INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

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Preprint No. 45-2016

PRAHA 2016

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Abstract

We consider the motion of a rigid body, governed by the Navier-Stokes equations in a bounded domain. Navier's condition is prescribed on the boundary of the body. We give the global in a time solvability result of weak solution. The result permits a possibility of collisions of the body with the boundary of the domain.

1. Presentation of the problem

We investigate the motion of a rigid body inside of a viscous incompressible fluid. Let Ω be a bounded domain of $\mathbb{R}^N$ for $N = 2$ or 3. At the initial moment $t = 0$ the body and the fluid occupy an open connected set $S_0 \subset \Omega$ and the set $F_0 = \Omega \backslash \overline{S_0}$, respectively. The motion of any point $\mathbf{y} = (y_1, .., y_N)^T \in S_0$ is described by a preserving orientation isometry

$$\mathbf{A}(t, \mathbf{y}) = \mathbf{q}(t) + \mathbb{Q}(t)(\mathbf{y} - \mathbf{q}(0)), \qquad t \in [0, T], \tag{1}$$

where $\mathbf{q} = \mathbf{q}(t)$ is the body mass center and $\mathbb{Q} = \mathbb{Q}(t)$ is the rotation matrix, such that $\mathbb{Q}(t)\mathbb{Q}(t)^T = \mathbb{I}$, $\mathbb{Q}(0) = \mathbb{I}$ with $\mathbb{I}$ being the identity matrix. Hence the body and the fluid occupy the sets $S(t) = A(t, S_0)$ and $F(t) = \Omega \backslash \overline{S(t)}$ at any time t. The velocity of the body is related with the isometry $\mathbf{A}$ by

$$\mathbf{u} = \mathbf{q}'(t) + \mathbb{P}(t)(\mathbf{x} - \mathbf{q}(t)) \qquad \text{for } \mathbf{x} \in S(t), \tag{2}$$

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Preprint submitted to Elsevier *7th September 2016*

where a matrix $\mathbb{P}(t)$ fulfills $\frac{d\mathbb{Q}}{dt}\mathbb{Q}^T = \mathbb{P}$, such that there exists a vector $\boldsymbol{\omega} = \boldsymbol{\omega}(t) \in \mathbb{R}^N$, satisfying $\mathbb{P}(t)\mathbf{x} = \boldsymbol{\omega}(t) \times \mathbf{x}, \ \forall \mathbf{x} \in \mathbb{R}^N$.

The motion of the fluid and the body is governed by the following system

$$
\begin{aligned}
&\partial_t \rho + (\mathbf{u}\cdot\nabla)\rho = 0, \qquad \mathrm{div}\mathbf{u} = 0, \qquad \rho(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) = \mathrm{div}P + \mathbf{g} \quad \text{for } \mathbf{x} \in F(t),\\
&\qquad m\mathbf{q}'' = \int_{\partial S(t)} P\mathsf{n}\, d\mathbf{x} + \int_{S(t)} \mathbf{g}\, d\mathbf{x},\\
&\qquad \frac{d(\mathbb{J}\boldsymbol{\omega})}{dt} = \int_{\partial S(t)} (\mathbf{x}-\mathbf{q}(t)) \times P\mathsf{n}\, d\mathbf{x} + \int_{S(t)} (\mathbf{x}-\mathbf{q}(t)) \times \mathbf{g}\, d\mathbf{x} \quad \text{for } \ \mathbf{x} \in S(t)
\end{aligned} \tag{3}
$$

with the initial conditions

$$
S = S_0, \quad \rho = \rho_0, \quad \mathbf{u} = \mathbf{u}_0 \quad \text{at } t = 0. \tag{4}
$$

Here $m = \int_{S(t)} \rho\, d\mathbf{x}-$ the mass of the body; ρ is the density in the body $S(t)$ and in the fluid $F(t)$; $P = -pI + 2\mu_f\, \mathbb{D}\mathbf{u}$ and $\mathbb{D}\mathbf{u} = \frac{1}{2}\{\nabla\mathbf{u} + (\nabla\mathbf{u})^T\}$ -the stress and the deformation-rate tensors of the fluid; p -the fluid pressure; $\mu_f > 0$ - the constant viscosity of the fluid; n -the unit outward normal to $\partial S(t)$; $\mathbb{J} = \int_{S(t)} \rho(|\mathbf{x}-\mathbf{q}(t)|^2\mathbb{I} - (\mathbf{x}-\mathbf{q}(t)) \otimes (\mathbf{x}-\mathbf{q}(t)))\, d\mathbf{x}$ -the matrix of the inertia moments of the body; $\mathbf{g}$-an external force.

The global existence of weak solution has been treated by many mathematicians: Hoffmann, Starovoitov, Conca, San Martín, Tucsnak, Feireisl, Nečasová, Hillairet, Bost, Cottet, Maitre; Desjardins, Esteban, Gunzburger, Lee, Seregin, Takahashi and etc.. All of these authors have considered non-slip boundary condition on boundaries of the body and the domain, but this boundary condition gives a paradoxical result of no collisions between the body and the boundary of the domain: Hesla [9], Hillairet [10], Starovoitov [15]. In the articles [7], [14], [15] the autors have studied the question of possible collisions with respect of the regularity of velocity and the regularity of boundaries. For instance, in [7] Gérard-Varet, Hillairet have demonstrated that under $C^{1,\alpha}$-boundaries the collision is possible in finite time if and only if $\alpha < 1/2$. These mentioned results have demonstrated that a more accurate model is needed for the description of the motion of bodies in a viscous incompressible fluid.

Neustupa, Penel [12] have investigated a prescribed collision of a ball with a wall, when the slippage is allowed on the boundaries of the ball and of the wall. The slippage is prescribed by Navier's boundary condition, having only the continuity of velocity field just in the normal component. This pioneer result [12] have shown that the slip boundary condition cleans the no-collision paradox. Recently Gérard-Varet, Hillairet [6] have proved a local-in-time existence result: up to collisions. The motion of a single body, moved in the whole space $\mathbb{R}^3$, have considered in [13]. The free fall of a ball above a wall with the slippage, prescribed on the boundaries, have been studied in [8], where it was shown that the ball touches the boundary of the wall in a finite time.

In this article we close system (3) by adding Navier's boundary condition

$$
\mathbf{u}_s \cdot \mathsf{n} = \mathbf{u}_f \cdot \mathsf{n}, \qquad (P_f \mathsf{n} + \gamma(\mathbf{u}_f - \mathbf{u}_s)) \cdot \mathsf{s} = 0 \quad \text{on } \partial S(t) \tag{5}
$$

and Dirichlet's boundary condition

$$
\mathbf{u} = 0 \quad \text{on } \partial\Omega. \tag{6}
$$

Here $\mathbf{u}_s$ and $\mathbf{u}_f$ are the trace values of the velocity $\mathbf{u}$ on $\partial S(t)$ from the rigid side $S(t)$ and from the fluid side $F(t)$, respectively; n and s are the external normal and arbitrary tangent vector to $\partial S(t)$; the constant $\gamma > 0$ is the friction coefficient of ∂S_0.

2. Weak solution of system (1)-(6) and the main result

To introduce the concept of weak solution for system (1)-(6), let us define some spaces of functions

$$V^{0,2}(\Omega) = \{\mathbf{v} \in L^2(\Omega) : \operatorname{div} \mathbf{v} = 0 \quad \text{in } \mathcal{D}'(\Omega), \quad \mathbf{v} \cdot \mathsf{n} = 0 \quad \text{in } H^{-1/2}(\partial\Omega)\},$$
$$BD_0(\Omega) = \left\{\mathbf{v} \in L^1(\Omega) : \mathbb{D}\mathbf{v} \in \mathcal{M}(\Omega), \quad \mathbf{v} = 0 \quad \text{on } \partial\Omega\right\},$$

where n is the unit normal to the boundary $\partial\Omega$ of Ω and $\mathcal{M}(\Omega)$ is the space of bounded Radon measures. Let S be an open connected subset of Ω. We consider the space

$$KB(S) = \{\mathbf{v} \in BD_0(\Omega) : \mathbb{D}\mathbf{v} \in L^2(\Omega \backslash \overline{S}), \ \ \mathbb{D}\mathbf{v} = 0 \ \text{ a.e. on } S, \ \ \operatorname{div}\mathbf{v} = 0 \quad \text{in } \mathcal{D}'(\Omega)\}.$$

Définition 2.1 *The triple $\{\mathbf{A}, \rho, \mathbf{u}\}$ is a weak solution of system* (1)-(6)*, if the following three conditions are fulfilled:*

1) The function $\mathbf{A}(t,\cdot) : \mathbb{R}^N \to \mathbb{R}^N$ is a preserving orientation isometry (1)*, such that the functions $\mathbf{q}$, $\mathbb{Q}$ are absolutely continuous on $[0,T]$. The isometry $\mathbf{A}$ is compatible with the rigid body velocity* (2) *on $S(t)$ and defines a time dependent set $S(t) = \mathbf{A}(t, S_0)$;*

2) The function $\rho \in L^\infty((0,T) \times \Omega)$ satisfies the integral equality

$$\int_0^T \int_\Omega \rho(\xi_t + (\mathbf{u}\cdot\nabla)\xi)\, dt d\mathbf{x} = -\int_\Omega \rho_0 \xi(0,\cdot)\, d\mathbf{x} \quad \text{for any } \ \xi \in C^1([0,T]\times\overline{\Omega}), \quad \xi(T,\cdot) = 0;$$

3) The function $\mathbf{u} \in L^2(0,T; KB(S(t))) \cap L^\infty(0,T; V^{0,2}(\Omega))$ satisfies the integral equality

$$\int_0^T \int_{\Omega \backslash \partial S(t)} \{\rho \mathbf{u} \boldsymbol{\psi}_t + \rho(\mathbf{u} \otimes \mathbf{u}) : \mathbb{D}\boldsymbol{\psi} - 2\mu_f\, \mathbb{D}\mathbf{u} : \mathbb{D}\boldsymbol{\psi} + \mathbf{g}\boldsymbol{\psi}\} d\mathbf{x} dt$$
$$= -\int_\Omega \rho_0 \mathbf{u}_0 \boldsymbol{\psi}(0,\cdot)\, d\mathbf{x} + \int_0^T \left\{ \int_{\partial S(t)} \beta(\mathbf{u}_s - \mathbf{u}_f)(\boldsymbol{\psi}_s - \boldsymbol{\psi}_f)\, d\mathbf{x} \right\} dt \tag{7}$$

for any $\boldsymbol{\psi} \in L^2(0,T; KB(S(t)))$, such that $\boldsymbol{\psi}_t \in L^2(0,T; L^2(\Omega \backslash \partial S(t)))$ and $\boldsymbol{\psi}(T,\cdot) = 0$. Here we denote the trace values of $\mathbf{u}$, $\boldsymbol{\psi}$ on $\partial S(t)$ from the rigid side $S(t)$ and the fluid side $F(t)$ by $\mathbf{u}_s(t,\cdot)$, $\boldsymbol{\psi}_s(t,\cdot)$ and $\mathbf{u}_f(t,\cdot)$, $\boldsymbol{\psi}_f(t,\cdot)$, respectively.

Our main result is the following theorem.

Theorem 2.1 *We assume that $S_0 \subset \Omega$, such that $dist[S_0, \Omega] > 0$. We admit that the boundaries $\partial\Omega \in C^{0,1}$ and $\partial S_0 \in C^2$. Let*

$$\rho_0(\mathbf{x}) = \begin{cases} \rho_s(\mathbf{x}) \geqslant const > 0, & \mathbf{x} \in S_0; \\ \rho_f = const > 0, & \mathbf{x} \in F_0, \end{cases} \qquad \rho_s \in L^\infty(S_0),$$

$$\mathbf{u}_0 \in V^{0,2}(\Omega), \quad \mathbb{D}\mathbf{u}_0 = 0 \quad \text{in } \ \mathcal{D}'(S_0) \quad \text{and} \quad \mathbf{g} \in L^2((0,T); (LD_0^2(\Omega))^*).$$

Then system (1)-(6) *possesses a weak solution $\{\mathbf{A}, \rho, \mathbf{u}\}$, such that the isometry $\mathbf{A}(t,\cdot)$ is Lipschitz continuous with respect to $t \in [0,T]$,*

$$\rho(t,\mathbf{x}) = \begin{cases} \rho_s(\mathbf{A}^{-1}(t,\mathbf{x})), & \mathbf{x} \in S(t); \\ \rho_f, & \mathbf{x} \in F(t), \end{cases} \qquad \text{for a.e. } t \in (0,T),$$

$\mathbf{u} \in C_{\rm weak}(0,T;V^{0,2}(\Omega))$ *and the following energy inequality holds*

$$\frac{1}{2}\int_\Omega \rho|\mathbf{u}|^2(r)\ d\mathbf{x} + \int_0^r \left\{\int_{F(t)} 2\mu_f\,|\mathbb{D}\,\mathbf{u}|^2\ d\mathbf{x} + \int_{\partial S(t)} \beta|\mathbf{u}_f - \mathbf{u}_s|^2\ d\mathbf{x}\right\} dt$$
$$\leqslant \frac{1}{2}\int_\Omega \rho_0|\mathbf{u}_0|^2\ d\mathbf{x} + \int_0^r <\mathbf{g},\mathbf{u}>\ dt \qquad \textit{for a.e. } r\in(0,T).$$

Let us point that in [8] it has been shown that the ball never touches the boundary of the wall for mixed boundary conditions (5), (6). Nevertheless of the result [8], the contacts of the body and the boundary of the domain are available in Theorem 2.1, due to the low regularity of the boundaries $\partial\Omega \in C^{0,1}$, $\partial S_0 \in C^2$. And moreover $\mathbf{g} \in L^2((0,T);(LD^2_0(\Omega))^*)$, we refer to the example constructed by Starovoitov [15]. In order to create a collision of the body with the boundary of the domain (in the case of non-slip conditions on the boundaries $\partial\Omega$ and ∂S_0), Starovoitov adds an external force from H^{-1}-space.

3. Sketch of the proof of Theorem

First we introduce an approximate scheme to system (1)-(6), using the idea that the "body+fluid" can be approximated by a *non-homogeneous* fluid, having different values of viscosity in three zones: approximation of "body", approximation of Navier's boundary condition (5) and "fluid" zone.

To construct such approximation problem we fix the following notations. For an open connected set $S \subset \mathbb{R}^N$, we define ${\rm dist}[\mathbf{x},S] = \inf_{\mathbf{y}\in S}|\mathbf{x}-\mathbf{y}|$, $d_S(\mathbf{x}) = {\rm dist}[\mathbf{x},\mathbb{R}^N\setminus S] - {\rm dist}[\mathbf{x},S]$ for any $\mathbf{x}\in\mathbb{R}^N$, $[S]_\delta = d_S^{-1}((\delta,+\infty))$ - the $\delta-$kernel of S and $]S[_\delta = d_S^{-1}((-\delta,+\infty))$ - the δ - neighborhood of S.

As Ω is a bounded domain, we assume that $\Omega \subset]-L,L[^N = \mathcal{T}$ for a certain $L>0$. Let us extend the functions ρ_0, $\mathbf{u}_0$ and $\mathbf{g}$ by zero values on $\mathcal{T}$. Let us consider the characteristic functions $\xi(\mathbf{x})$, $\varphi_0(\mathbf{x})$ and $\chi_0^\delta(\mathbf{x})$ of the sets $\mathcal{T}\setminus\overline{[\Omega]_{2\tau}}$, S_0 and $]S_0[_\delta\setminus\overline{S_0}$, defined on the whole $\mathcal{T}$. Finally we define $\rho_0^{\varepsilon\delta} = (1-\chi_0^\delta)\rho_0 + \varepsilon\chi_0^\delta$.

The approximation problem to system (1)-(6) consists from the transport equations

$$\partial_t\rho + (\overline{\mathbf{u}}\cdot\nabla)\rho = 0,\quad \partial_t\varphi + (\overline{\mathbf{u}}\cdot\nabla)\varphi = 0,\quad \partial_t\chi + (\overline{\mathbf{u}}\cdot\nabla)\chi = 0 \quad \text{in } (0,T)\times]\mathcal{T}[_\tau,$$
$$\rho(0) = \rho_0^{\varepsilon\delta},\qquad \varphi(0) = \varphi_0,\qquad \chi(0) = \chi_0^\delta \qquad \text{in }]\mathcal{T}[_\tau, \tag{8}$$

and the momentum equation

$$\int_0^T\{\int_{\mathcal{T}}[\rho\mathbf{u}\partial_t\boldsymbol{\psi} + \rho\mathbf{u}(\overline{\mathbf{u}}\cdot\nabla)\boldsymbol{\psi} - \xi_\varepsilon\mathbf{u}\boldsymbol{\psi} - \mu_\delta\mathbb{D}\mathbf{u}:\mathbb{D}\boldsymbol{\psi} + \rho\mathbf{g}\boldsymbol{\psi}]\ d\mathbf{x}$$
$$- \int_{]\mathcal{T}[_\tau}\zeta_\varepsilon\mathbb{D}\overline{\mathbf{u}}:\mathbb{D}\overline{\boldsymbol{\psi}}\ d\mathbf{x}\}\ dt = -\int_{\mathcal{T}}\rho_0^{\varepsilon\delta}\mathbf{u}_0\boldsymbol{\psi}(0,\cdot)\ d\mathbf{x}, \tag{9}$$

which is valid for any test function $\boldsymbol{\psi} \in L^{2(N-1)}(0,T;V^{1,2}(\mathcal{T}))\cap H^1((0,T)\times\mathcal{T})$: $\boldsymbol{\psi}(T,\cdot) = 0$. Here

$$\xi_\varepsilon = \frac{1}{\varepsilon}\xi,\quad \mu_\delta = \varphi + 2\mu_f\theta + \beta_0\chi\int_{]\mathcal{T}[_\tau}\chi\ d\mathbf{x},\quad \zeta_\varepsilon = \frac{1}{\varepsilon}\varphi,\quad \theta = 1-(\varphi+\chi)$$

with the constants $\beta_0 = \frac{\beta}{|\partial S_0|}$, $|\partial S_0| = \int_{\partial S_0} 1 d\mathbf{x}$. The function $\overline{\mathbf{u}}$ is the standard mollification of $\mathbf{u}$ on the parameter τ.

In relation (9) the "viscosity" term ζ_ε is an analog of penalization, introduced in [11], where the rigid body is replaced by a fluid, having high viscosity value. The second "viscosity" μ_δ defines a mixture region between the fluid and the "body", which approximates the jump boundary term on $\partial S(t)$ in (7). The penalization ξ_ε was developed in [5], which is used here just for technical purposes. The solvability of this approximation problem (8)-(9) can be shown by Galerkin's method and theoretical results for transport equations (see [1], [2], [16]).

Next in the approximation problem we have to pass on limits with respect of the parameters ε, δ. These limits are based on the results for the transport equations [2].

– The first limit on $\varepsilon \to 0$ is related with a so-called *"solidification"* procedure in the zone of the non-homogeneous fluid, corresponding to the "body". This limit can be treated as in [11], [14]. In the limit we obtain the motion of the rigid body in a viscous fluid, which occupies the domain $[\Omega]_{2\tau}$;
– In the second limit on $\delta \to 0$, we obtain the motion of the body already with a prescribed Navier's boundary condition. Firstly we need to construct an appropriate set of test functions, depending on δ. Then, using embedding results in cusp domains, we show that the third term of μ_δ converges to the jump boundary term on $\partial S(t)$ in (7). The embedding results allow also to apply a compactness result in the convective term of (9) by using the approach of Proposition 6.1 in [14].

The demonstration of Theorem 2.1 is a quite lengthy and technical one. The details can be found in [4].

4. Acknowledgment

The work of Š. Nečasová was supported by Grant No. 16-03230S of GAČ in the framework of RVO 67985840.

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