Solvability of problems involving inviscid fluids

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Driven Euler system

Field equations

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho \mathbf{u})\mathrm{d}t = 0$$

$$\mathrm{d}(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u})\mathrm{d}t + \nabla_x p(\varrho)\mathrm{d}t = \varrho \mathbf{G}(\varrho, \varrho \mathbf{u})\mathrm{d}W,$$

Stochastic forcing

$$\varrho \mathbf{G}(\varrho, \varrho \mathbf{u})\mathrm{d}W = \sum_{k=1}^{\infty} \varrho \mathbf{G}_k(\varrho, \varrho \mathbf{u})\mathrm{d}\beta_k$$

Iconic examples

$$\varrho \mathbf{G}(\varrho, \varrho \mathbf{u})\mathrm{d}W = \varrho \sum_{k=1}^{\infty} \mathbf{G}_k(x)\mathrm{d}\beta_k, \ \ \varrho \mathbf{G}(\varrho, \varrho \mathbf{u})\mathrm{d}W = \lambda \varrho \mathbf{u} \mathrm{d}\beta$$

Weak formulation

Field equations

$$\left[\int_\Omega \varrho\phi \, \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \varrho \mathbf{u} \cdot \nabla_x \phi \, \mathrm{d}x\mathrm{d}t,$$

$$\left[\int_\Omega \varrho \mathbf{u} \cdot \phi \, \mathrm{d}x\right]_{t=0}^{t=\tau} - \int_0^\tau \int_\Omega \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \phi + p(\varrho) \mathrm{div}_x \phi \, \mathrm{d}x\mathrm{d}t$$

$$= \boxed{\int_0^\tau \left(\int_\Omega \varrho \mathbf{G} \cdot \phi \, \mathrm{d}x \right) \mathrm{d}W}$$

$\phi = \phi(x) -$ a smooth test function

Stochastic integral (Itô's formulation)

$$\int_0^\tau \left(\int_\Omega \varrho \mathbf{G} \cdot \phi \, \mathrm{d}x \right) \mathrm{d}W = \sum_{k=1}^\infty \int_0^\tau \left(\int_\Omega \varrho \mathbf{G}_k \cdot \phi \, \mathrm{d}x \right) \mathrm{d}\beta_k$$

Existence theory

Local existence [Breit, EF, Hofmanová [2017]]

If the initial data are smooth, then the problem admits local-in-time smooth solutions. Solutions exist up to a (maximal) positive *stopping time*. The life-span is a random variable.

Weak–strong uniqueness [Breit, EF, Hofmanová [2016]]

A weak and strong solutions coincide as long as the latter exists. More specifically, their *laws* are the same provided the laws of the initial data are the same

Semi-deterministic approach - additive noise

"Additive noise" problem

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \varrho \sum_{k=1}^{\infty} \mathbf{G}_k \partial_t \beta_k$$

$$\varrho \sum_{k=1}^{\infty} \mathbf{G}_k \partial_t \beta_k = \varrho \mathbf{G} \mathrm{d} W$$

Additive noise, Step I

Step I

$$\partial_t(\varrho\mathbf{u}-\varrho\mathbf{G}W)+\mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u})+\nabla_x p(\varrho)=-\partial_t\varrho\mathbf{G}W=\mathrm{div}_x(\varrho\mathbf{u})\mathbf{G}W$$

Tranformed system I

$$\mathbf{w}=\varrho\mathbf{u}-\varrho\mathbf{G}W$$

$$\partial_t\varrho+\mathrm{div}_x(\mathbf{w}+\varrho\mathbf{G}W)=0$$

$$\partial_t\mathbf{w}+\mathrm{div}_x\left(\frac{(\mathbf{w}+\varrho\mathbf{G}W)\otimes(\mathbf{w}+\varrho\mathbf{G}W)}{\varrho}\right)+\nabla_x p(\varrho)$$

$$=\mathrm{div}_x(\mathbf{w}+\varrho\mathbf{G}W)\mathbf{G}W$$

Additive noise, Step II

Step II

$$\mathbf{w} = \mathbf{v} + \mathbf{V} + \nabla_x \Phi, \ \mathrm{div}_x \mathbf{v} = 0, \int_\Omega \mathbf{v} \ \mathrm{d}x = 0, \ \mathbf{V} = \mathbf{V}(t)$$

Tranformed system II

$$\mathbf{w} = \varrho \mathbf{u} - \varrho \mathbf{G} W$$

$$\partial_t \varrho + \mathrm{div}_x (\nabla_x \Phi + \varrho \mathbf{G} W) = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{V} + \nabla_x \Phi + \varrho \mathbf{G} W) \otimes (\mathbf{v} + \mathbf{V} + \nabla_x \Phi + \varrho \mathbf{G} W)}{\varrho} \right)$$

$$+ \nabla_x p(\varrho) + \nabla_x \partial_t \Phi = \mathrm{div}_x (\nabla_x \Phi + \varrho \mathbf{G} W) \mathbf{G} W - \partial_t \mathbf{V}$$

Additive noise, Step III

Step III

Fix Φ, ϱ, $\mathbf{V}$ so that

$$\varrho(0,\cdot) = \varrho_0, \ \mathbf{V}(0) = \frac{1}{|\Omega|}\int_\Omega \mathbf{u}_0 \ \mathrm{d}x, \ \nabla_x \Phi = \mathbf{H}^{\perp}[\mathbf{u}_0]$$

$$\partial_t \varrho + \mathrm{div}_x(\nabla_x \Phi + \varrho \mathbf{G} W) = 0$$

$$\mathbf{V}(0) = \frac{1}{|\Omega|}\int_\Omega \mathbf{u}_0 \ \mathrm{d}x$$

$$\partial_t \mathbf{V} = \frac{1}{|\Omega|}\mathrm{div}_x(\nabla_x \Phi + \varrho \mathbf{G} W)\mathbf{G} W$$

$$\mathrm{div}_x \left(\nabla_x \mathbf{M} + \nabla_x \mathbf{M}^{\perp} - \frac{2}{N}\mathrm{div}_x \mathbf{M} \right)$$

$$= \mathrm{div}_x(\nabla_x \Phi + \varrho \mathbf{G} W)\mathbf{G} W - \partial_t \mathbf{V}$$

Additive noise, Step IV

Step IV

Fix Φ, ϱ, $\mathbf{V}$ so that

$$\mathbf{h} = \mathbf{V} + \nabla_x \Phi + \varrho \mathbf{G} W,\ \mathbb{H} = \nabla_x \mathbf{M} + \nabla_x^t \mathbf{M} - \frac{2}{N} \mathrm{div}_x \mathbf{M} \mathbb{I} \in R^{N \times N}_{0,\mathrm{sym}}$$

Tranformed system III

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{h}) \otimes (\mathbf{v} + \mathbf{h})}{\varrho} - \mathbb{H} + p(\varrho)\mathbb{I} + \partial_t \Phi \mathbb{I} \right) = 0$$

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0] - \frac{1}{|\Omega|} \int_\Omega \mathbf{u}_0 \ \mathrm{d}x$$

Additive noise, Step V

Prescribing the kinetic energy

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} = e = \Lambda - \frac{N}{2}\left(p(\varrho) + \partial_t \Phi\right), \ \Lambda = \Lambda(t)$$

Tranformed system IV

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v}+\mathbf{h}) \otimes (\mathbf{v}+\mathbf{h})}{\varrho} - \frac{1}{N}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho}\mathbb{I} - \mathbf{M} \right) = 0$$

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} = e$$

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0$$

Multiplicative noise

"Multiplicative noise" problem

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \lambda \varrho \mathbf{u} \mathrm{d} W$$

Multiplicative noise, Step I

Step I

$$\exp\left(-\lambda W\right)\left(\mathrm{d}(\varrho\mathbf{u}) - \lambda\varrho\mathbf{u}\,\mathrm{d}W\right)$$
$$= -\exp\left(-\lambda W\right)\left(\mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p(\varrho)\right)\mathrm{d}t$$

Itô's chain rule

$$\mathrm{d}\exp\left(-\lambda W\right) = -\lambda\exp\left(-\lambda W\right)\mathrm{d}W + \frac{\lambda^2}{2}\exp\left(-\lambda W\right)\mathrm{d}t$$

Transformed system I

$$\mathrm{d}\left(\varrho\mathbf{u}\exp\left(-\lambda W\right)\right) + \lambda^2\varrho\mathbf{u}\exp\left(-\lambda W\right)\mathrm{d}t$$
$$= -\exp\left(-\lambda W\right)\left(\mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p(\varrho)\right)\mathrm{d}t$$

Multiplicative noise, Step II

Step II

$$\mathbf{w} = \varrho \mathbf{u} \exp\left(-\lambda W\right)$$

Transformed system II

$$\mathrm{d}\varrho + \mathrm{div}_x \left(\exp\left(\lambda W\right) \mathbf{w}\right) = 0$$

$$\mathrm{d}\mathbf{w} + \frac{\lambda^2}{2} \mathbf{w} \mathrm{d}t + \exp(\lambda W) \left[\mathrm{div}_x \frac{\mathbf{w} \otimes \mathbf{w}}{\varrho} + \nabla_x p(\varrho) \right] \mathrm{d}t = 0$$

Multiplicative noise, Step III

Step III

$$\mathbf{w} = \mathbf{v} + \mathbf{V} + \nabla_x \Phi, \ \mathrm{div}_x \mathbf{v} = 0$$

$$\mathrm{d}\varrho + \mathrm{div}_x \left(\exp\left(\lambda W\right) \nabla_x \Phi \right) = 0$$

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} + \frac{\lambda^2}{2}\mathbf{V} = 0, \ \mathbf{V}(0) = \frac{1}{|\Omega|} \int_\Omega \mathbf{u}_0 \ \mathrm{d}x = 0$$

Transformed system III

$$\mathrm{d}\mathbf{v} + \frac{\lambda^2}{2}\mathbf{v}\mathrm{d}t + \nabla_x \partial_t \Phi + \frac{\lambda^2}{2} \nabla_x \Phi \mathrm{d}t$$

$$+ \exp(\lambda W) \left[\mathrm{div}_x \frac{(\mathbf{v} + \mathbf{V} + \nabla_x \Phi) \otimes (\mathbf{v} + \mathbf{V} + \nabla_x \Phi)}{\varrho} + \nabla_x p(\varrho) \right] \mathrm{d}t = 0$$

Multiplicative noise, Step IV

Step IV

$$\mathbf{h} = \mathbf{V} + \nabla_x \Phi$$

$$\mathrm{div}_x \left(\mathbf{M} + \mathbf{M}^{\perp} - \frac{1}{N} \mathrm{div}_x \mathbf{M} \mathbb{I} \right) = - \frac{\lambda^2}{2} \boxed{\mathbf{v}}$$

Transformed system IV

$$\partial_t \mathbf{v} +$$
$$+ \exp(\lambda W) \left[\mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{h}) \otimes (\mathbf{v} + \mathbf{h})}{\varrho} + \left(p(\varrho) + \partial_t \Phi + \lambda^2 / 2 \Phi \right) \mathbb{I} \right) \right]$$
$$- \mathrm{div}_x \mathbb{M}[\mathbf{v}] \mathrm{d}t = 0$$

Multiplicative noise, Step V

Step V

$$r = \exp(-\lambda W)\varrho$$

Transformed system V

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v}+\mathbf{h}) \otimes (\mathbf{v}+\mathbf{h})}{r} - \frac{1}{N}\frac{|\mathbf{v}+\mathbf{h}|^2}{r}\mathbb{I} - \mathbb{M}[\mathbf{v}] \right)$$

$$\frac{1}{2}\frac{|\mathbf{v}|^2}{r} = e = \Lambda(t) - \frac{N}{2}\exp(\lambda W)p(\varrho) - \partial_t \Phi - \frac{\lambda^2}{2}\Phi$$

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0] - \frac{1}{|\Omega|}\int_\Omega \mathbf{u}_0 \ \mathrm{d}x$$

Abstract formulation

Variable coefficients "Euler system"

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{H}[\mathbf{v}]) \odot (\mathbf{v} + \mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} + \mathbb{M}[\mathbf{v}] \right) = 0$$

$$\mathrm{div}_x \mathbf{v} = 0,$$

Kinetic energy

$$\frac{1}{2} \frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]} = E[\mathbf{v}]$$

Data

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{v}(T, \cdot) = \mathbf{v}_T$$

Abstract operators

Boundedness

b maps bounded sets in $L^\infty((0,T) \times \Omega; R^N)$ on bounded sets in $C_b(Q, R^M)$

Continuity

$$b[\mathbf{v}_n] \to b[\mathbf{v}] \text{ in } C_b(Q; R^M) \text{ (uniformly for } (t,x) \in Q \text{)}$$

whenever

$$\mathbf{v}_n \to \mathbf{v} \text{ in } C_{\mathrm{weak}}([0,T]; L^2(\Omega; R^N))$$

Causality

$\mathbf{v}(t,\cdot) = \mathbf{w}(t,\cdot)$ for $0 \le t \le \tau \le T$ implies $b[\mathbf{v}] = b[\mathbf{w}]$ in $[(0,\tau] \times \Omega]$

Subsolutions

Field equations, differential constraints

$$\partial_t \mathbf{v} + \mathrm{div}_x \mathbb{F} = 0, \ \mathrm{div}_x \mathbf{v} = 0$$

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0, \ \mathbf{v}(T,\cdot) = \mathbf{v}_T$$

Non-linear constraint

$$\mathbf{v} \in C(Q; R^N), \ \mathbb{F} \in C(Q; R^{N\times N}_{\mathrm{sym},0}),$$

$$\frac{N}{2}\lambda_{\max}\left[\frac{(\mathbf{v}+\mathbf{H}[\mathbf{v}])\otimes(\mathbf{v}+\mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} - \mathbb{F} + \mathbb{M}[\mathbf{v}]\right] < E[\mathbf{v}]$$

Subsolution relaxation

Algebraic inequality

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]} \leq \frac{N}{2}\lambda_{\max}\left[\frac{(\mathbf{v}+\mathbf{H}[\mathbf{v}])\otimes(\mathbf{v}+\mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} - \mathbb{F} + \mathbb{M}[\mathbf{v}]\right]$$

$$< E[\mathbf{v}]$$

Solutions

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]} = E[\mathbf{v}]$$

$$\Rightarrow$$

$$\mathbb{F} = \frac{(\mathbf{v}+\mathbf{H}[\mathbf{v}])\odot(\mathbf{v}+\mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} + \mathbb{M}[\mathbf{v}]$$

Oscillatory lemma

Hypotheses:

$$U \subset R \times R^N, \ N = 2, 3 \text{ bounded open set}$$

$$\tilde{\mathbf{h}} \in C(U; R^N), \ \tilde{\mathbb{H}} \in C(U; R^{N \times N}_{\mathrm{sym},0}), \ \tilde{e}, \ \tilde{r} \in C(U), \ \tilde{r} > 0, \ \tilde{e} \le \overline{e} \text{ in } U$$

$$\frac{N}{2} \lambda_{\max} \left[\frac{\tilde{\mathbf{h}} \otimes \tilde{\mathbf{h}}}{\tilde{r}} - \tilde{\mathbb{H}} \right] < \tilde{e} \text{ in } U.$$

Conclusion:

$$\mathbf{w}_n \in C_c^\infty(U; R^N),\ \mathbb{G}_n \in C_c^\infty(U; R^{N\times N}_{\mathrm{sym},0}),\ n = 0, 1, \ldots$$

$$\partial_t \mathbf{w}_n + \mathrm{div}_x \mathbb{G}_n = 0,\ \mathrm{div}_x \mathbf{w}_n = 0 \text{ in } R \times R^N,$$

$$\frac{N}{2} \lambda_{\max} \left[\frac{(\tilde{\mathbf{h}} + \mathbf{w}_n) \otimes (\tilde{\mathbf{h}} + \mathbf{w}_n)}{\tilde{r}} - (\tilde{\mathbb{H}} + \mathbb{G}_n) \right] < \tilde{e} \text{ in } U,$$

$$\mathbf{w}_n \to 0 \text{ weakly in } L^2(U; R^N)$$

$$\liminf_{n\to\infty} \int_U \frac{|\mathbf{w}_n|^2}{\tilde{r}} \ \mathrm{d}x\mathrm{d}t \geq \Lambda(\overline{e}) \int_U \left(\tilde{e} - \frac{1}{2} \frac{|\tilde{\mathbf{h}}|^2}{\tilde{r}} \right)^2 \ \mathrm{d}x\mathrm{d}t$$

Basic ideas of proof

Localization

Localizing the result of DeLellis and Széhelyhidi to “small” cubes by means of scaling arguments

Linearization

Replacing all continuous functions by their means on any of the “small” cubes

Eliminating singular sets

Applying Whitney’s decomposition lemma to the non-singular sets (e.g. out of the vacuum $\{h = 0\}$)

Energy and other coefficients depending on solutions

Applying compactness of the abstract operators in C

Results

Result (A)

The set of subsolutions is non-empty $\Rightarrow$ there exists infinitely many weak solutions of the problem with the same initial data

Initial energy jump

$$\frac{1}{2}\frac{|\mathbf{v_0}+\mathbf{H}[\mathbf{v_0}]|^2}{h[\mathbf{v_0}]} \boxed{<} \liminf_{t\to 0}\frac{1}{2}\frac{|\mathbf{v}+\mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]}$$

Result (B)

The set of subsolutions is non-empty $\Rightarrow$ there exists a dense set of times t_0 such that the values $\mathbf{v}(t)$ give rise to non-empty subsolution set with

$$\frac{1}{2}\frac{|\mathbf{v_0}+\mathbf{H}[\mathbf{v_0}]|^2}{h[\mathbf{v_0}]} \boxed{=} \liminf_{t\to t_0}\frac{1}{2}\frac{|\mathbf{v}+\mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]}$$