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Abstract

We study the Riemann problem for the multidimensional compressible isentropic Euler equations. Using the framework developed in [6] and based on the techniques of De Lellis and Székelyhidi [11], we extend the results of [8] and prove that whenever the initial Riemann data give rise to a self-similar solution consisting of one admissible shock and one rarefaction wave and are not too far from lying on a simple shock wave, the problem admits also infinitely many admissible weak solutions.

1 Introduction

In this note we consider the Euler system of isentropic gas dynamics in two space dimensions

$$\left\{\begin{array}{l} \partial_t\rho + \operatorname{div}_x(\rho v) \;=\; 0 \\ \partial_t(\rho v) + \operatorname{div}_x(\rho v\otimes v) + \nabla_x[p(\rho)] \;=\; 0 \\ \rho(\cdot,0) \;=\; \rho^0 \\ v(\cdot,0) \;=\; v^0\,, \end{array}\right. \tag{1.1}$$

where the unknowns (ρ, v) denote the density and the velocity of the gas respectively. The pressure p is a given function of ρ satisfying the hyperbolicity condition $p' > 0$. We will work with pressure laws $p(\rho) = \rho^\gamma$ with constant $\gamma \geq 1$. We also denote the space variable as $x = (x_1, x_2) \in \mathbb{R}^2$.

Being a hyperbolic system of conservation laws, the system (1.1) admits a single (mathematical) entropy, namely the physical total energy. Denoting $\varepsilon(\rho)$ the internal energy related to the pressure through $p(r) = r^2\varepsilon'(r)$ the *entropy (energy) inequality* reads as

$$\partial_t\left(\rho\varepsilon(\rho) + \rho\frac{|v|^2}{2}\right) + \operatorname{div}_x\left[\left(\rho\varepsilon(\rho) + \rho\frac{|v|^2}{2} + p(\rho)\right)v\right] \;\leq\; 0. \tag{1.2}$$

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We consider bounded weak solutions of (1.1) which satisfy (1.1) in the usual distributional sense. Moreover, we say that a weak solution to (1.1) is *admissible*, when it satisfies (1.2) in the sense of distributions; we also call such solutions *entropy solutions*. More precisely, admissible/entropy solutions are required to satisfy a slightly stronger condition, i.e., a form of (1.2) which involves also the initial data (see Definition 3 in [8]). We refer to the monographs [1] and [10] for detailed treatises of the related background literature.

In the last years, jointly with Camillo De Lellis, we could prove a surprising series of results concerning non–uniqueness of admissible solutions to the isentropic Euler equations in more than one space dimension (see [5], [6], [8] and also [7]), thereby showing that the most popular concept of admissible solution, the entropy inequality, fails even under quite strong assumption on the initial data. Non–uniqueness originates from the construction of *non–standard* rapidly oscillating solutions to (1.1) which are also admissible and are built via subsequent versions of the method of convex integration originally developed by De Lellis and Székelyhidi [11] for the incompressible Euler equations (see also [17]). The results we present here arise as a continuation of the work done in [6] and [8].

We are concerned with the Riemann problem for the system (1.1), more specifically we consider initial data of the following particular form

$$(\rho^0(x), v^0(x)) := \begin{cases} (\rho_-, v_-) & \text{if } x_2 < 0 \\ \\ (\rho_+, v_+) & \text{if } x_2 > 0, \end{cases} \tag{1.3}$$

where $\rho_\pm, v_\pm$ are constants. The Riemann problem (1.1)-(1.3) has been the building block in the construction of non–unique entropy solutions for the isentropic Euler equations starting from Lipschitz initial data in [6]) and also in [8] for the investigation on the effectiveness of the entropy dissipation rate criterion, as proposed by Dafermos in [9], for the same system of equations (see also [14] for complemetary results on the Dafermos criterion).

It is well known that the Riemann problem (1.1)-(1.3) admits self–similar solutions $(\rho, v)(x,t) := (r, w)(x_2/t)$ and that uniqueness holds in the class of admissible solutions if we require them to be self-similar and to have locally bounded variation. On the other hand, both in [6] and in [8] it is illustrated that, once these hypotheses are removed, uniqueness of admissible solutions can fail. In particular, in [8] it was proven that any Riemann data whose associated self–similar solution consists of two shocks admit also infinitely many non–standard solutions, which are admissible too and are genuinely two–dimensional (depend non–trivially on x_1).

In this note we aim at better understanding the relation between the structure of the Riemann data (1.3) and the formation of admissible non–standard solutions originating from such data. As detailed in Section 2 of [8], if we search for self–similar solutions $(\rho, v)(x,t) := (r, w)(x_2/t)$ of the Riemann problem (1.1)-(1.3), then, depending on the values of the constants $\rho_\pm, v_\pm$, we encounter different cases. In particular, if we choose $v_{-1} = v_{+1}$, then the first component of the self–similar velocity will remain constant for all positive times and the relation between the left state (ρ_-, v_{-2}) and the right state (ρ_+, v_{+2}) determines the form of the self–similar solution. If the right state lies on a *simple wave* going through the left state (see Fig, 1), then the self–similar solution consists of either a single *shock* or a single *rarefaction wave* as explained in Lemma 2.3 in [8]. We refer to

[10] for the precise definitions of shock and rarefaction waves. In Fig. 1 we denote by $S_{1,3}$ and by $R_{1,3}$ the $1,3$- shock and $1,3$-rarefaction waves through the point (ρ_-, v_{-2}).

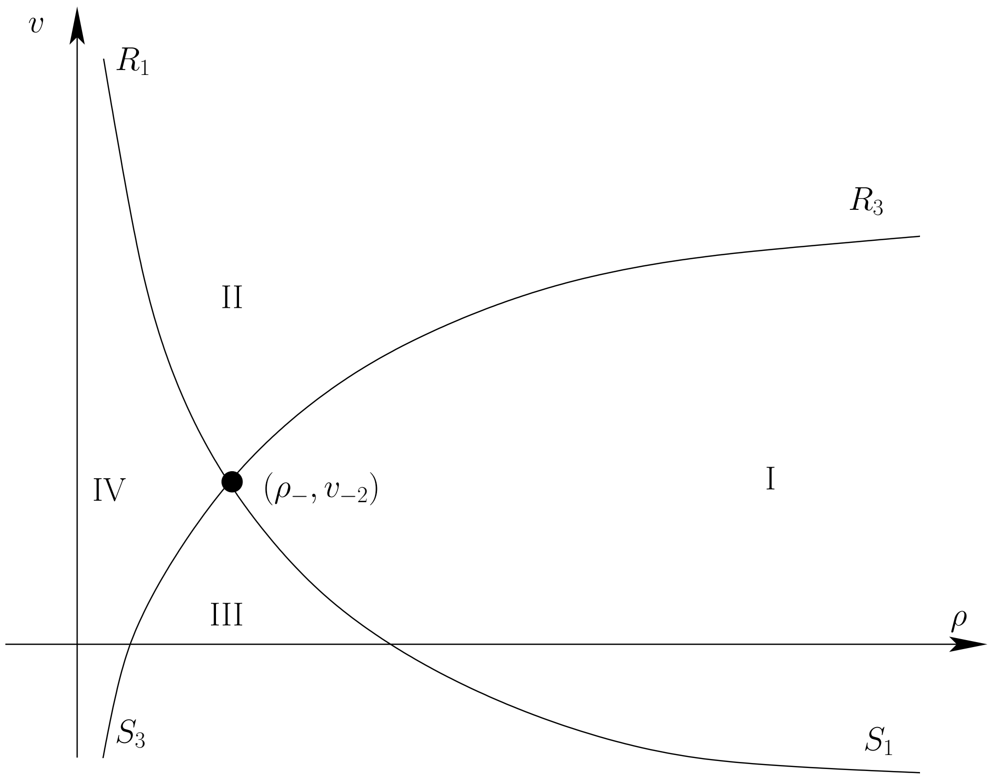


Figure 1: Shocks and rarefaction curves through the point (ρ_-, v_-).

If (ρ_-, v_{-2}) and (ρ_+, v_{+2}) do not lie on any simple wave, then we can distinguish four situations:

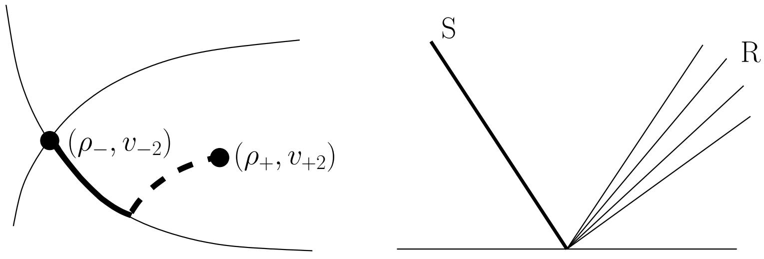
CASE 1: $(\rho_+, v_{+2}) \in$ "region I": the solution consists of a 1-shock and a 3-rarefaction;

CASE 2: $(\rho_+, v_{+2}) \in$ "region II": the solution consists of two rarefaction waves;

CASE 3: $(\rho_+, v_{+2}) \in$ "region III": the solution consists of two shocks;

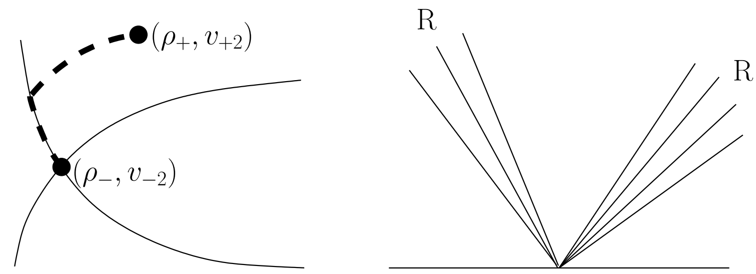
CASE 4: $(\rho_+, v_{+2}) \in$ "region IV": the solution consists of a 1-rarefaction wave and a 3-shock

In Fig. 2–5, we describe schematically how these four cases look like placing side by side the wave curves plots and the pattern of the self–similar solution in the $x_2 - t$ plane. When the self–similar solution contains no discontinuities, i.e. when it consists of rarefaction waves only (CASE 2), the Riemann problem (1.1)-(1.3) enjoys uniqueness as was shown first by Chen and Chen [2]. The same result was obtained in [15], the authors not being aware of the result of Chen and Chen. Similar results are contained also in the work of Serre [16] and related are also the work of DiPerna [13] and the works of Chen and Frid [3] and [4]. Oppositely, when the self–similar solutions consists of two shocks (CASE 3), the Riemann problem (1.1)-(1.3) admits also infinitely many non–standard solutions as proven in [8].



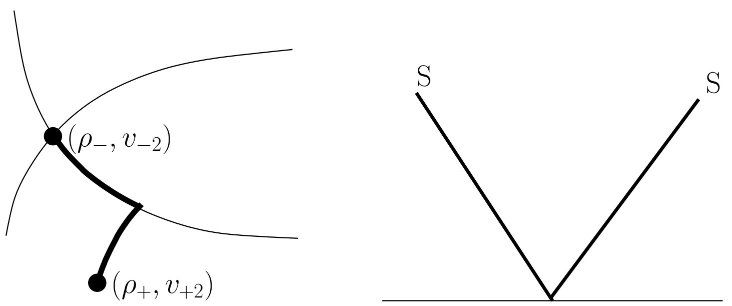
(a) Wave curves (b) Wave fan

Figure 2: Case 1.



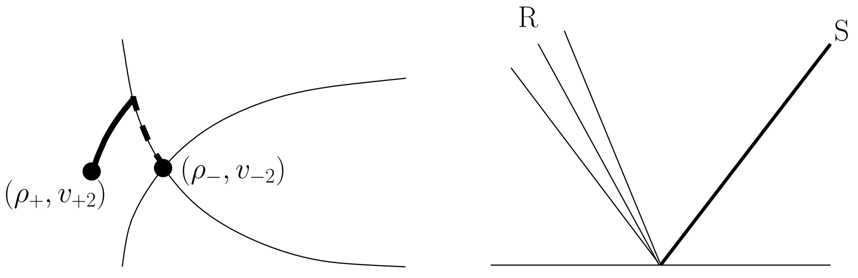
(a) Wave curves (b) Wave fan

Figure 3: Case 2.



(a) Wave curves (b) Wave fan

Figure 4: Case 3.



(a) Wave curves (b) Wave fan

Figure 5: Case 4.

In this note we investigate whether such non–uniqueness of admissible solutions can be obtained in CASE 1 and in CASE 4 as well, at least when we are close enough (in a suitable sense) to CASE 3 and far from CASE 2. We do not discuss the case of Riemann data lying on a simple shock wave even if we expect that non–uniqueness should hold in this case.

Our main result proves that non–uniqueness of admissible solutions of the Riemann problem (1.1)-(1.3) indeed occurs at least for right data (ρ_+, v_{+2}) belonging to subregions of region I and IV which are adjacent to region III and detached from region II. The result is independent of the specific choice for the constant v_1. The precise statement of the result is as follows.

Theorem 1 *Let* $p(\rho) = \rho^\gamma$, $\gamma > 1$. *Let* $\rho_- \neq \rho_+$, $\rho_\pm > 0$ *and* $v_{+2} \in \mathbb{R}$ *be given. There exists* $V = V(\rho_-, \rho_+, v_{+2}, \gamma) < \sqrt{\frac{(\rho_+-\rho_-)(p(\rho_+)-p(\rho_-))}{\rho_+\rho_-}}$ *such that for all* v_{-2} *satisfying* $V < v_{-2} - v_{+2} < \sqrt{\frac{(\rho_+-\rho_-)(p(\rho_+)-p(\rho_-))}{\rho_+\rho_-}}$ *there exists infinitely many bounded admissible weak solutions to the Euler equations* (1.1) *with Riemann initial data* (1.3).

Remark 1.1 The theorem is stated for the two-dimensional case but it naturally extends to any dimension $d > 1$.

We remark that the upper bound $v_{-2} - v_{+2} < \sqrt{\frac{(\rho_+-\rho_-)(p(\rho_+)-p(\rho_-))}{\rho_+\rho_-}}$ characterizes regions I and IV, or else said characterizes Riemann data allowing for self–similar solutions consisting of a shock and a rarefaction wave (cf. [8]). The existence of $V = V(\rho_-, \rho_+, v_{+2}, \gamma) < \sqrt{\frac{(\rho_+-\rho_-)(p(\rho_+)-p(\rho_-))}{\rho_+\rho_-}}$ and the corresponding lower bound for $v_{-2} - v_{+2}$ guarantee instead the existence of subregions inside I and IV where non–uniqueness can arise. In Fig. 6 we give a qualitative picture of such subregions in blue, while we describe in red the area where non–uniqueness holds due to [8].

2 Preliminaries

We start with three important definitions taken from [6].

Definition 1 (Fan partition) *A* fan partition *of* $\mathbb{R}^2 \times (0, \infty)$ *consists of three open sets* P_-, P_1, P_+ *of the following form*

$$P_- = \{(x,t) : t > 0 \quad and \quad x_2 < \nu_- t\} \tag{2.1}$$
$$P_1 = \{(x,t) : t > 0 \quad and \quad \nu_- t < x_2 < \nu_+ t\} \tag{2.2}$$
$$P_+ = \{(x,t) : t > 0 \quad and \quad x_2 > \nu_+ t\}, \tag{2.3}$$

where $\nu_- < \nu_+$ *is an arbitrary couple of real numbers.*

Definition 2 (Fan subsolution) *A* fan subsolution *to the compressible Euler equations* (1.1) *with initial data* (1.3) *is a triple* $(\overline{\rho}, \overline{v}, \overline{u}) : \mathbb{R}^2 \times (0, \infty) \to (\mathbb{R}^+, \mathbb{R}^2, \mathcal{S}_0^{2\times 2})$ *of piecewise constant functions satisfying the following requirements.*

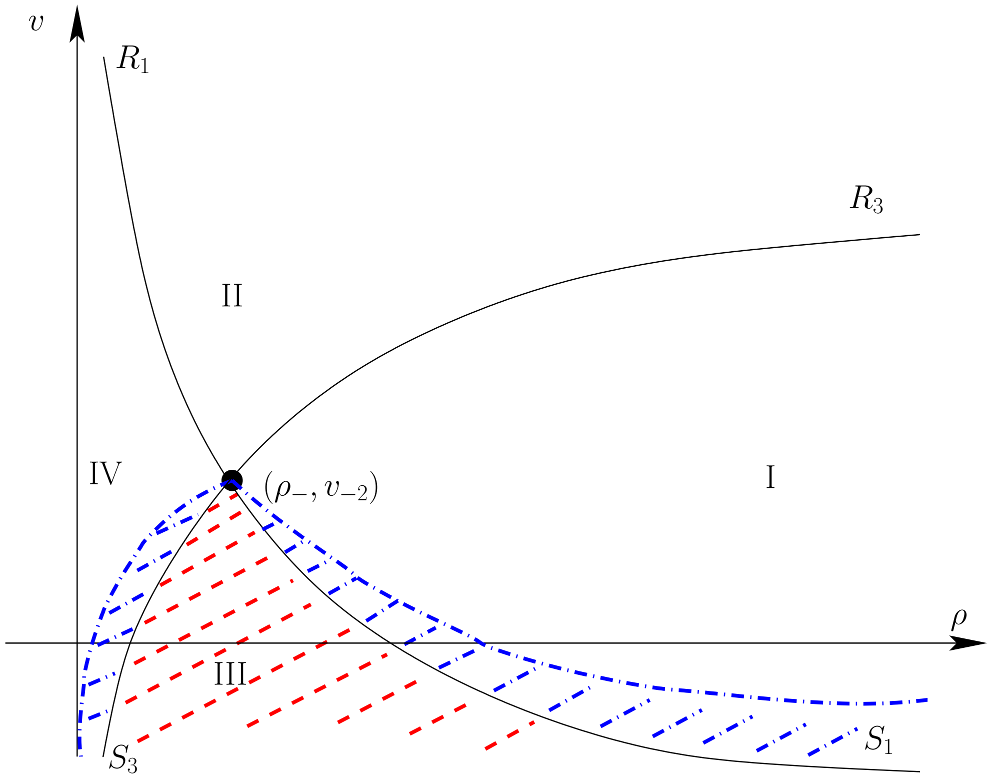


Figure 6: Regions where non–uniqueness holds.

(i) *There is a fan partition* P_-, P_1, P_+ *of* $\mathbb{R}^2 \times (0, \infty)$ *such that*

$$(\overline{\rho}, \overline{v}, \overline{u}) = (\rho_-, v_-, u_-)\mathbf{1}_{P_-} + (\rho_1, v_1, u_1)\mathbf{1}_{P_1} + (\rho_+, v_+, u_+)\mathbf{1}_{P_+}$$

where ρ_1, v_1, u_1 *are constants with* $\rho_1 > 0$ *and* $u_\pm = v_\pm \otimes v_\pm - \frac{1}{2}|v_\pm|^2\mathrm{Id}$*;*

(ii) *There exists a positive constant* C *such that*

$$v_1 \otimes v_1 - u_1 < \frac{C}{2}\mathrm{Id}\,; \tag{2.4}$$

(iii) *The triple* $(\overline{\rho}, \overline{v}, \overline{u})$ *solves the following system in the sense of distributions:*

$$\partial_t \overline{\rho} + \mathrm{div}_x(\overline{\rho}\,\overline{v}) \;=\; 0 \tag{2.5}$$

$$\partial_t(\overline{\rho}\,\overline{v}) + \mathrm{div}_x\,(\overline{\rho}\,\overline{u}) + \nabla_x \left(p(\overline{\rho}) + \frac{1}{2}\left(C\rho_1 \mathbf{1}_{P_1} + \overline{\rho}|\overline{v}|^2 \mathbf{1}_{P_+ \cup P_-} \right) \right) = 0. \tag{2.6}$$

Definition 3 (Admissible fan subsolution) *A fan subsolution* $(\overline{\rho}, \overline{v}, \overline{u})$ *is said to be* admissible *if it satisfies the following inequality in the sense of distributions*

$$\begin{aligned} &\partial_t\left(\overline{\rho}\varepsilon(\overline{\rho})\right) + \mathrm{div}_x\left[(\overline{\rho}\varepsilon(\overline{\rho}) + p(\overline{\rho}))\,\overline{v}\right] + \partial_t\left(\overline{\rho}\frac{|\overline{v}|^2}{2}\mathbf{1}_{P_+\cup P_-}\right) + \mathrm{div}_x\left(\overline{\rho}\frac{|\overline{v}|^2}{2}\overline{v}\mathbf{1}_{P_+\cup P_-}\right) \\ &\quad + \left[\partial_t\left(\rho_1\,\frac{C}{2}\,\mathbf{1}_{P_1}\right) + \mathrm{div}_x\left(\rho_1\,\overline{v}\,\frac{C}{2}\,\mathbf{1}_{P_1}\right)\right] \;\le\; 0\,. \end{aligned} \tag{2.7}$$

The existence of infinitely many admissible weak solutions is related to the existence of a single admissible fan subsolution through the following proposition.

Proposition 2.1 *Let p be any C^1 function and $(\rho_\pm, v_\pm)$ be such that there exists at least one admissible fan subsolution $(\overline{\rho}, \overline{v}, \overline{u})$ of* (1.1) *with initial data* (1.3)*. Then there are infinitely many bounded admissible solutions (ρ, v) to* (1.1),(1.2), (1.3) *such that $\rho = \overline{\rho}$ and $|v|^2 \mathbf{1}_{P_1} = C$.*

The core of the proof of Proposition 2.1 is the following fundamental Lemma.

Lemma 2.2 *Let $(\tilde{v}, \tilde{u}) \in \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}$ and $C_0 > 0$ be such that $\tilde{v} \otimes \tilde{v} - \tilde{u} < \frac{C_0}{2}\mathrm{Id}$. For any open set $\Omega \subset \mathbb{R}^2 \times \mathbb{R}$ there are infinitely many maps $(\underline{v}, \underline{u}) \in L^\infty(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2})$ with the following property*

- *(i)* $\underline{v}$ *and* $\underline{u}$ *vanish identically outside* Ω*;*
- *(ii)* $\mathrm{div}_x \underline{v} = 0$ *and* $\partial_t \underline{v} + \mathrm{div}_x \underline{u} = 0$*;*
- *(iii)* $(\tilde{v} + \underline{v}) \otimes (\tilde{v} + \underline{v}) - (\tilde{u} + \underline{u}) = \frac{C_0}{2}\mathrm{Id}$ *a.e. on* Ω*.*

Having Lemma 2.2 at hand, solutions to the Euler equations (1.1) are created by adding to the single subsolution infinitely many maps as in Lemma 2.2 in the region P_1. More precisely we use the Lemma 2.2 with $\Omega = P_1, (\tilde{v}, \tilde{u}) = (v_1, u_1)$ and $C_0 = C$. One can easily check that each couple $(\overline{\rho}, \overline{v} + \underline{v})$ is indeed an admissible weak solution to (1.1). For a complete proof of Proposition 2.1, we refer to [6, Section 3.3].

The proof of the Lemma 2.2 can be found in [6, Section 4] and is essentially based on the theory of De Lellis and Székelyhidi [11] for the incompressible Euler system. We will not present the proof here.

As we explained above, our goal is now to find an admissible fan subsolution for Riemann initial data as in Theorem 1. In order to do that, similarly as in [8] we introduce the real numbers $\alpha, \beta, \gamma_1, \gamma_2, v_{-1}, v_{-2}, v_{+1}, v_{+2}$ such that

$$v_1 = (\alpha, \beta), \tag{2.8}$$

$$v_- = (v_{-1}, v_{-2}) \tag{2.9}$$

$$v_+ = (v_{+1}, v_{+2}) \tag{2.10}$$

$$u_1 = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}. \tag{2.11}$$

As in [8], we can easily check the existence of a fan subsolution thanks to the following Proposition.

Proposition 2.3 *Let P_-, P_1, P_+ be a fan partition as in Definition 1. The constants $v_1, v_-, v_+, u_1, \rho_-, \rho_+, \rho_1$ as in* (2.8)-(2.11) *define an admissible fan subsolution as in Definitions 2-3 if and only if the following identities and inequalities hold:*

- *Rankine-Hugoniot conditions on the left interface:*

$$\nu_-(\rho_- - \rho_1) = \rho_- v_{-2} - \rho_1 \beta \tag{2.12}$$

$$\nu_-(\rho_- v_{-1} - \rho_1 \alpha) = \rho_- v_{-1} v_{-2} - \rho_1 \gamma_2 \tag{2.13}$$

$$\nu_-(\rho_- v_{-2} - \rho_1 \beta) = \rho_- v_{-2}^2 + \rho_1 \gamma_1 + p(\rho_-) - p(\rho_1) - \rho_1 \frac{C}{2}\,; \tag{2.14}$$

- *Rankine-Hugoniot conditions on the right interface:*

$$\nu_+(\rho_1 - \rho_+) = \rho_1\beta - \rho_+ v_{+2} \tag{2.15}$$
$$\nu_+(\rho_1\alpha - \rho_+ v_{+1}) = \rho_1\gamma_2 - \rho_+ v_{+1}v_{+2} \tag{2.16}$$
$$\nu_+(\rho_1\beta - \rho_+ v_{+2}) = -\rho_1\gamma_1 - \rho_+ v_{+2}^2 + p(\rho_1) - p(\rho_+) + \rho_1\frac{C}{2}\,; \tag{2.17}$$

- *Subsolution condition:*

$$\alpha^2 + \beta^2 < C \tag{2.18}$$
$$\left(\frac{C}{2} - \alpha^2 + \gamma_1\right)\left(\frac{C}{2} - \beta^2 - \gamma_1\right) - (\gamma_2 - \alpha\beta)^2 > 0\,; \tag{2.19}$$

- *Admissibility condition on the left interface:*

$$\begin{aligned} &\nu_-(\rho_-\varepsilon(\rho_-) - \rho_1\varepsilon(\rho_1)) + \nu_-\left(\rho_-\frac{|v_-|^2}{2} - \rho_1\frac{C}{2}\right) \\ &\le [(\rho_-\varepsilon(\rho_-) + p(\rho_-))v_{-2} - (\rho_1\varepsilon(\rho_1) + p(\rho_1))\beta] + \left(\rho_- v_{-2}\frac{|v_-|^2}{2} - \rho_1\beta\frac{C}{2}\right)\,; \end{aligned} \tag{2.20}$$

- *Admissibility condition on the right interface:*

$$\begin{aligned} &\nu_+(\rho_1\varepsilon(\rho_1) - \rho_+\varepsilon(\rho_+)) + \nu_+\left(\rho_1\frac{C}{2} - \rho_+\frac{|v_+|^2}{2}\right) \\ &\le [(\rho_1\varepsilon(\rho_1) + p(\rho_1))\beta - (\rho_+\varepsilon(\rho_+) + p(\rho_+))v_{+2}] + \left(\rho_1\beta\frac{C}{2} - \rho_+ v_{+2}\frac{|v_+|^2}{2}\right)\,. \end{aligned} \tag{2.21}$$

The existence of a fan subsolution is then equivalent to the existence of real numbers $\nu_- < \nu_+, \rho_1 > 0, \alpha, \beta, \gamma_1, \gamma_2, C > 0$ solving the set of identities and inequalities (2.12)–(2.21). We start with the following observation.

Lemma 2.4 *Let $v_{-1} = v_{+1}$. Then $\alpha = v_{-1} = v_{+1}$ and $\gamma_2 = \alpha\beta$.*

Proof. See [8, Lemma 4.2] □

The set of identities and inequalities from Proposition 2.3 then simplifies as follows

- Rankine-Hugoniot conditions on the left interface:

$$\nu_-(\rho_- - \rho_1) = \rho_- v_{-2} - \rho_1\beta \tag{2.22}$$
$$\nu_-(\rho_- v_{-2} - \rho_1\beta) = \rho_- v_{-2}^2 - \rho_1(\frac{C}{2} - \gamma_1) + p(\rho_-) - p(\rho_1)\,; \tag{2.23}$$

- Rankine-Hugoniot conditions on the right interface:

$$\nu_+(\rho_1 - \rho_+) = \rho_1\beta - \rho_+ v_{+2} \tag{2.24}$$
$$\nu_+(\rho_1\beta - \rho_+ v_{+2}) = \rho_1(\frac{C}{2} - \gamma_1) - \rho_+ v_{+2}^2 + p(\rho_1) - p(\rho_+)\,; \tag{2.25}$$

- Subsolution condition:

$$\alpha^2 + \beta^2 < C \tag{2.26}$$

$$\left(\frac{C}{2} - \alpha^2 + \gamma_1\right)\left(\frac{C}{2} - \beta^2 - \gamma_1\right) > 0\,; \tag{2.27}$$

with admissibility conditions (2.20) and (2.21) same as above and $\alpha = v_{-1} = v_{+1}$. Next we reformulate the conditions for the matrix $u_1 + \frac{C}{2}\mathrm{Id} - v_1 \otimes v_1$ to be positive definite.

Lemma 2.5 *A necessary condition for* (2.26)-(2.27) *to be satisfied is* $\frac{C}{2} - \gamma_1 > \beta^2$.

Proof. See [8, Lemma 4.3] □

Following the strategy developed in [8], we introduce ε_1 and ε_2 as

$$0 < \varepsilon_1 := \frac{C}{2} - \gamma_1 - \beta^2 \tag{2.28}$$

$$0 < \varepsilon_2 := C - \alpha^2 - \beta^2 - \varepsilon_1 \tag{2.29}$$

and further reformulate the set of identities and inequalities as follows.

Lemma 2.6 *In the case* $v_{-1} = v_{+1} = \alpha$ *and with* ε_1, ε_2 *as defined above, the set of algebraic identities and inequalities* (2.22)-(2.27) *together with* (2.20)-(2.21) *is equivalent to*

- *Rankine-Hugoniot conditions on the left interface:*

$$\nu_-(\rho_- - \rho_1) = \rho_- v_{-2} - \rho_1\beta \tag{2.30}$$

$$\nu_-(\rho_- v_{-2} - \rho_1\beta) = \rho_- v_{-2}^2 - \rho_1(\beta^2 + \varepsilon_1) + p(\rho_-) - p(\rho_1)\,; \tag{2.31}$$

- *Rankine-Hugoniot conditions on the right interface:*

$$\nu_+(\rho_1 - \rho_+) = \rho_1\beta - \rho_+ v_{+2} \tag{2.32}$$

$$\nu_+(\rho_1\beta - \rho_+ v_{+2}) = \rho_1(\beta^2 + \varepsilon_1) - \rho_+ v_{+2}^2 + p(\rho_1) - p(\rho_+)\,; \tag{2.33}$$

- *Subsolution condition:*

$$\varepsilon_1 > 0 \tag{2.34}$$

$$\varepsilon_2 > 0\,; \tag{2.35}$$

- *Admissibility condition on the left interface:*

$$(\beta - v_{-2})\left(p(\rho_-) + p(\rho_1) - 2\rho_-\rho_1\frac{\varepsilon(\rho_-) - \varepsilon(\rho_1)}{\rho_- - \rho_1}\right) \le \varepsilon_1\rho_1(v_{-2} + \beta) - (\varepsilon_1 + \varepsilon_2)\frac{\rho_-\rho_1(\beta - v_{-2})}{\rho_- - \rho_1}\,; \tag{2.36}$$

- *Admissibility condition on the right interface:*

$$
\begin{aligned}
&(v_{+2}-\beta)\left(p(\rho_1)+p(\rho_+)-2\rho_1\rho_+\frac{\varepsilon(\rho_1)-\varepsilon(\rho_+)}{\rho_1-\rho_+}\right)\\
&\le -\varepsilon_1\rho_1(v_{+2}+\beta)+(\varepsilon_1+\varepsilon_2)\frac{\rho_1\rho_+(v_{+2}-\beta)}{\rho_1-\rho_+}\,. \qquad (2.37)
\end{aligned}
$$

Proof. See [8, Lemma 4.4] □

Let us emphasize that the expressions

$$P(\rho_-,\rho_1) := \left(p(\rho_-)+p(\rho_1)-2\rho_-\rho_1\frac{\varepsilon(\rho_-)-\varepsilon(\rho_1)}{\rho_--\rho_1}\right) \qquad (2.38)$$

$$P(\rho_1,\rho_+) := \left(p(\rho_1)+p(\rho_+)-2\rho_1\rho_+\frac{\varepsilon(\rho_1)-\varepsilon(\rho_+)}{\rho_1-\rho_+}\right) \qquad (2.39)$$

appearing on the left hand sides of (2.36) and (2.37) are both positive for $p(\rho)=\rho^\gamma$ with $\gamma\ge 1$ as a consequence of [8, Lemma 2.1].

3 Proof

After reducing the existence of a fan subsolution to Lemma 2.6 we are now ready to prove Theorem 1.

Recall that the quantities $\rho_\pm, v_{\pm 2}$ are considered to be given as the initial data. Therefore the system of relations (2.30)–(2.37) consists of 4 equations and 4 inequalities for 6 unknowns $\nu_\pm,\rho_1,\beta,\varepsilon_1,\varepsilon_2$, with ε_2 appearing only in the inequalities. Similarly as in [8] we choose ρ_1 as a parameter and using the equations (2.30)–(2.33) we express $\nu_\pm,\beta$ and ε_1 in terms of the initial data and of the chosen parameter ρ_1.

We use the following notation for functions of initial data

$$R := \rho_- - \rho_+ \qquad (3.1)$$

$$A := \rho_- v_{-2} - \rho_+ v_{+2} \qquad (3.2)$$

$$H := \rho_- v_{-2}^2 - \rho_+ v_{+2}^2 + p(\rho_-) - p(\rho_+). \qquad (3.3)$$

$$u := v_{+2} - v_{-2} \qquad (3.4)$$

$$B := A^2 - RH = \rho_-\rho_+ u^2 - (\rho_+-\rho_-)(p(\rho_+)-p(\rho_-)) \qquad (3.5)$$

and recall that we study the problem with

$$v_{-2} - v_{+2} < \sqrt{\frac{(\rho_+-\rho_-)(p(\rho_+)-p(\rho_-))}{\rho_+\rho_-}} \qquad (3.6)$$

which translates into $B<0$.

3.1 The case R < 0

Let us first assume that $\rho_+ > \rho_-$, i.e. $R < 0$.

Following the calculations of [8, Section 4], we recover

$$\nu_- = \frac{A - \nu_+(\rho_1 - \rho_+)}{\rho_- - \rho_1}. \tag{3.7}$$

and

$$\nu_+ = \frac{A}{R} \pm \frac{1}{R}\sqrt{B\frac{\rho_1 - \rho_-}{\rho_1 - \rho_+}}. \tag{3.8}$$

with the correct sign chosen such that $\nu_- < \nu_+$. Since we assume $B < 0$, the necessary condition to follow is $\rho_1 \in (\rho_-, \rho_+)$ and we find that

$$\nu_- = \frac{A}{R} + \frac{\sqrt{-B}}{R}\sqrt{\frac{\rho_+ - \rho_1}{\rho_1 - \rho_-}} \tag{3.9}$$

$$\nu_+ = \frac{A}{R} - \frac{\sqrt{-B}}{R}\sqrt{\frac{\rho_1 - \rho_-}{\rho_+ - \rho_1}}. \tag{3.10}$$

Further we express β from (2.32) as

$$\beta = \frac{\rho_+ v_{+2}}{\rho_1} - \frac{(\rho_+ - \rho_1)A}{R\rho_1} + \frac{\sqrt{-B}}{R\rho_1}\sqrt{(\rho_1 - \rho_-)(\rho_+ - \rho_1)} \tag{3.11}$$

and finally we use (2.33) to express ε_1 as a function of ρ_1 and of the initial data. We have

$$\varepsilon_1(\rho_1) = \frac{p(\rho_+) - p(\rho_1)}{\rho_1} - \frac{\rho_+(\rho_+ - \rho_1)}{\rho_1^2}(\nu_+ - v_{+2})^2 \tag{3.12}$$

and further plugging in (3.10) we get

$$\varepsilon_1(\rho_1) = \frac{p(\rho_+) - p(\rho_1)}{\rho_1} - \frac{\rho_+}{\rho_1}\left(\frac{\sqrt{-B}}{-R}\sqrt{1 - \frac{\rho_-}{\rho_1}} - \frac{\rho_- u}{R}\sqrt{\frac{\rho_+}{\rho_1} - 1}\right)^2 \tag{3.13}$$

For simplicity we further denote

$$K := \frac{\rho_- u}{R} \tag{3.14}$$

$$L := \frac{\sqrt{-B}}{-R} \tag{3.15}$$

and recall that both $K, L > 0$. Thus we have

$$\varepsilon_1(\rho_1) = \frac{p(\rho_+) - p(\rho_1)}{\rho_1} - \frac{\rho_+}{\rho_1}\left(L\sqrt{1 - \frac{\rho_-}{\rho_1}} - K\sqrt{\frac{\rho_+}{\rho_1} - 1}\right)^2 \tag{3.16}$$

Lemma 3.1 *There exists a unique $\overline{\rho}$ such that*

$$\varepsilon_1 > 0 \qquad \text{for} \quad \rho_1 \in (\rho_-, \overline{\rho}) \tag{3.17}$$

$$\varepsilon_1 < 0 \qquad \text{for} \quad \rho_1 \in (\overline{\rho}, \rho_+). \tag{3.18}$$

Moreover, $\overline{\rho} \to \rho_+$ as $u \to -\sqrt{\frac{(\rho_+ - \rho_-)(p(\rho_+) - p(\rho_-))}{\rho_+ \rho_-}}$

Proof. First observe that $\varepsilon_1(\rho_-) > 0$. Indeed we have

$$\varepsilon_1(\rho_-) = \frac{p(\rho_+) - p(\rho_-)}{\rho_-} - K^2 \frac{\rho_+(\rho_+ - \rho_-)}{\rho_-^2} = \frac{p(\rho_+) - p(\rho_-)}{\rho_-} - \frac{\rho_+ u^2}{\rho_+ - \rho_-}$$

and this expression is positive due to (3.6). Next it is easy to see that $\varepsilon_1(\rho_+) < 0$.

We denote

$$\tilde{\rho} := \frac{K^2 \rho_+ + L^2 \rho_-}{K^2 + L^2} \in (\rho_-, \rho_+), \tag{3.19}$$

i.e. $\tilde{\rho}$ is the zero of the expression $L\sqrt{1 - \frac{\rho_-}{\rho_1}} - K\sqrt{\frac{\rho_+}{\rho_1} - 1}$. Obviously $\varepsilon_1(\tilde{\rho}) > 0$. It is easy to observe that the function $\varepsilon_1(\rho_1)$ is decreasing on the interval $(\tilde{\rho}, \rho_+)$ thus yielding the existence of a single zero of ε_1 on the interval $(\tilde{\rho}, \rho_+)$ which is indeed the $\overline{\rho}$ claimed in the Lemma.

Our final goal is thus to ensure that there are no zeros of $\varepsilon_1(\rho_1)$ on the interval $(\rho_-, \tilde{\rho})$. This is equivalent to say that there are no zeros of the function $\tilde{\varepsilon_1}(\rho_1) = \rho_1 \varepsilon_1(\rho_1)$ on this interval. For this purpose it is enough to show that $\tilde{\varepsilon_1}(\rho_1)$ is a concave function on $(\rho_-, \tilde{\rho})$.

We have

$$\tilde{\varepsilon_1}(\rho_1) = p(\rho_+) - p(\rho_1) - \rho_+ \left(L\sqrt{1 - \frac{\rho_-}{\rho_1}} - K\sqrt{\frac{\rho_+}{\rho_1} - 1} \right)^2$$

$$\tilde{\varepsilon_1}'(\rho_1) = -p'(\rho_1) + \rho_+ \rho_1^{-2} \left(K^2 \rho_+ - L^2 \rho_- + KL \left(\rho_- \sqrt{\frac{\rho_+ - \rho_1}{\rho_1 - \rho_-}} - \rho_+ \sqrt{\frac{\rho_1 - \rho_-}{\rho_+ - \rho_1}} \right) \right)$$

and finally

$$\begin{aligned} \tilde{\varepsilon_1}''(\rho_1) = {} & -p''(\rho_1) - 2\rho_+ \rho_1^{-3} \left(K^2 \rho_+ - L^2 \rho_- + KL \left(\rho_- \sqrt{\frac{\rho_+ - \rho_1}{\rho_1 - \rho_-}} - \rho_+ \sqrt{\frac{\rho_1 - \rho_-}{\rho_+ - \rho_1}} \right) \right) \\ & + \frac{\rho_+ KL}{2\rho_1^2} \left(-\frac{\rho_-(\rho_+ - \rho_-)}{(\rho_1 - \rho_-)^2} \sqrt{\frac{\rho_1 - \rho_-}{\rho_+ - \rho_1}} - \frac{\rho_+(\rho_+ - \rho_-)}{(\rho_+ - \rho_1)^2} \sqrt{\frac{\rho_+ - \rho_1}{\rho_1 - \rho_-}} \right). \end{aligned} \tag{3.20}$$

The first and third terms on the right hand side are clearly nonpositive, so to conclude our proof we need to show that

$$K^2 \rho_+ - L^2 \rho_- + KL \left(\rho_- \sqrt{\frac{\rho_+ - \rho_1}{\rho_1 - \rho_-}} - \rho_+ \sqrt{\frac{\rho_1 - \rho_-}{\rho_+ - \rho_1}} \right) > 0$$

for $\rho_1 \in (\rho_-, \tilde{\rho})$. Denoting

$$x = \sqrt{\frac{\rho_+ - \rho_1}{\rho_1 - \rho_-}} \geq 0$$

we search where

$$K^2\rho_+ - L^2\rho_- + KL\rho_- x - KL\rho_+ x^{-1} > 0.$$

Simple computations yield that this is satisfied for $x > L/K$ which means $\rho_1 < \tilde{\rho}$. The proof of the first claim of Lemma 3.1 is complete.

In order to prove the second claim we observe that $L \to 0$ and $\varepsilon_1(\rho_+) \to 0$ as $u \to -\sqrt{\frac{(\rho_+-\rho_-)(p(\rho_+)-p(\rho_-))}{\rho_+\rho_-}}$. In particular also $\tilde{\rho} \to \rho_+$ and since obviously $\tilde{\rho} < \overline{\rho} < \rho_+$ we conclude that $\overline{\rho} \to \rho_+$. □

It remains to study the admissibility conditions. Let us denote

$$P(r,s) := p(r) + p(s) - 2rs\frac{\varepsilon(r)-\varepsilon(s)}{r-s} \tag{3.21}$$

and thus the admissibility conditions (2.36)–(2.37) can be rewritten as follows

$$(\beta - v_{-2})P(\rho_-,\rho_1) \leq \varepsilon_1\rho_1(v_{-2}+\beta) - (\varepsilon_1+\varepsilon_2)\frac{\rho_-\rho_1(\beta - v_{-2})}{\rho_- - \rho_1} \tag{3.22}$$

$$(v_{+2} - \beta)P(\rho_1,\rho_+) \leq -\varepsilon_1\rho_1(v_{+2}+\beta) + (\varepsilon_1+\varepsilon_2)\frac{\rho_1\rho_+(v_{+2} - \beta)}{\rho_1 - \rho_+}. \tag{3.23}$$

To prove the main theorem, we will now rewrite some of the above expressions in a different way. We denote

$$T = \frac{(\rho_+ - \rho_-)(p(\rho_+) - p(\rho_-))}{\rho_+\rho_-} \tag{3.24}$$

and recall that we are interested in the cases where $u + \sqrt{T} > 0$. In particular we will study the limits as $u \to -\sqrt{T}$.

We have

$$B = \rho_-\rho_+(u^2 - T) = \rho_-\rho_+(u - \sqrt{T})(u + \sqrt{T}),$$

where the middle term is negative. Rewriting (3.9) and (3.10) we get

$$\nu_- = v_{+2} + \frac{\rho_-\sqrt{T}}{R} - \frac{\rho_-}{R}(u+\sqrt{T}) + \sqrt{u+\sqrt{T}}\frac{\sqrt{(\sqrt{T}-u)\rho_-\rho_+}}{R}\sqrt{\frac{\rho_+ - \rho_1}{\rho_1 - \rho_-}} \tag{3.25}$$

$$\nu_+ = v_{+2} + \frac{\rho_-\sqrt{T}}{R} - \frac{\rho_-}{R}(u+\sqrt{T}) - \sqrt{u+\sqrt{T}}\frac{\sqrt{(\sqrt{T}-u)\rho_-\rho_+}}{R}\sqrt{\frac{\rho_1 - \rho_-}{\rho_+ - \rho_1}}. \tag{3.26}$$

In particular we observe that for any positive $u + \sqrt{T}$ we have $\nu_- < \nu_+$, but both ν_-, ν_+ have the same limit as $u \to -\sqrt{T}$.

Next we reformulate the expression (3.11). We have

$$\begin{aligned}\beta = v_{+2} &+ \frac{\rho_-(\rho_1 - \rho_+)\sqrt{T}}{R\rho_1} - (u+\sqrt{T})\frac{\rho_-(\rho_1-\rho_+)}{R\rho_1} \\ &+ \sqrt{u+\sqrt{T}}\frac{\sqrt{(\sqrt{T}-u)\rho_-\rho_+}}{R\rho_1}\sqrt{(\rho_1-\rho_-)(\rho_+-\rho_1)}\end{aligned} \tag{3.27}$$

and denoting $\overline{\beta} = \lim_{u \to -\sqrt{T}} \beta$ we observe

$$\overline{\beta} = v_{+2} + \frac{\rho_-(\rho_1 - \rho_+)\sqrt{T}}{R\rho_1} \tag{3.28}$$

We will not rewrite in detail the expression (3.16) for ε_1 and instead we introduce $\overline{\varepsilon_1} = \lim_{u \to -\sqrt{T}} \varepsilon_1$

$$\overline{\varepsilon_1} = \frac{p(\rho_+) - p(\rho_1)}{\rho_1} - \frac{\rho_+\rho_-^2(\rho_+ - \rho_1)T}{\rho_1^2 R^2} \tag{3.29}$$

Using Lemma 3.1 we have that $\overline{\varepsilon_1}(\rho_-) = \overline{\varepsilon_1}(\rho_+) = 0$ and $\overline{\varepsilon_1} > 0$ for all $\rho_1 \in (\rho_-, \rho_+)$.

In order to examine the admissibility inequalities we first study the signs of the expressions $\beta - v_{-2}$ and $v_{+2} - \beta$. Recalling $R < 0$ we see from (3.28) that

$$v_{+2} - \overline{\beta} = -\frac{\rho_-(\rho_1 - \rho_+)\sqrt{T}}{R\rho_1}, \tag{3.30}$$

so it is negative for $\rho_1 \in (\rho_-, \rho_+)$ with zero value in $\rho_1 = \rho_+$ and a strictly negative value in $\rho_1 = \rho_-$. Using a continuity argument we conclude $v_{+2} - \beta < 0$ at least on some interval $(\rho_-, \rho_+ - \varepsilon)$ with ε small for u close to $-\sqrt{T}$.

Similarly we have

$$\overline{\beta} - v_{-2} = \frac{\rho_+(\rho_1 - \rho_-)\sqrt{T}}{R\rho_1} \tag{3.31}$$

and again, this expression is negative for $\rho_1 \in (\rho_-, \rho_+)$ with zero value in $\rho_1 = \rho_+$ and a strictly negative value in $\rho_1 = \rho_-$. Thus we conclude $\beta - v_{-2} < 0$ at least on some interval $(\rho_- + \varepsilon, \rho_+)$ with ε small for u close to $-\sqrt{T}$.

Using this information we reformulate (3.22) and (3.23) as

$$\varepsilon_2 \le \frac{\rho_1 - \rho_-}{\rho_1\rho_-} P(\rho_-, \rho_1) - \varepsilon_1\rho_1 \frac{\beta + v_{-2}}{\beta - v_{-2}} \frac{\rho_1 - \rho_-}{\rho_1\rho_-} - \varepsilon_1 := M_1 \tag{3.32}$$

$$\varepsilon_2 \ge -\frac{\rho_+ - \rho_1}{\rho_+\rho_1} P(\rho_1, \rho_+) - \varepsilon_1\rho_1 \frac{v_{+2} + \beta}{v_{+2} - \beta} \frac{\rho_+ - \rho_1}{\rho_+\rho_1} - \varepsilon_1 := M_2. \tag{3.33}$$

To finish our proof we proceed as follows. We show first that in the limit $u \to -\sqrt{T}$ we have $M_1 > M_2$ for all $\rho_1 \in (\rho_-, \rho_+)$ and then that we can find some $s \in (\rho_-, \rho_+)$ for which $M_1(s) > 0$. By a continuity argument we then conclude that at least for u sufficiently close to $-\sqrt{T}$ we will still have $v_{+2} - \beta(s) < 0$, $\beta(s) - v_{-2} < 0$, $M_1(s) > M_2(s)$ and $M_1(s) > 0$, thus there exist ε_2 satisfying both (3.32) and (3.33) while $\varepsilon_1(s) > 0$.

Let us now therefore first express $\overline{M_1} = \lim_{u \to -\sqrt{T}} M_1$. In order to do this we start with $\frac{\overline{\beta} + v_{-2}}{\overline{\beta} - v_{-2}}$. We have

$$\frac{\overline{\beta} + v_{-2}}{\overline{\beta} - v_{-2}} = \frac{v_{+2} + v_{-2} + \frac{\rho_-\sqrt{T}(\rho_1 - \rho_+)}{\rho_1 R}}{v_{+2} - v_{-2} + \frac{\rho_-\sqrt{T}(\rho_1 - \rho_+)}{\rho_1 R}} = \frac{2v_{+2} + \sqrt{T}\left(\frac{2\rho_- - \rho_+}{R} - \frac{\rho_-\rho_+}{\rho_1 R}\right)}{\sqrt{T}\frac{\rho_+(\rho_1 - \rho_-)}{\rho_1 R}}$$

$$= \frac{2v_{+2}\rho_1 R}{\sqrt{T}\rho_+(\rho_1 - \rho_-)} + \frac{(2\rho_- - \rho_+)\rho_1 - \rho_-\rho_+}{\rho_+(\rho_1 - \rho_-)}$$

Plugging this into (3.32) we obtain

$$\varepsilon_2 \leq \frac{\rho_1 - \rho_-}{\rho_1 \rho_-} P(\rho_-, \rho_1) - \frac{\overline{\varepsilon_1} \rho_1}{\rho_- \rho_+} \left(2\rho_- - \rho_+ + \frac{2Rv_{+2}}{\sqrt{T}} \right) = \overline{M_1} \tag{3.34}$$

with $\overline{\varepsilon_1}$ given by (3.29). Similarly we handle $\overline{M_2} = \lim_{u \to -\sqrt{T}} M_2$. We have

$$\frac{v_{+2} + \overline{\beta}}{v_{+2} - \overline{\beta}} = \frac{2v_{+2} + \frac{\rho_- \sqrt{T}(\rho_1 - \rho_+)}{\rho_1 R}}{-\frac{\rho_- \sqrt{T}(\rho_1 - \rho_+)}{\rho_1 R}} = -1 - \frac{2v_{+2}\rho_1 R}{\sqrt{T}\rho_-(\rho_1 - \rho_+)}$$

and inserting this into (3.33) we get

$$\varepsilon_2 \geq -\frac{\rho_+ - \rho_1}{\rho_+ \rho_1} P(\rho_1, \rho_+) - \frac{\overline{\varepsilon_1} \rho_1}{\rho_- \rho_+} \left(\rho_- + \frac{2Rv_{+2}}{\sqrt{T}} \right) = \overline{M_2} \tag{3.35}$$

Now it is easy to show that $\overline{M_1} > \overline{M_2}$ just by comparing the two expressions as we have

$$\frac{\rho_1 - \rho_-}{\rho_1 \rho_-} P(\rho_-, \rho_1) + \frac{\rho_+ - \rho_1}{\rho_+ \rho_1} P(\rho_1, \rho_+) > \frac{\overline{\varepsilon_1} \rho_1}{\rho_- \rho_+} (\rho_- - \rho_+),$$

which obviously holds for all $\rho_1 \in (\rho_-, \rho_+)$ since the left hand side is positive and the right hand side is negative. This means we can always find ε_2 satisfying both (3.34) and (3.35).

Next we have to assure that such ε_2 can be chosen positive. This, however, follows from the fact that $\overline{M_1}(\rho_+) = \frac{\rho_+ - \rho_-}{\rho_+ \rho_-} P(\rho_-, \rho_+) > 0$. In particular, we can always find $s < \rho_+$ such that $\overline{M_1}(s) > \frac{\rho_+ - \rho_-}{2\rho_+ \rho_-} P(\rho_-, \rho_+)$ and then use the continuity argument to show that for u sufficiently close to $-\sqrt{T}$ there exists an admissible fan subsolution and therefore also infinitely many admissible weak solutions to the Euler equations (1.1).

3.2 The case R > 0

Now let us treat the case $R > 0$, i.e. $\rho_- > \rho_+$. Note that in the case $\rho_- = \rho_+$ the self-similar solution to the Riemann problem can consist only of two rarefaction waves or of two admissible shocks, in particular it is not possible for the self-similar solution to consist of one shock and one rarefaction wave.

The case $R > 0$ is on one hand quite similar to the case $R < 0$, on the other hand we have to use different equations to obtain the same result, therefore we emphasize here how to proceed in this case.

First, the expressions for ν_+ and ν_- have to be modified in order to obtain $\nu_- < \nu_+$. Instead of (3.9)-(3.10) we now have

$$\nu_- = \frac{A}{R} - \frac{\sqrt{-B}}{R} \sqrt{\frac{\rho_1 - \rho_+}{\rho_- - \rho_1}} \tag{3.36}$$

$$\nu_+ = \frac{A}{R} + \frac{\sqrt{-B}}{R} \sqrt{\frac{\rho_- - \rho_1}{\rho_1 - \rho_+}}. \tag{3.37}$$

In order to express β and ε_1 we now choose to work rather with the equation on the left interface (2.30)-(2.31) than on the right interface (2.32)-(2.33). Thus we obtain the expression for β as

$$\beta = \frac{\rho_- v_{-2}}{\rho_1} - \frac{(\rho_- - \rho_1)A}{R\rho_1} + \frac{\sqrt{-B}}{R\rho_1}\sqrt{(\rho_- - \rho_1)(\rho_1 - \rho_+)}. \tag{3.38}$$

and next

$$\varepsilon_1(\rho_1) = \frac{p(\rho_-) - p(\rho_1)}{\rho_1} - \frac{\rho_-(\rho_- - \rho_1)}{\rho_1^2}(\nu_- - v_{-2})^2, \tag{3.39}$$

and consequently using (3.36)

$$\varepsilon_1(\rho_1) = \frac{p(\rho_-) - p(\rho_1)}{\rho_1} - \frac{\rho_-}{\rho_1}\left(\frac{\sqrt{-B}}{R}\sqrt{1 - \frac{\rho_+}{\rho_1}} + \frac{\rho_+ u}{R}\sqrt{\frac{\rho_-}{\rho_1} - 1}\right)^2. \tag{3.40}$$

Now, similarly as in the case $\rho_- < \rho_+$, we can denote

$$K := \frac{\rho_+ u}{-R} \tag{3.41}$$

$$L := \frac{\sqrt{-B}}{R} \tag{3.42}$$

so that both $K, L > 0$ and obtain

$$\varepsilon_1(\rho_1) = \frac{p(\rho_-) - p(\rho_1)}{\rho_1} - \frac{\rho_-}{\rho_1}\left(L\sqrt{1 - \frac{\rho_+}{\rho_1}} - K\sqrt{\frac{\rho_-}{\rho_1} - 1}\right)^2. \tag{3.43}$$

Notice that expression (3.43) is the same as (3.16) just with switched indices + and −. In particular, we can use the proof of Lemma 3.1 to deduce that in the case $\rho_- > \rho_+$ we have

Lemma 3.2 *There exists a unique $\overline{\rho}$ such that*

$$\varepsilon_1 > 0 \quad \text{for} \quad \rho_1 \in (\rho_+, \overline{\rho}) \tag{3.44}$$

$$\varepsilon_1 < 0 \quad \text{for} \quad \rho_1 \in (\overline{\rho}, \rho_-). \tag{3.45}$$

Moreover, $\overline{\rho} \to \rho_-$ as $u \to -\sqrt{\frac{(\rho_- - \rho_+)(p(\rho_-) - p(\rho_+))}{\rho_+\rho_-}}$.

Concerning the admissibility conditions, the expressions (3.22)-(3.23) stay exactly the same. Instead of (3.25)-(3.26) we have

$$\nu_- = v_{+2} + \frac{\rho_-\sqrt{T}}{R} - \frac{\rho_-}{R}(u + \sqrt{T}) - \sqrt{u + \sqrt{T}}\frac{\sqrt{(\sqrt{T} - u)\rho_-\rho_+}}{R}\sqrt{\frac{\rho_1 - \rho_+}{\rho_- - \rho_1}} \tag{3.46}$$

$$\nu_+ = v_{+2} + \frac{\rho_-\sqrt{T}}{R} - \frac{\rho_-}{R}(u + \sqrt{T}) + \sqrt{u + \sqrt{T}}\frac{\sqrt{(\sqrt{T} - u)\rho_-\rho_+}}{R}\sqrt{\frac{\rho_- - \rho_1}{\rho_1 - \rho_+}}. \tag{3.47}$$

The expression (3.27) for β actually stays the same

$$\beta = v_{+2} + \frac{\rho_-(\rho_1-\rho_+)\sqrt{T}}{R\rho_1} - (u+\sqrt{T})\frac{\rho_-(\rho_1-\rho_+)}{R\rho_1} + \sqrt{u+\sqrt{T}}\frac{\sqrt{(\sqrt{T}-u)\rho_-\rho_+}}{R\rho_1}\sqrt{(\rho_1-\rho_-)(\rho_+-\rho_1)} \tag{3.48}$$

and in particular we also have $\overline{\beta} = \lim_{u\to-\sqrt{T}} \beta$ as

$$\overline{\beta} = v_{+2} + \frac{\rho_-(\rho_1-\rho_+)\sqrt{T}}{R\rho_1}. \tag{3.49}$$

Next we express $\overline{\varepsilon_1} = \lim_{u\to-\sqrt{T}} \varepsilon_1$ as

$$\overline{\varepsilon_1} = \frac{p(\rho_-)-p(\rho_1)}{\rho_1} - \frac{\rho_-\rho_+^2(\rho_- - \rho_1)T}{\rho_1^2 R^2} \tag{3.50}$$

and again use Lemma 3.2 to observe that $\overline{\varepsilon_1}(\rho_-) = \overline{\varepsilon_1}(\rho_+) = 0$ and $\overline{\varepsilon_1} > 0$ for all $\rho_1 \in (\rho_+, \rho_-)$.

The signs of $v_{+2} - \beta$ and $\beta - v_{-2}$ can be again shown to be negative at least on intervals $(\rho_+ + \varepsilon, \rho_-)$ and $(\rho_+, \rho_- - \varepsilon)$ respectively. Reverse order of ρ_- and ρ_+ causes the admissibility conditions to change to

$$\varepsilon_2 \geq -\frac{\rho_- - \rho_1}{\rho_1\rho_-}P(\rho_-, \rho_1) + \varepsilon_1\rho_1\frac{\beta + v_{-2}}{\beta - v_{-2}}\frac{\rho_- - \rho_1}{\rho_1\rho_-} - \varepsilon_1 := N_1 \tag{3.51}$$

$$\varepsilon_2 \leq \frac{\rho_1 - \rho_+}{\rho_+\rho_1}P(\rho_1, \rho_+) + \varepsilon_1\rho_1\frac{v_{+2} + \beta}{v_{+2} - \beta}\frac{\rho_1 - \rho_+}{\rho_+\rho_1} - \varepsilon_1 := N_2. \tag{3.52}$$

The rest of the proof is now following the same steps as in the case $R < 0$ with N_2 playing the role of M_1 and vice versa. We show that $\overline{N_2} > \overline{N_1}$ and since for $\rho_1 = \rho_-$ we have $\overline{N_2} > 0$ we again conclude the existence of an admissible subsolution. The proof is finished.

4 Concluding remarks

We mention here some more remarks about the problem, concentrating on the case $R < 0$. The consequences of the admissibility inequalities (3.22)-(3.23) heavily depend on the signs of the expressions $\beta - v_{-2}$ and $v_{+2} - \beta$. Whereas it can be shown that $\beta - v_{-2}$ is always negative (not only in the limit $\overline{\beta} - v_{-2}$ as was shown in (3.31)), this is not the case of $v_{+2} - \beta$, in fact we have $v_{+2} - \beta < 0$ on (ρ_-, ρ_T) and $v_{+2} - \beta > 0$ on (ρ_T, ρ_+), where $\rho_T = \frac{\rho_-\rho_+T}{\rho_-u^2+\rho_+(T-u^2)}$. In particular for fixed $u > -\sqrt{T}$ one can search for a subsolution in two regions. On the interval (ρ_-, ρ_T) the admissibility condition (3.23) transfers to (3.33) as stated in the previous section. However, on the interval (ρ_T, ρ_+) the sign in (3.33) is opposite and M_2 becomes the upper bound for ε_2, not a lower bound.

Note also that the special case $v_{+2} = \beta$, i.e. $\rho = \rho_T$, considerably simplifies the admissibility condition (3.23). In particular the inequality (3.23) becomes just $0 \leq -\varepsilon_1(\rho_T)\rho_T v_{+2}$ which is satisfied if and only if $v_{+2} \leq 0$. In this case the subsolution exists if $M_1(\rho_T) > 0$. As it turns out in the examples below, this is not the optimal strategy and for $v_{+2} < 0$ there are subsolutions with $\rho_1 > \rho_T$ in the case when $M_1(\rho_T) < 0$.

In the case $v_{+2} > 0$, there are no subsolutions with $\rho_1 \in (\rho_T, \rho_+)$ as the expression on the right hand side of (3.33) is negative on this interval. This no longer holds for $v_{+2} < 0$, where that expression becomes positive at least on some part of the interval (ρ_T, ρ_+) and there may exist subsolutions with such density ρ_1.

Finally let us discuss the special case $v_{+2} = 0$. In this case the admissibility inequality (3.23) simplifies to

$$\varepsilon_2 \geq -\frac{\rho_+ - \rho_1}{\rho_+\rho_1} P(\rho_1, \rho_+) - \frac{\rho_1}{\rho_+}\varepsilon_1 \qquad \text{for } \rho_1 \in (\rho_-, \rho_T) \tag{4.1}$$

$$\varepsilon_2 \leq -\frac{\rho_+ - \rho_1}{\rho_+\rho_1} P(\rho_1, \rho_+) - \frac{\rho_1}{\rho_+}\varepsilon_1 \qquad \text{for } \rho_1 \in (\rho_T, \rho_+) \tag{4.2}$$

and is trivially satisfied in $\rho_1 = \rho_T$. In particular it is easy to observe that the expression on the right hand side of (4.1) and (4.2) is negative whenever ε_1 is positive. This means that there cannot be any subsolution on (ρ_T, ρ_+), on the other hand the inequality (4.1) imposes no further restriction on ε_2 on (ρ_-, ρ_T).

In order to illustrate how large is the set of Riemann initial data for which the existence of infinitely many admissible weak solutions is proved in this note, we provide here some examples of Riemann data allowing for existence of infinitely many solutions.

Let us take similarly as the example in [6] the pressure law $p(\rho) = \rho^2$ and let $\rho_- = 1$, $\rho_+ = 4$. In this case we have $T = \frac{(\rho_+ - \rho_-)(p(\rho_+) - p(\rho_-))}{\rho_+\rho_-} = \frac{45}{4}$ and thus we are interested in Riemann data satisfying $v_{-2} - v_{+2} < \frac{\sqrt{45}}{2} \sim 3.35$. The case of the example in [6] was taken as $v_{+2} = 0$ and $v_{-2} = 2\sqrt{2}(\sqrt{\rho_+} - \sqrt{\rho_-}) = 2.83$.

Detail analysis of the problem gives us the following thresholds for various positive values of v_{+2}:

$$\begin{aligned} v_{+2} = 0.1 \quad &\Longrightarrow \quad V \sim 2.75 \\ v_{+2} = 1 \quad &\Longrightarrow \quad V \sim 2.955 \\ v_{+2} = 2 \quad &\Longrightarrow \quad V \sim 3.05 \end{aligned}$$

In this case the subsolution is obtained such that $v_{+2} - \beta$ is indeed negative and there are no subsolutions in the region where $v_{+2} - \beta$ is positive. Moreover it seems that V is increasing with v_{+2}.

For $v_{+2} = 0$ the value of V is approximately 2.7.

For negative values of v_{+2} we get even lower values of V meaning larger set of initial data for which nonuniqueness holds.

$$\begin{aligned} v_{+2} = -0.1 \quad &\Longrightarrow \quad V \sim 2.65 \\ v_{+2} = -1 \quad &\Longrightarrow \quad V \sim 1.8 \\ v_{+2} = -2 \quad &\Longrightarrow \quad V \sim 1.02 \end{aligned}$$

In this case the subsolutions near the critical values of V are obtained in the region where $v_{+2} - \beta$ is negative. Moreover it seems V is decreasing with $|v_{+2}|$ increasing.

Finally, for $R > 0$ similar remarks hold switching v_{-2} and v_{+2} and several inequality signs.

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