourhood of any map of type 2^∞ is proved. Then generically chaotic interval maps on the interval are characterized. Further, the topological entropy for nonautonomous dynamical systems on a compact metric space is defined and its properties are studied. Finally, minimal maps are studied. Among others, it is shown that any minimal map on a compact metric space is almost one-to-one and that the torus admits noninvertible minimal maps.

8 RESUMEN

La disertación versa sobre sistemas dinámicos discretos generados por aplicaciones de un espacio métrico compacto en sí mismo. En el intervalo se resuelven tres problemas sobre puntos periódicos: se caracterizan las órbitas periódicas minimales, se prueba la no existencia de aplicaciones lineales a trozos de tipo 2^∞, y se prueba la existencia de aplicaciones de tipo menor que 2^∞ en cualquier entorno de toda aplicación de tipo 2^∞. A continuación se caracterizan las aplicaciones del intervalo que son genéricamente caóticas. Además, se define la entropía topológica de los sistemas dinámicos no autónomos en espacios métricos compactos y se estudian sus propiedades. Por último se estudian las aplicaciones minimales. Entre otros resultados, se demuestra que cualquier aplicación minimal en un espacio métrico compacto es casi inyectiva y que el toro admite aplicaciones minimales no invertibles.

9 APPENDIX: LIST OF SELECTED PUBLICATIONS OF THE AUTHOR

The papers included into the dissertation are marked by the symbol ⇒. The other papers which are related to the dissertation are marked by the symbol →. (Papers in professional journals, lecture notes for students, translation, jubilee article and abstracts are not included.)

Scientific papers in domestic journals (CD)

[CD 1] J. Smítal, Ľ. Snoha: *Generalization of a Theorem of S. Piccard* , Acta Math. Univ. Comenianae XXXVII (1980), 173-181; contribution 50 %

[CD 2] Ľ. Snoha: *On connectivity points*, Math. Slovaca 33 (1983), 59-67

⇒ [CD 3] Ľ. Snoha: *Characterization of potentially minimal periodic orbits of continuous mappings of an interval*, Acta Math. Univ. Comenianae LII-LIII (1987), 111-124

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Scientific papers in foreign journals (CZ)

[CZ 1] S. F. Kolyada, Ľ. Snoha: *On ω-limit sets of triangular maps*, Real Analysis Exchange, Vol. 18(1), (1992/93), 115-130; contribution 50 %

[CZ 2] J. Bobok, Ľ. Snoha: *Periodic points and Little Fermat Theorem*, Nieuw Archief voor Wiskunde, Vol. 10, No. 1-2 (1992), 33-35; contribution 50 %

→ [CZ 3] Ll. Alsedà, S. F. Kolyada, Ľ. Snoha: *On topological entropy of triangular maps of the square*, Bull. Austral. Math. Soc., 48(1993), 55-67; contribution 33 %

⇒ [CZ 4] S. Kolyada, Ľ. Snoha: *Topological entropy of nonautonomous dynamical systems*, Random & Computational Dynamics, 4(2 & 3), 205-233 (1996); contribution 50 %

→ [CZ 5] V. Jiménez López, Ľ. Snoha: *Full cascades of simple periodic orbits on the interval*, Ukrain. Math. J., 48 (1996), No. 12, 1843 - 1851 (published also in Ukrainskii Matematicheskii Zhurnal, 48 (1996), No. 12, 1628 - 1637); contribution 50 %

⇒ [CZ 6] V. Jiménez López, Ľ. Snoha: *There are no piecewise linear maps of type 2^∞*, Trans. Amer. Math. Soc. 349(1997), 1377-1387; contribution 50 %

$\Rightarrow$ [CZ 7] V. Jiménez López, Ľ. Snoha: *All maps of type 2^∞ are boundary maps*, Proc. Amer. Math. Soc. 125(1997), 1667-1673; contribution 50 %

$\rightarrow$ [CZ 8] Ll. Alsedà, V. Jiménez López, Ľ. Snoha: *All solenoids of piecewise smooth maps are period doubling*, Fund. Math. 157(1998), 121–138; contribution 33 %

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$\rightarrow$ [CZ 10] S. Kolyada, M. Misiurewicz, Ľ. Snoha: *Topological entropy of nonautonomous piecewise monotone dynamical systems on the interval*, Fund. Math. 160 (1999), 161–181; contribution 33 %

$\rightarrow$ [CZ 11] M. Grinč, R. Hric and Ľ. Snoha: *The structure of the space $C(I, I)$ from the point of view of Sharkovsky stratification*, Topology 39(2000), 937-946; contribution 33 %

[CZ 12] M. Grinč and Ľ. Snoha: *Jungck theorem for triangular maps and related results*, Applied General Topology, Vol.1, No.1 (2000), 83–92; contribution 50 %

$\Rightarrow$ [CZ 13] S. Kolyada, Ľ. Snoha, S. Trofimchuk: *Noninvertible minimal maps*, Fund. Math. 168(2001), 141–163; contribution 33 %

$\rightarrow$ [CZ 14] Ll. Alsedà, S. Kolyada, J. Llibre, Ľ. Snoha: *Axiomatic definition of the topological entropy on the interval*, Aequationes Math. 65(2003), 113–132; contribution 25 %

$\rightarrow$ [CZ 15] H. Bruin, S. Kolyada, Ľ. Snoha: *Minimal nonhomogeneous continua*, Colloq. Math. 95(2003), 123-132; contribution 33 %

$\rightarrow$ [CZ 16] V. Jiménez López, Ľ. Snoha: *Stroboscopical property in topological dynamics*, Topology Appl. 129/3(2003), 301-316; contribution 50 %

$\rightarrow$ [CZ 17] F. Balibrea, Ľ. Snoha: *Topological entropy of Devaney chaotic maps*, Topology Appl. 133(2003), 225-239; contribution 50 %

$\rightarrow$ [CZ 18] S. Kolyada, Ľ. Snoha, S. Trofimchuk: *On minimality of nonautonomous dynamical systems*, Neliniini Kolyvannya 7(2004), no 1, 86 - 92 (published also in Nonlinear Oscillations 7(2004), No.1, 83-89); contribution 33 %

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$\rightarrow$ [CZ 20] J. Auslander, S. Kolyada, Ľ. Snoha: *Functional envelope of a dynamical system*, Max-Planck Institut für Mathematik Preprint Series 2004(131), Bonn (29 pages); contribution 33 %

→ [CZ 21] T. Downarowicz, P. Maličký, Ľ. Snoha, V. Špitalský: *Measure of noninvertibility of minimal maps*, submitted; contribution 25 %

→ [CZ 22] Ľ. Snoha, V. Špitalský: *A nonhomogeneous minimal system with positive entropy and a solution of a problem in triangular dynamics*, submitted; contribution 50 %

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→ [CZ 24] A. Linero, Ľ. Snoha: *Minimal sets of permutation–product maps*, preprint; contribution 50 %

→ [CZ 25] S. Kolyada, Ľ. Snoha, S. Trofimchuk: *Minimal sets in fibred systems*, preprint; contribution 33 %

→ [CZ 26] F. Blanchard, W. Huang, Ľ. Snoha: *The topological size of scrambled sets*, preprint; contribution 33 %

Invited lectures at foreign conferences (RZ)

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[RZ 4] V. Jiménez López, Ľ. Snoha: *Stroboscopical property, equicontinuity and weak mixing*, Grazer Math. Ber., Bericht Nr. 346(2004), 235-244; contribution 50 %

Contributions at foreign conferences (PZ)

→ [PZ 1] Ľ. Snoha: *On functions having periodic orbits which are not potentially minimal*, Proceedings of the Conference Ergodic Theory and Related Topics II, Teubner Texte zur Mathematik, Band 94, 185-189, Leipzig 1987

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