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Abstract

We consider the inviscid incompressible limits of the rotating compressible Navier-Stokes system for a barotropic fluid. We show that the limit system is represented by the rotating incompressible Euler equation on the whole space.

Key words: compressible Navier-Stokes system, rotating fluids, incompressible limit, inviscid limit.

Contents

1 Introduction

In this paper we consider the scaled compressible Navier-Stokes system for a barotropic rotating fluid occupying an arbitrary open set $\Omega \subset \mathbb{R}^3$, namely

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

mass

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) = -\frac{1}{\mathcal{M}a^2} \nabla_x p(\varrho) + \frac{1}{\mathcal{R}e} \mathrm{div}_x S(\nabla_x \mathbf{u}) - (\varrho \mathbf{u} \times \omega), \tag{1.2}$$

momentum

$$S = S(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbf{I} \right) + \eta \, \mathrm{div}_x \mathbf{u} \mathbf{I}, \quad \mu > 0, \quad \eta \geq 0, \tag{1.3}$$

stress

with the following far field conditions for the density and the velocity field

$$\lim_{|x| \to \infty} \varrho(x,t) = 1, \quad \lim_{|x| \to \infty} \mathbf{u}(x,t) = 0. \tag{1.4}$$

bound2

There are two unknowns: the density $\varrho = \varrho(x,t)$ and the velocity field $\mathbf{u} = \mathbf{u}(x,t)$ of the fluid, functions of the spatial position $x \in \mathbb{R}^3$ and the time $t \in \mathbb{R}$. The scaled system contains two characteristic numbers: $\mathcal{M}a$, the Mach number and $\mathcal{R}e$, the Reynolds number.

The Mach number is the ratio between the characteristic speed of the fluid and the speed of sound. The low Mach number limit means that the fluids becomes incompressible. The Reynolds number is defined as the ratio of inertial forces to viscous forces. By the high Reynolds number limit the viscosity of fluid becomes negligible.

The shear viscosity coefficient μ and the bulk viscosity coefficient η are assumed to be constant, p is a scalar function termed pressure, given function of the density $p = p(\varrho)$, $\omega = [0,0,1]$ is the angular velocity and the quantity $(\varrho \mathbf{u} \times \omega)$ represents the Coriolis force. The effect of the centrifugal force is neglected. This is a standard simplification adopted, for instance, in models of atmosphere or astrophysics (see [22, 23, 24]).

We will consider the case when $\mathcal{M}a = \varepsilon$ and $\mathcal{R}e = \varepsilon^{-1}$. Our aim is to identify the system of equations in the limit of $\varepsilon \to 0$, meaning the inviscid, incompressible limit. More precisely, we want to show that the weak solution of the Navier-Stokes system converges to the classical solution of the corresponding rotating incompressible Euler system, namely

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{v} \times \omega + \nabla_x \Pi = 0, \quad \mathrm{div}_x \mathbf{v} = 0. \tag{1.5}$$

euler

The incompressible inviscid limit was investigated by Lions and Masmoudi [31] in the case of well-prepared initial data (see also [32, 33]). A different approach by using Strichartz's estimates for the linear wave equation has been developed by Schochet [39]. For the application of the Strichartz's estimates in the case of low Mach number limit the reader can refer to the work of Desjardins and Grenier [5]. Incompressible inviscid limit in the case of ill-prepared initial data was study by Masmoudi [34] in the whole space case and also in the case with periodic boundary conditions.

We consider a fluid in a rotating frame only in the whole space $\mathbb{R}^3$. Comparing with article of Masmoudi [34] we used a different technique based on the relative energy inequality and weak-strong uniqueness, see [15].

The relative energy inequality was introduced by Dafermos [3] and in the fluid context was introduced by Germain [21]. Deriving the relative energy inequality for sufficient smooth test functions and proving the weak-strong uniqueness it gives us very powerful and elegant tool for the purpose of measuring the stability of a solution compared to another solution with a better regularity. This method was developed by E. Feireisl, A. Novotný and co-workers in the framework of singular limits problems (see for example [11], [13],[15] and [20] and references therein). For the using of the relative energy inequality in other contexts, the reader can refer to [1], [3], [14], [16], [17], [21], [35], [38], [41] and reference therein.

The paper is organized as follows. Section 2 will be devoted to the weak and classical solutions of the Navier-Stokes and Euler systems, respectively. In Section 3 we will discuss the acoustic waves generated by the compressibility of the fluid. In Section 4 we prove the convergence of the weak solution of the Navier-Stokes system to the classical solution of the Euler system through the use of the relative energy inequality.

2 Weak and classical solutions

In this section we introduce the definition of weak solutions for the compressible Navier-Stokes system (1.1 - 1.3). In particular, we define the so-called bounded energy weak solution (see [9], [18] and [37]) and we discuss the global-in-time existence. Finally, we discuss the global existence of the classical solution of the incompressible Euler system (1.5).

The introduction of the bounded energy weak solution is motivated by the following discussion. In [4] it was shown the existence of weak solutions to the compressible Navier-Stokes equations on unbounded domain satisfying the differential form of the energy inequality (and consequently the integral form) for a barotropic fluid with finite mass. While the existence of weak solutions for a fluid with infinite mass *remains an open question.* Weak solutions satisfying the differential form of the energy inequality are usually termed finite energy weak solutions (see [1], [15], [19], [26] and [37]), while weak solutions satisfying the integral form of the energy inequality are usually termed bounded energy weak solutions (see [9], [18] and [37]).

Because our analysis will be performed in the whole space $\mathbb{R}^3$ under the condition that the mass of the fluid is infinite (see relation 1.4), we have to use the integral form of the energy inequality and consequently to deal with bounded energy weak solutions.

2.1 Bounded energy weak solution

Multiplying (formally) the equation (1.2) by $\mathbf{u}$ and integrating by parts, we deduce the energy inequality in its integral form

$$E(\tau) + \int_0^\tau \int_\Omega S\left(\nabla_x \mathbf{u}\right) : \nabla_x \mathbf{u} \,\mathrm{d}x\mathrm{d}t \leq E_0 \tag{2.1}$$

ei

where the total energy E is given by the formula

$$E = E\left[\varrho, \mathbf{u}\right](t) = \int_\Omega \frac{1}{2}\varrho \left|\mathbf{u}\right|^2 + H(\varrho)\mathrm{d}x, \tag{2.2}$$

e

with E_0 the initial energy, and

$$H(\varrho) = \frac{1}{\gamma - 1}\left(\varrho^\gamma - \gamma\varrho + \gamma - 1\right) \tag{2.3}$$

h

the Helmholtz free energy (see [9], [11] and [37]). The parameter γ is the adiabatic index or heat capacity ratio.

Now, we define the so-called bounded energy weak solution of the compressible Navier-Stokes system (1.1 - 1.3) (see [18] and [37]).

be **Definition 1.** (Bounded energy weak solution) Let $\Omega \subset \mathbb{R}^3$ be an arbitrary open set. We say that $[\varrho, \mathbf{u}]$ is a bounded energy weak solution of the compressible Navier-Stokes system (1.1 - 1.3) in the time-space cylinder $(0,T) \times \Omega$ if

$$\varrho \in L^\infty\left((0,T), L^1(\Omega)\right), \quad \varrho \geq 0 \quad \text{a.e. in} \quad (0,T) \times \Omega,$$

$$H(\varrho) \in L^\infty((0,T), L^1(\Omega)),$$

$$\mathbf{u} \in L^2\left((0,T), \left(D_0^{1,2}(\Omega)\right)^3\right), \quad \varrho\left|\mathbf{u}\right|^2 \in L^\infty\left((0,T), L^1(\Omega)\right).$$

The continuity equation (1.1) holds in $\mathcal{D}'((0,T)\times\Omega)$. The momentum equation (1.2) holds in $\mathcal{D}'((0,T)\times\Omega)$. The energy inequality (2.1) holds for a.a. $\tau \in (0,T)$ with E defined by

$$E = \int_\Omega \frac{1}{2}\frac{\left|\varrho\mathbf{u}\right|^2}{\varrho} 1_{\{x;\varrho>0\}} + H(\varrho)\mathrm{d}x \tag{2.4}$$

e_r

and E_0 defined by

$$E_0 = \int_\Omega \frac{1}{2}\frac{\left|\varrho_0\mathbf{u}_0\right|^2}{\varrho_0} 1_{\{x;\varrho_0>0\}} + H(\varrho_0)\mathrm{d}x. \tag{2.5}$$

e_r0

Remark 2. Here, the space $D_0^{1,2}(\Omega)$ is a completion of $\mathcal{D}(\Omega)$ - the space of smooth functions compactly supported in Ω - with respect to the norm

$$\|v\|^2_{D_0^{1,2}(\Omega)} = \int_\Omega \left|\nabla v\right|^2 \mathrm{d}x.$$

Now, the following theorem concerns with the global-in-time existence of bounded energy weak solution (see [9] and [18]).

thm: 1 **Theorem 3.** *(Global-in-time existence of bounded energy weak solution) Let $\Omega \subset \mathbb{R}^3$ be an arbitrary open set. Let the pressure p be given by a general constitutive law satisfying*

$$p \in C^1[0,\infty), \quad p(0) = 0, \quad \frac{1}{a}\varrho^{\gamma-1} - b \le p'(\varrho) \le a\varrho^{\gamma-1} + b, \quad \text{for all} \quad \varrho > 0 \tag{2.6}$$

pressure

with

$$a > 0, \quad b \ge 0, \quad \gamma > \frac{3}{2}.$$

Let the initial data ϱ_0, $\mathbf{u}_0$ satisfy

$$\varrho_0 \in L^1(\Omega), \quad H(\varrho_0) \in L^1(\Omega), \quad \varrho_0 \ge 0 \quad a.e. \quad in \quad \Omega,$$

$$\varrho_0\mathbf{u}_0 \in \left(L^1(\Omega)\right)^3 \quad such \quad that \quad \frac{|\varrho_0\mathbf{u}_0|^2}{\varrho_0} 1_{\{x;\varrho_0>0\}} \in L^1(\Omega)$$

$$and \quad such \quad that \quad \varrho_0\mathbf{u}_0 = 0 \quad whenever \quad x \in \{\varrho_0 = 0\}. \tag{2.7}$$

id

Then the problem (1.1 - 1.3) admits at least one bounded energy weak solution $[\varrho, \mathbf{u}]$ on $(0,T) \times \Omega$ in the sense of Definition 1. Moreover $[\varrho, \mathbf{u}]$ satisfy the energy inequality (2.1).

Remark 4. The first existence result for problem (1.1 - 1.3) was obtained by Lions [30] in the case when $\Omega \subset \mathbb{R}^3$ is a domain with smooth and compact boundary and flow is isentropic $p(\varrho) \approx \varrho^\gamma$ with $\gamma \ge \frac{9}{5}$. This result was extended to more physical case to $\gamma > \frac{3}{2}$ in [19] in the case when Ω is a bounded smooth domain. Existence for certain classes of unbounded domains was shown in [37] (see also [30]).

Remark 5. Theorem 3 in [9] and [18] concerns the existence of a bounded energy weak solution for a given external force $\mathbf{f}(x,t)$, bounded and measurable function of the time $t \in (0,T)$ and the spatial coordinate $x \in \Omega$. Its validity in the presence of the Coriolis force is maintained since this force is considered a sort of perturbation for the compressible Navier-Stokes system (see for example [14, 16] and reference therein).

2.2 Classical solutions to the Euler system - target system

For the solvability of the system (1.5) with the initial data $\mathbf{v}(0) = \mathbf{v}_0$, we report the following result (see [42]):

thm: 2 **Theorem 6.** *Let $s \in \mathbb{R}$ satisfy $s > \frac{3}{2} + 1$. Then, for $0 < T < \infty$ and $\mathbf{v}_0 \in W^{s,2}\left(\mathbb{R}^3\right)$ satisfying $div_x \mathbf{v}_0 = 0$, there exists a positive parameter $\Omega_0 = \Omega_0(s, T, \|\mathbf{v}_0\|_{W^{s,2}})$ such that if $|\omega| \geq \Omega_0$ then the system (1.5) possesses a unique classical solution $\mathbf{v}$ satisfying*

$$\mathbf{v} \in C\left([0,T]\,; W^{s,2}(\mathbb{R}^3;\mathbb{R}^3)\right),$$

$$\partial_t \mathbf{v} \in C\left([0,T]\,; W^{s-1,2}(\mathbb{R}^3;\mathbb{R}^3)\right),$$

$$\nabla \Pi \in C\left([0,T]\,; W^{s,2}(\mathbb{R}^3;\mathbb{R}^3)\right). \tag{2.8}$$ reg

Remark 7. The global existence stated above was proved in [25] for the initial data in $W^{s,2}\left(\mathbb{R}^3\right)$ with $s > 7/2$.

Remark 8. Theorem 6 deals with inviscid flows in a rotating frame under the condition of fast rotation. In terms of scale analysis (see [36]), if we define by U and L the characteristic velocity and length scale of the fluid, we can estimate the order of magnitude of the non-linear term and the rotational term in the equation (1.5) as follows

$$\mathbf{v} \cdot \nabla \mathbf{v} \sim O\left(\frac{U^2}{L}\right), \tag{2.9}$$ vel

$$\mathbf{v} \times \omega \sim O\left(\Omega U\right), \tag{2.10}$$ om

where

$$\omega \sim O\left(\Omega\right) \sim O\left(\frac{U}{L}\right), \tag{2.11}$$ omega

with Ω characteristic angular velocity. Comparing (2.9) and (2.10), we have

$$\frac{U}{L} \sim \Omega. \tag{2.12}$$ comp

Fast rotation implies

$$\frac{U}{\Omega L} \ll 1 \tag{2.13}$$ fast

and we can neglect the non-linear term in (1.5), obtaining

$$\partial_t \mathbf{v} + \mathbf{v} \times \omega + \nabla_x \Pi = 0, \quad \mathrm{div}_x \mathbf{v} = 0. \tag{2.14}$$ euler_lin

These are linear equations. In other words, fast rotation leads to averaging mechanism that weakens the nonlinear effects. This of course prevents singularity allowing the life span of the solution to extend (see [2] and references therein).

3 Acoustic waves

An oscillatory motion with small amplitude in a compressible fluid is called an acoustic wave. Hence, the compressibility of the fluid allows the propagation of acoustic waves. Consequently, acoustic waves should definitely disappear in the incompressible limit regime. In the following, we introduce the acoustic system related to the equations (1.1) and (1.2) and we discuss the decay of acoustic waves in the limit of Mach number tends to zero introducing the so-called dispersive estimate (see [5], [12], [34] and [40]).

For a physical explanations of acoustic wave, the reader can consult Falkovich [6] and Landau-Lifshitz [27].

3.1 Acoustic system

Assuming the perturbation of the density is small, we can write the acoustic system related to the equations (1.1) and (1.2) by the following linear relations (see [11], [17] and [28, 29]):

$$\varepsilon \partial_t s + \triangle \Psi = 0, \quad \varepsilon \partial_t \nabla \Psi + a \nabla_x s = 0, \; a = p'(1) > 0, \tag{3.1}$$

ac_1

with the initial data

$$s(0) = \varrho_0^{(1)}, \quad \nabla_x \Psi(0) = \nabla_x \Psi_0 = \mathbf{u}_0 - \mathbf{v}_0 \tag{3.2}$$

ac_2

where $\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0]$ and $\mathbf{H}$ denotes the Helmholtz projection into the space of solenoidal functions.

3.2 Regularization

For the purpose of our work (see the next Section) and the use of the estimates (3.6) and (3.7) for the acoustic waves decay, it is convenient to regularize the initial data (3.2) in the following way

$$\varrho_0^{(1)} = \varrho_{0,\eta}^{(1)} = \chi_\eta \star \left(\psi_\eta \varrho_0^{(1)} \right), \quad \nabla_x \Psi_0 = \nabla_x \Psi_{0,\eta} = \chi_\eta \star \left(\psi_\eta \nabla_x \Psi_0 \right), \quad \eta > 0, \tag{3.3}$$

smooth

where $\{\chi_\eta\}$ is a family of regularazing kernels and $\psi_\eta \in C_0^\infty(\mathbb{R}^3)$ are standards cut-off functions. Consequently, the acoustic system posses a (unique) smooth solution $[s, \Psi]$ and the quantities $\nabla_x \Psi$ and s are compactly supported in $\mathbb{R}^3$ (see [12]).

3.3 Energy and decay of acoustic waves

The total change in energy of the fluid caused by the acoustic wave is given by the integral

$$\int_{\mathbb{R}^3} \left(\frac{1}{2} a \, |s|^2 + \frac{1}{2} \, |\nabla_x \Psi|^2 \right) \mathrm{d}x, \tag{3.4}$$

den_ac

where the integrand may be regarded as the density of sound energy (see [27]). It is easy to verify (see [27]) that the density of sound energy is conserved in time, namely

$$\left[\int_{\mathbb{R}^3}\left(\frac{1}{2}a\,|s|^2+\frac{1}{2}\,|\nabla_x\Psi|^2\right)(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau}=0. \tag{3.5}$$ ac_en

In addition, we have the following energy estimates (see [12])

$$\|\nabla_x\Psi\,(t,\cdot)\|_{W^{k,2}(\mathbb{R}^3;\mathbb{R}^3)}+\|s\,(t,\cdot)\|_{W^{k,2}(\mathbb{R}^3)}$$

$$\leq c\left(\|\nabla_x\Psi_0\|_{W^{k,2}(\mathbb{R}^3;\mathbb{R}^3)}+\left\|\varrho_0^{(1)}\right\|_{W^{k,2}(\mathbb{R}^3)}\right),\ k=0,1,..., \tag{3.6}$$ en_est

for any $t>0$. Instead, concerning the decay of the acoustic waves in the incompressible limit, the following dispersive estimates hold (see [5], [12], [34] and [40])

$$\|\nabla_x\Psi\,(t,\cdot)\|_{W^{k,p}(\mathbb{R}^3;\mathbb{R}^3)}+\|s\,(t,\cdot)\|_{W^{k,p}(\mathbb{R}^3)}$$

$$\leq c(1+\frac{t}{\varepsilon})^{-\left(\frac{1}{q}-\frac{1}{p}\right)}\left(\|\nabla_x\Psi_0\|_{W^{k,q}(\mathbb{R}^3;\mathbb{R}^3)}+\left\|\varrho_0^{(1)}\right\|_{W^{k,q}(\mathbb{R}^3)}\right), \tag{3.7}$$ disp_est

$$2\leq p\leq\infty,\ \frac{1}{p}+\frac{1}{q}=1,\ k=0,1,....$$

4 Convergence

In this section we introduce the relative energy functional with the associated relative energy inequality. Then, we present the main result and a priori estimates. Finally, we perform the convergence to the incompressible Euler system. Fixing $\eta>0$, our goal is to perform the limit for $\varepsilon\to 0$.

4.1 Relative energy inequality

We introduce the relative energy functional

$$\mathcal{E}(\varrho,\mathbf{u}\mid r,\mathbf{U})=\int_{\mathbb{R}^3}\left[\frac{1}{2}\varrho\,|(\mathbf{u}-\mathbf{U})|^2\right.$$

$$\left.+\frac{1}{\varepsilon^2}\left(H\left(\varrho\right)-H'\left(r\right)\left(\varrho-r\right)-H\left(r\right)\right)\right]\mathrm{d}x \tag{4.1}$$ entr_funct

along with the relative energy inequality associated to the Navier-Stokes system (1.1 - 1.3)

$$\left[\mathcal{E}(\varrho,\mathbf{u}\mid r,\mathbf{U})\right]_{t=0}^{t=\tau}$$

$$+\varepsilon \int_0^\tau \int_{\mathbb{R}^3} S\left(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}\right) : \left(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}\right) \mathrm{d}x\mathrm{d}t \leq \int_0^\tau \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U})\mathrm{d}t, \quad (4.2)$$

entr_ineq

where the remainder $\mathcal{R}$ is expressed as follows

$$\mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) = \int_{\mathbb{R}^3} \varrho \left(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U}\right) \cdot (\mathbf{U} - \mathbf{u})\mathrm{d}x$$

$$+\varepsilon \int_{\mathbb{R}^3} S(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{u})\mathrm{d}x$$

$$+\frac{1}{\varepsilon^2} \int_{\mathbb{R}^3} \left((r - \varrho)\, \partial_t H'(r) + \nabla_x H'(r) \cdot (r\mathbf{U} - \varrho\mathbf{u})\right) \mathrm{d}x$$

$$-\frac{1}{\varepsilon^2} \int_{\mathbb{R}^3} \left(p(\varrho) - p(r)\right) \mathrm{div}_x \mathbf{U}\mathrm{d}x$$

$$+\int_{\mathbb{R}^3} \left(\varrho\mathbf{u} \times \omega\right) \cdot \left(\mathbf{U} - \mathbf{u}\right) \mathrm{d}x := I_1 + ... + I_5 \quad (4.3)$$

rem

and for all smooth functions r, $\mathbf{U}$ such that

$$r > 0, \quad r - 1 \in C_c^\infty\left([0,T] \times \mathbb{R}^3\right), \quad \mathbf{U} \in C_c^\infty\left([0,T] \times \mathbb{R}^3; \mathbb{R}^3\right). \quad (4.4)$$

test

It can be shown (see [15]) that any bounded energy weak solution $[\varrho, \mathbf{u}]$ to the compressible Navier-Stokes system (1.1 - 1.3) satisfies the relative energy inequality for any pair of sufficiently smooth test functions r, $\mathbf{U}$ as in (4.4). As mentioned before, the relative energy functional (4.1) can be used to measure the stability of the solutions $[\varrho, \mathbf{u}]$ as compared to the test functions $[r, \mathbf{U}]$. The particular choice of $[r, \mathbf{U}]$ will be clarified later.

4.2 Main result

The main result of the present paper can be stated as follows:

thm: 3

Theorem 9. *Let $M > 0$ be a constant. Let the pressure p satisfy the hypothesis (2.6) and, in addition, assume $p \in C^3(0, \infty)$. Let the initial data $[\varrho_0, \mathbf{u}_0]$ for the Navier-Stokes system (1.1 - 1.3) be of the following form*

$$\varrho(0) = \varrho_{0,\varepsilon} = 1 + \varepsilon \varrho_{0,\varepsilon}^{(1)}, \ \mathbf{u}(0) = \mathbf{u}_{0,\varepsilon}, \quad (4.5)$$

well data

$$\left\| \varrho_{0,\varepsilon}^{(1)} \right\|_{L^2 \cap L^\infty(\mathbb{R}^3)} + \left\| \mathbf{u}_{0,\varepsilon} \right\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \leq M \quad (4.6)$$

data bound

and satisfying the assumption (3.3). Let all the requirements of Theorem 6 be satisfied with the initial datum for the Euler system $\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0]$. Let $[s, \Psi]$ be the solution of the acoustic system (3.1) with the initial data (3.3). Then,

$$\|\sqrt{\varrho}\,(\mathbf{u}-\mathbf{v}-\nabla\Psi)\,(\tau,\cdot)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)}$$

$$+\left\|\frac{\varrho-1}{\varepsilon}(\tau,\cdot)-s(\tau,\cdot)\right\|^2_{L^2(\mathbb{R}^3)}+\left\|\frac{\varrho-1}{\varepsilon^{2/\gamma}}(\tau,\cdot)-\frac{s(\tau,\cdot)}{\varepsilon^{(2/\gamma)-1}}\right\|^{\gamma}_{L^{\gamma}(\mathbb{R}^3)}$$

$$\leq c\left(\|\mathbf{u}_{0,\varepsilon}-\mathbf{v}_0\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)}+\left\|\varrho^{(1)}_{0,\varepsilon}-\varrho^{(1)}_0\right\|^2_{L^2(\mathbb{R}^3)}\right),\quad \tau\in[0,T]\,, \tag{4.7}$$ th

for any weak solutions $[\varrho,\mathbf{u}]$ *of the compressible Navier-Stokes system (1.1 - 1.3).*

Thanks to the compactness of the acoustic waves and their decay due to the dispersive estimates, a consequence of the above Theorem is the following:

cor: 4 **Corollary 10.** *Let all the requirements of Theorem 9 be satisfied. Assume that*

$$\varrho^{(1)}_{0,\varepsilon}\to\varrho^{(1)}_0 \quad \textit{in} \quad L^2(\mathbb{R}^3),\quad \mathbf{u}_{0,\varepsilon}\to\mathbf{u}_0 \quad \textit{in} \quad L^2(\mathbb{R}^3;\mathbb{R}^3) \quad \textit{when} \quad \varepsilon\to 0.$$

Then

$$\operatorname{ess}\sup_{\tau\in[0,T]}\|\sqrt{\varrho}\,(\mathbf{u}-\mathbf{v})\,(\tau,\cdot)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\to 0 \quad \textit{when} \quad \varepsilon\to 0,$$

$$\operatorname{ess}\sup_{\tau\in[0,T]}\|\varrho-1\|^2_{L^2(\mathbb{R}^3)}\to 0 \quad \textit{when} \quad \varepsilon\to 0,$$

$$\operatorname{ess}\sup_{\tau\in[0,T]}\|\varrho-1\|^{\gamma}_{L^{\gamma}(\mathbb{R}^3)}\to 0 \quad \textit{when} \quad \varepsilon\to 0,$$

for any weak solutions $[\varrho,\mathbf{u}]$ *of the compressible Navier-Stokes system (1.1 - 1.3) and* $[r,\mathbf{U}]$ *sufficiently smooth test functions.*

Here and hereafter, the symbol c will denote a positive generic constant, independent by ε, usually found in inequalities, that will not have the same value when used in different parts of the text. The rest of the paper is devoted to the proof of Theorem 9.

In accordance with the energy inequality (2.1), we have

$$\operatorname{ess}\sup_{\tau\in[0,T]}\|\varrho(\tau,\cdot)\|_{L^{\gamma}\cap L^1(\mathbb{R}^3)}\leq c(M), \tag{4.8}$$ unif_bound0

$$\operatorname{ess}\sup_{\tau\in[0,T]}\|\sqrt{\varrho}\mathbf{u}(\tau,\cdot)\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\leq c(M), \tag{4.9}$$ unif_bound1

From (4.8) and (4.9), we obtain

$$\|\varrho\mathbf{u}(\tau,\cdot)\|_{L^q(\mathbb{R}^3;\mathbb{R}^3)}=\|\sqrt{\varrho}\sqrt{\varrho}\mathbf{u}(\tau,\cdot)\|_{L^q(\mathbb{R}^3;\mathbb{R}^3)}$$

$$\leq \|\sqrt{\varrho}(\tau,\cdot)\|_{L^{2\gamma}(\mathbb{R}^3)} \, \|\sqrt{\varrho}\mathbf{u}(\tau,\cdot)\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \,, \tag{4.10}$$

interp

with

$$q = \frac{2\gamma}{\gamma+1}. \tag{4.11}$$

q

We conclude that

$$\operatorname{ess}\sup_{\tau\in[0,T]} \|\varrho\mathbf{u}(\tau,\cdot)\|_{L^q(\mathbb{R}^3;\mathbb{R}^3)} \leq c(M), \ \ q = \frac{2\gamma}{\gamma+1}. \tag{4.12}$$

unif_bound2

Moreover, introducing (see [21])

$$I(\varrho,r) = H\left(\varrho\right) - H'\left(r\right)\left(\varrho - r\right) - H\left(r\right), \tag{4.13}$$

I

we observe that the map $\varrho \to I(\varrho,r)$ is, for any fixed $r>0$, a strictly convex function on $(0,\infty)$ with global minimum equal to 0 at $\varrho = r$, which grows at infinity with the rate ϱ^γ. Consequently, the integral $\int_\Omega I\left(\varrho,r\right)(\tau,x)\,\mathrm{d}x$ in (4.2) provides a control of $(\varrho - r)(\tau,\cdot)$ in L^2 over the sets $\{x : |\varrho - r|(\tau,x) < 1\}$ and in L^γ over the sets $\{x : |\varrho - r|(\tau,x) \geq 1\}$ So, for any r in a compact set $(0,\infty)$, there holds

$$I(\varrho,r) \approx |\varrho - r|^2 \, 1_{\{|\varrho - r|<1\}} + |\varrho - r|^\gamma \, 1_{\{|\varrho - r|\geq 1\}}, \quad \forall \varrho \geq 0, \tag{4.14}$$

I_2

in the sense that $I(\varrho,r)$ gives an upper and lower bound in term of the right-hand side quantity (see [1], [20] and [41]). Therefore, we have the following uniform bounds

$$\operatorname{ess}\sup_{\tau\in[0,T]} \left\| \left[\left(\varrho-1\right)(\tau,\cdot)\right] 1_{\{|\varrho-1|<1\}} \right\|_{L^2(\mathbb{R}^3)} \leq c(M)\varepsilon, \tag{4.15}$$

unif_bound3

$$\operatorname{ess}\sup_{\tau\in[0,T]} \left(\left\| \left[\left(\varrho-1\right)(\tau,\cdot)\right] 1_{\{|\varrho-1|\geq 1\}} \right\|_{L^\gamma(\mathbb{R}^3)} \right) \leq c(M)\varepsilon^{2/\gamma}, \tag{4.16}$$

unif_bound4

where we have set $r=1$ and $\mathbf{U}=0$ in the relative energy inequality (4.2). Now, the basic idea is to apply (4.2) to $[r,\mathbf{U}] = [1+\varepsilon s, \mathbf{v} + \nabla_x \Psi]$. We fix η. For the initial data we have

$$[\mathcal{E}(\varrho,\mathbf{u} \mid r,\mathbf{U})](0) = \int_{\mathbb{R}^3} \frac{1}{2} \varrho_{0,\varepsilon} \left|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\right|^2 \mathrm{d}x$$

$$+ \int_{\mathbb{R}^3} \frac{1}{\varepsilon^2} \left[H\left(1+\varepsilon\varrho_{0,\varepsilon}^{(1)}\right) - \varepsilon H'\left(1+\varepsilon\varrho_0^{(1)}\right)\left(\varrho_{0,\varepsilon}^{(1)} - \varrho_0^{(1)}\right) - H\left(1+\varepsilon\varrho_0^{(1)}\right)\right] \mathrm{d}x, \tag{4.17}$$

initial data conv

where $\mathbf{u}_0 = \mathbf{H}[\mathbf{u}_0] + \nabla\Psi_0$. Given (4.5) and (4.6), for the first term on the right hand side of the equality (4.17) we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \frac{1}{2} \varrho_{0,\varepsilon} \left| \mathbf{u}_{0,\varepsilon} - \mathbf{u}_0 \right|^2 \mathrm{d}x \\
&\leq \int_{\mathbb{R}^3} \frac{1}{2} \left| 1 + \varepsilon \varrho_{0,\varepsilon}^{(1)} \right| \left| \mathbf{u}_{0,\varepsilon} - \mathbf{u}_0 \right|^2 \mathrm{d}x \\
&\leq \int_{\mathbb{R}^3} \frac{1}{2} \left| \mathbf{u}_{0,\varepsilon} - \mathbf{u}_0 \right|^2 \mathrm{d}x + \int_{\mathbb{R}^3} \frac{1}{2} \left| \varepsilon \varrho_{0,\varepsilon}^{(1)} \right| \left| \mathbf{u}_{0,\varepsilon} - \mathbf{u}_0 \right|^2 \mathrm{d}x \\
&\leq \int_{\mathbb{R}^3} \frac{1}{2} \left| \mathbf{u}_{0,\varepsilon} - \mathbf{u}_0 \right|^2 \mathrm{d}x + \varepsilon \left\| \varrho_{0,\varepsilon}^{(1)} \right\|_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} \frac{1}{2} \left| \mathbf{u}_{0,\varepsilon} - \mathbf{u}_0 \right|^2 \mathrm{d}x \\
&\leq c(M) \left(1 + \varepsilon \right) \left\| \mathbf{u}_{0,\varepsilon} - \mathbf{u}_0 \right\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)} . \qquad (4.18)
\end{aligned}
$$

initial data conv1

For the second term on the right hand side of the equality (4.17), setting $a = 1 + \varepsilon \varrho_{0,\varepsilon}^{(1)}$ and $b = 1 + \varepsilon \varrho_0^{(1)}$ and observing that

$$
H(a) = H(b) + H'(b)(a-b) + \frac{1}{2} H''(\xi)(a-b)^2, \;\; \xi \in (a,b) \, ,
$$

$$
\left| H(a) - H'(b)(a-b) - H(b) \right| \leq c \left| a - b \right|^2 ,
$$

we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \frac{1}{\varepsilon^2} \left[H\left(1 + \varepsilon \varrho_{0,\varepsilon}^{(1)}\right) - \varepsilon H'\left(1 + \varepsilon \varrho_0^{(1)}\right) \left(\varrho_{0,\varepsilon}^{(1)} - \varrho_0^{(1)} \right) - H\left(1 + \varepsilon \varrho_0^{(1)}\right) \right] \mathrm{d}x \\
&\qquad \leq c(M) \int_{\mathbb{R}^3} \frac{1}{\varepsilon^2} \left(\left| \varepsilon \left(\varrho_{0,\varepsilon}^{(1)} - \varrho_0^{(1)} \right) \right|^2 \right) \mathrm{d}x \\
&\qquad \leq c(M) \left\| \varrho_{0,\varepsilon}^{(1)} - \varrho_0^{(1)} \right\|^2_{L^2(\mathbb{R}^3)} . \qquad (4.19)
\end{aligned}
$$

initial data conv2

Finally, we can conclude

$$
[\mathcal{E}(\varrho, \mathbf{u} \mid r, \mathbf{U})](0) \leq c(M)[(1+\varepsilon) \left\| \mathbf{u}_{0,\varepsilon} - \mathbf{u}_0 \right\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)} + \left\| \varrho_{0,\varepsilon}^{(1)} - \varrho_0^{(1)} \right\|^2_{L^2(\mathbb{R}^3)}].
$$

Now, we decompose I_1 into

$$
\begin{aligned}
I_1 = & \int_0^\tau \int_{\mathbb{R}^3} \varrho \left[\left(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} \right) \cdot \left(\mathbf{U} - \mathbf{u} \right) \right] \mathrm{d}x \mathrm{d}t \\
& - \int_0^\tau \int_{\mathbb{R}^3} \varrho \nabla_x \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) \cdot (\mathbf{U} - \mathbf{u}) \mathrm{d}x \mathrm{d}t. \qquad (4.20)
\end{aligned}
$$

conv

For the second term on the right hand side of (4.20), thanks to the Sobolev imbedding theorem, the Minkowski inequality, (2.8) and the dispersive estimate (3.7), we have

$$\int_0^\tau \int_{\mathbb{R}^3} \varrho \nabla_x \mathbf{U} \cdot (\mathbf{U}-\mathbf{u}) \cdot (\mathbf{U}-\mathbf{u}) \mathrm{d}x\mathrm{d}t$$

$$\leq \int_0^\tau \int_{\mathbb{R}^3} \varrho \left|\nabla_x \mathbf{U}\right| \cdot \left|(\mathbf{U}-\mathbf{u})\right|^2 dx\mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{E} \left\|\nabla_x \mathbf{v} + \nabla_x^2 \Psi\right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t,$$

$$\leq \int_0^\tau \mathcal{E} \left\|\nabla_x \mathbf{v}\right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t + \int_0^\tau \mathcal{E} \left\|\nabla_x^2 \Psi\right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t,$$

$$\leq c \int_0^\tau \mathcal{E} \mathrm{d}t + c(M) \left[\varepsilon \left(\log\left(\varepsilon+\tau\right) - \log\left(\varepsilon\right)\right)\right], \tag{4.21}$$

The first term on the right hand side of (4.20) can be rewritten as follows

$$\int_0^\tau \int_{\mathbb{R}^3} \varrho \left[(\partial_t \mathbf{U} + \mathbf{U}\cdot\nabla_x \mathbf{U}) \cdot (\mathbf{U}-\mathbf{u})\right] \mathrm{d}x\mathrm{d}t$$

$$= \int_0^\tau \int_{\mathbb{R}^3} \varrho (\mathbf{U}-\mathbf{u}) \cdot \left(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla_x \mathbf{v}\right) \mathrm{d}x\mathrm{d}t$$

$$+ \int_0^\tau \int_{\mathbb{R}^3} \varrho (\mathbf{U}-\mathbf{u}) \cdot \partial_t \nabla_x \Psi \mathrm{d}x\mathrm{d}t$$

$$+ \int_0^\tau \int_{\mathbb{R}^3} \varrho (\mathbf{U}-\mathbf{u}) \otimes \nabla_x \Psi : \nabla_x \mathbf{v} \mathrm{d}x\mathrm{d}t$$

$$+ \int_0^\tau \int_{\mathbb{R}^3} \varrho (\mathbf{U}-\mathbf{u}) \otimes \mathbf{v} : \nabla_x^2 \Psi \mathrm{d}x\mathrm{d}t$$

$$+ \int_0^\tau \int_{\mathbb{R}^3} \varrho (\mathbf{U}-\mathbf{u}) \cdot \nabla_x \left|\nabla_x \Psi\right|^2 \mathrm{d}x\mathrm{d}t. \tag{4.22}$$

conv3

In view of uniform bound (4.12), (2.8) and dispersive estimate (3.7), the last three integrals can be estimated as follows

$$\int_0^\tau \int_{\mathbb{R}^3} \varrho (\mathbf{U}-\mathbf{u}) \otimes \nabla_x \Psi : \nabla_x \mathbf{v} \mathrm{d}x\mathrm{d}t = \int_0^\tau \int_{\mathbb{R}^3} \varrho (\mathbf{v} + \nabla_x \Psi - \mathbf{u}) \otimes \nabla_x \Psi : \nabla_x \mathbf{v} \mathrm{d}x\mathrm{d}t$$

$$= \int_0^\tau \int_{\mathbb{R}^3} (\varrho \mathbf{v}) \otimes \nabla_x \Psi : \nabla_x \mathbf{v} \mathrm{d}x\mathrm{d}t$$

$$+ \int_0^\tau \int_{\mathbb{R}^3} (\varrho \nabla_x \Psi) \otimes \nabla_x \Psi : \nabla_x \mathbf{v} \mathrm{d}x\mathrm{d}t$$

$$- \int_0^\tau \int_{\mathbb{R}^3} (\varrho \mathbf{u}) \otimes \nabla_x \Psi : \nabla_x \mathbf{v} \mathrm{d}x\mathrm{d}t$$

$$\le c\int_0^\tau \|\varrho\|_{L^1}\|\mathbf{v}\|_{L^\infty}\|\nabla_x\Psi\|_{L^\infty}\|\nabla_x\mathbf{v}\|_{L^\infty}\,\mathrm{d}t$$

$$+c\int_0^\tau \|\varrho\|_{L^1}\|\nabla_x\Psi\|_{L^\infty}\|\nabla_x\Psi\|_{L^\infty}\|\nabla_x\mathbf{v}\|_{L^\infty}\,\mathrm{d}t$$

$$+c\int_0^\tau \|\varrho\mathbf{u}\|_{L^{\frac{2\gamma}{\gamma+1}}}\|\nabla_x\Psi\|_{L^{\frac{2\gamma}{\gamma-1}}}\|\nabla_x\mathbf{v}\|_{L^\infty}\,\mathrm{d}t$$

$$\le c(M)\left[\varepsilon\left(\log\left(\varepsilon+\tau\right)-\log\left(\varepsilon\right)\right)+\left(\frac{\varepsilon^2}{\varepsilon+\tau}-\varepsilon\right)+\left(\frac{\gamma\left(\varepsilon+\tau\right)\left(\frac{\varepsilon+\tau}{\varepsilon}\right)^{-1/\gamma}}{\gamma-1}-\frac{\gamma\varepsilon}{\gamma-1}\right)\right]; \tag{4.23}$$ 1th

similarly to (4.23),

$$\int_0^\tau\int_{\mathbb{R}^3}\varrho(\mathbf{U}-\mathbf{u})\otimes\mathbf{v}:\nabla_x^2\Psi\mathrm{d}x\mathrm{d}t=\int_0^\tau\int_{\mathbb{R}^3}\varrho(\mathbf{v}+\nabla_x\Psi-\mathbf{u})\otimes\mathbf{v}:\nabla_x^2\Psi\mathrm{d}x\mathrm{d}t$$

$$\le c(M)\left[\varepsilon\left(\log\left(\varepsilon+\tau\right)-\log\left(\varepsilon\right)\right)+\left(\frac{\varepsilon^2}{\varepsilon+\tau}-\varepsilon\right)+\left(\frac{\gamma\left(\varepsilon+\tau\right)\left(\frac{\varepsilon+\tau}{\varepsilon}\right)^{-1/\gamma}}{\gamma-1}-\frac{\gamma\varepsilon}{\gamma-1}\right)\right] \tag{4.24}$$ 2th

and

$$\int_0^\tau\int_{\mathbb{R}^3}\varrho(\mathbf{U}-\mathbf{u})\cdot\nabla_x\left|\nabla_x\Psi\right|^2\mathrm{d}x\mathrm{d}t=\int_0^\tau\int_{\mathbb{R}^3}\varrho(\mathbf{v}+\nabla_x\Psi-\mathbf{u})\cdot\nabla_x\left|\nabla_x\Psi\right|^2\mathrm{d}x\mathrm{d}t$$

$$\le c(M)\left[\left(\frac{\varepsilon^2}{\varepsilon+\tau}-\varepsilon\right)+\left(\frac{\varepsilon^3}{2\left(\varepsilon+\tau\right)^2}-\varepsilon\right)+\left(\varepsilon\gamma\left(\frac{\varepsilon+\tau}{\varepsilon}\right)^{-1/\gamma}-\varepsilon\gamma\right)\right]. \tag{4.25}$$ 3th

Using (1.5), for the first term of (4.22), we have

$$\int_0^\tau\int_{\mathbb{R}^3}\varrho(\mathbf{U}-\mathbf{u})\cdot\left(\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla_x\mathbf{v}\right)\mathrm{d}x\mathrm{d}t$$

$$=-\int_0^\tau\int_{\mathbb{R}^3}\varrho(\mathbf{U}-\mathbf{u})\cdot\nabla_x\Pi\mathrm{d}x\mathrm{d}t-\int_0^\tau\int_{\mathbb{R}^3}(\mathbf{U}-\mathbf{u})\cdot\left(\omega\times\varrho\mathbf{v}\right)\mathrm{d}x\mathrm{d}t$$

$$=\int_0^\tau\int_{\mathbb{R}^3}\varrho\mathbf{u}\cdot\nabla_x\Pi\mathrm{d}x\mathrm{d}t-\int_0^\tau\int_{\mathbb{R}^3}\varrho\mathbf{U}\cdot\nabla_x\Pi\mathrm{d}x\mathrm{d}t-\int_0^\tau\int_{\mathbb{R}^3}(\mathbf{U}-\mathbf{u})\cdot\left(\omega\times\varrho\mathbf{v}\right)\mathrm{d}x\mathrm{d}t. \tag{4.26}$$ conv4

Regarding the first integral on the right hand side of (4.26), as a consequence of the estimate (4.12), we have

$$\varrho\mathbf{u} \to \mathbf{w} \text{ weakly-}(*) \text{ in } L^{\infty}\left(0,T;L^{2\gamma/\gamma+1}(\mathbb{R}^3;\mathbb{R}^3)\right), \tag{4.27}$$

press_conv2

where $\mathbf{w}$ denotes the weak limit of the composition. Now, taking the limit in the weak formulation of the continuity equation

$$\varepsilon\int_0^{\tau}\int_{\mathbb{R}^3}\left(\frac{\varrho-1}{\varepsilon}\right)\partial_t\varphi \mathrm{d}x\mathrm{d}t + \int_0^{\tau}\int_{\mathbb{R}^3}\varrho\mathbf{u}\nabla_x\varphi \mathrm{d}x\mathrm{d}t = 0 \tag{4.28}$$

weak_cont

for sufficiently smooth φ, thanks to the estimate (4.15) and (4.16) we deduce that

$$\int_0^{\tau}\int_{\mathbb{R}^3}\mathbf{w}\cdot\nabla_x\varphi \mathrm{d}x\mathrm{d}t = 0 \tag{4.29}$$

weak_cont_0

when $\varepsilon \to 0$. We may infer that

$$\int_0^{\tau}\int_{\mathbb{R}^3}\varrho\mathbf{u}\cdot\nabla_x\Pi \mathrm{d}x\mathrm{d}t \to \int_0^{\tau}\int_{\mathbb{R}^3}\mathbf{w}\cdot\nabla_x\Pi \mathrm{d}x\mathrm{d}t = 0. \tag{4.30}$$

conv_0

For the second integral on the right hand side of (4.26), we have

$$\left|\int_0^{\tau}\int_{\mathbb{R}^3}\varrho\mathbf{U}\cdot\nabla_x\Pi \mathrm{d}x\mathrm{d}t\right| \leq \left|\int_0^{\tau}\int_{\mathbb{R}^3}(\varrho-1)\cdot\mathbf{U}\cdot\nabla_x\Pi \mathrm{d}x\mathrm{d}t\right| + \left|\int_0^{\tau}\int_{\mathbb{R}^3}\mathbf{U}\cdot\nabla_x\Pi \mathrm{d}x\mathrm{d}t\right|. \tag{4.31}$$

split

For the first integral on the right-hand side of (4.31), thanks to (2.8), the estimate (3.7) and the uniform bounds (4.15) and (4.16), we have

$$\int_0^{\tau}\int_{\mathbb{R}^3}(\varrho-1)\cdot\mathbf{U}\cdot\nabla_x\Pi \mathrm{d}x\mathrm{d}t$$

$$\leq c\varepsilon\int_0^{\tau}\left\|[\frac{\varrho-1}{\varepsilon}]1_{\{|\varrho-1|<1\}}\right\|_{L^2(\mathbb{R}^3)}\cdot\|\mathbf{v}+\nabla_x\Psi\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\cdot\|\nabla_x\Pi\|_{L^{\infty}(\mathbb{R}^3;\mathbb{R}^3)}\,\mathrm{d}t$$

$$\leq c\varepsilon\int_0^{\tau}\left\|[\frac{\varrho-1}{\varepsilon}]1_{\{|\varrho-1|<1\}}\right\|_{L^2(\mathbb{R}^3)}\cdot\|\mathbf{v}\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\cdot\|\nabla_x\Pi\|_{L^{\infty}(\mathbb{R}^3;\mathbb{R}^3)}\,\mathrm{d}t$$

$$+c\varepsilon\int_0^{\tau}\left\|[\frac{\varrho-1}{\varepsilon}]1_{\{|\varrho-1|<1\}}\right\|_{L^2(\mathbb{R}^3)}\cdot\|\nabla_x\Psi\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\cdot\|\nabla_x\Pi\|_{L^{\infty}(\mathbb{R}^3;\mathbb{R}^3)}\,\mathrm{d}t \leq c(M)\varepsilon \tag{4.32}$$

press_conv2-1

and

$$\int_0^\tau \int_{\mathbb{R}^3} (\varrho - 1) \cdot \mathbf{U} \cdot \nabla_x \Pi \mathrm{d}x\mathrm{d}t$$

$$\leq c \int_0^\tau \left\| [\varrho - 1] 1_{\{|\varrho-1|\geq 1\}} \right\|_{L^\gamma(\mathbb{R}^3)} \cdot \| (\mathbf{v} + \nabla_x \Psi) \cdot \nabla_x \Pi \|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t$$

$$\leq c \int_0^\tau \left\| [\varrho - 1] 1_{\{|\varrho-1|\geq 1\}} \right\|_{L^\gamma(\mathbb{R}^3)} \cdot \| \mathbf{v} \cdot \nabla_x \Pi \|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t$$

$$+ c \int_0^\tau \left\| [\varrho - 1] 1_{\{|\varrho-1|\geq 1\}} \right\|_{L^\gamma(\mathbb{R}^3)} \cdot \| \nabla_x \Psi \cdot \nabla_x \Pi \|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t. \qquad (4.33)$$

press_conv2-2

Thanks to the following interpolation inequalities

$$\| \nabla_x \Psi \cdot \nabla_x \Pi \|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)} \leq \| \nabla_x \Psi \cdot \nabla_x \Pi \|_{L^1(\mathbb{R}^3;\mathbb{R}^3)}^{\frac{\gamma-1}{\gamma}} \| \nabla_x \Psi \cdot \nabla_x \Pi \|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1-\frac{\gamma-1}{\gamma}}$$

$$\leq \| \nabla_x \Psi \|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}^{\frac{\gamma-1}{\gamma}} \| \nabla_x \Pi \|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}^{\frac{\gamma-1}{\gamma}} \| \nabla_x \Psi \cdot \nabla_x \Pi \|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma}$$

$$\leq c(M) \| \nabla_x \Psi \cdot \nabla_x \Pi \|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma} \leq c(M) \| \nabla_x \Psi \|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma} \cdot \| \nabla_x \Pi \|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma}$$

$$\leq c(M) \| \nabla_x \Psi \|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma}, \qquad (4.34)$$

int_1

$$\| \mathbf{v} \cdot \nabla_x \Pi \|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)} \leq \| \mathbf{v} \cdot \nabla_x \Pi \|_{L^1(\mathbb{R}^3;\mathbb{R}^3)}^{\frac{\gamma-1}{\gamma}} \| \mathbf{v} \cdot \nabla_x \Pi \|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1-\frac{\gamma-1}{\gamma}}$$

$$\leq \| \mathbf{v} \|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}^{\frac{\gamma-1}{\gamma}} \| \nabla_x \Pi \|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}^{\frac{\gamma-1}{\gamma}} \| \mathbf{v} \cdot \nabla_x \Pi \|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma}$$

$$\leq c \| \mathbf{v} \cdot \nabla_x \Pi \|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma} \leq c \| \mathbf{v} \|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma} \cdot \| \nabla_x \Pi \|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma} \leq c, \qquad (4.35)$$

int_2

and the estimate (3.7), for the integral in (4.33) we have,

$$\int_0^\tau \left\| [\varrho - 1] 1_{\{|\varrho-1|\geq 1\}} \right\|_{L^\gamma(\mathbb{R}^3)} \cdot \| \mathbf{v} \cdot \nabla_x \Pi \|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t$$

$$+ \int_0^\tau \left\| [\varrho - 1] 1_{\{|\varrho-1|\geq 1\}} \right\|_{L^\gamma(\mathbb{R}^3)} \cdot \| \nabla_x \Psi \cdot \nabla_x \Pi \|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t$$

$$\leq c(M) \varepsilon^{2/\gamma} + c(M) \varepsilon^{2/\gamma} \int_0^\tau \| \nabla_x \Psi \|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma} \mathrm{d}t$$

$$\leq c(M)\varepsilon^{2/\gamma} + c(M)\varepsilon^{2/\gamma}\left(\frac{\gamma\left(\varepsilon+\tau\right)\left(\frac{\varepsilon+\tau}{\varepsilon}\right)^{-1/\gamma}}{\gamma-1} - \frac{\gamma\varepsilon}{\gamma-1}\right). \tag{4.36}$$ press_conv2-3

For the second integral on the right-hand side of (4.31), we have

$$\int_0^\tau \int_{\mathbb{R}^3} \mathbf{U}\cdot\nabla_x\Pi \mathrm{d}x\mathrm{d}t = \int_0^\tau \int_{\mathbb{R}^3} \mathbf{v}\cdot\nabla_x\Pi \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\mathbb{R}^3} \nabla_x\Psi\cdot\nabla_x\Pi \mathrm{d}x\mathrm{d}t. \tag{4.37}$$ press_conv2-4

Performing integration by parts in the first term on the right-hand side of (4.37), we have

$$\int_0^\tau \int_{\mathbb{R}^3} \mathrm{div}_x\mathbf{v}\cdot\Pi \mathrm{d}x\mathrm{d}t = 0$$

thanks to incompressibility condition, $\mathrm{div}_x\mathbf{v} = 0$. For the second term on the right-hand side of (4.37) using integration by parts and acoustic equation (3.1), we have

$$\begin{aligned}\int_0^\tau \int_{\mathbb{R}^3} \nabla_x\Psi\cdot\nabla_x\Pi \mathrm{d}x\mathrm{d}t &= -\int_0^\tau \int_{\mathbb{R}^3} \triangle\Psi\cdot\Pi \mathrm{d}x\mathrm{d}t \\ &= \varepsilon\int_0^\tau \int_{\mathbb{R}^3} \partial_t s\cdot\Pi \mathrm{d}x\mathrm{d}t \\ &= \varepsilon\left[\int_{\mathbb{R}^3} s\cdot\Pi \mathrm{d}x\right]_{t=0}^{t=\tau} - \varepsilon\int_0^\tau \int_{\mathbb{R}^3} s\cdot\partial_t\Pi \mathrm{d}x\mathrm{d}t, \end{aligned} \tag{4.38}$$ phi_p

that it goes to zero for $\varepsilon \to 0$. For the second term of (4.22), we have

$$\begin{aligned}&\int_0^\tau \int_{\mathbb{R}^3} \varrho(\mathbf{U}-\mathbf{u})\cdot\partial_t\nabla_x\Psi \mathrm{d}x\mathrm{d}t \\ &= -\int_0^\tau \int_{\mathbb{R}^3} \varrho\mathbf{u}\cdot\partial_t\nabla_x\Psi \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\mathbb{R}^3} \varrho\mathbf{v}\cdot\partial_t\nabla_x\Psi \mathrm{d}x\mathrm{d}t \\ &\quad + \frac{1}{2}\int_0^\tau \int_{\mathbb{R}^3} \varrho\partial_t\left|\nabla_x\Psi\right|^2 \mathrm{d}x\mathrm{d}t, \end{aligned} \tag{4.39}$$ u_phi_1

where

$$\begin{aligned}&\int_0^\tau \int_{\mathbb{R}^3} \varrho\mathbf{v}\cdot\partial_t\nabla_x\Psi \mathrm{d}x\mathrm{d}t \\ &= \int_0^\tau \int_{\mathbb{R}^3} \left(\varrho-1\right)\mathbf{v}\cdot\partial_t\nabla_x\Psi \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\mathbb{R}^3} \mathbf{v}\cdot\partial_t\nabla_x\Psi \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{4.40}$$ u_phi_2

We use the acoustic equation (3.1) to rewrite the first term above as follows

$$\int_0^\tau \int_{\mathbb{R}^3} (\varrho - 1)\, \mathbf{v} \cdot \partial_t \nabla_x \Psi \mathrm{d}x\mathrm{d}t$$
$$= -a \int_0^\tau \int_{\mathbb{R}^3} \frac{\varrho - 1}{\varepsilon} \mathbf{v} \cdot \nabla_x s \mathrm{d}x\mathrm{d}t, \tag{4.41}$$ s_phi

where, thanks to (2.8), (3.7), (4.15) and (4.16), we have

$$\int_0^\tau \int_{\mathbb{R}^3} \frac{\varrho - 1}{\varepsilon} \mathbf{v} \cdot \nabla_x s \mathrm{d}x\mathrm{d}t$$
$$\leq \int_0^\tau \left\| [\frac{\varrho - 1}{\varepsilon}] 1_{\{|\varrho - 1| < 1\}} \right\|_{L^2(\mathbb{R}^3)} \|\mathbf{v}\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \|\nabla_x s\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t \leq c(M)\varepsilon \left(\log(\varepsilon + \tau) - \log(\varepsilon)\right) \tag{4.42}$$ s_rho_v

and

$$\int_0^\tau \int_{\mathbb{R}^3} \frac{\varrho - 1}{\varepsilon} \mathbf{v} \cdot \nabla_x s \mathrm{d}x\mathrm{d}t$$
$$\leq \int_0^\tau \left\| [\frac{\varrho - 1}{\varepsilon}] 1_{\{|\varrho - 1| \geq 1\}} \right\|_{L^\gamma(\mathbb{R}^3)} \|\mathbf{v}\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)} \|\nabla_x s\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t$$
$$\leq c(M)\varepsilon^{\frac{2}{\gamma}} \left(\log(\varepsilon + \tau) - \log(\varepsilon)\right), \tag{4.43}$$ vs

where we used the following interpolation inequality for $\mathbf{v}$

$$\|\mathbf{v}\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)} \leq \|\mathbf{v}\|_{L^1(\mathbb{R}^3;\mathbb{R}^3)}^{\frac{\gamma-1}{\gamma}} \|\mathbf{v}\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1-\frac{\gamma-1}{\gamma}}$$
$$\leq \|\mathbf{v}\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}^{\frac{\gamma-1}{\gamma}} \|\mathbf{v}\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}^{\frac{\gamma-1}{\gamma}} \|\mathbf{v}\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma} \leq c.$$

For the second term in (4.40), performing integration by parts, we have

$$\int_0^\tau \int_{\mathbb{R}^3} \mathrm{div}_x \mathbf{v} \cdot \partial_t \Psi \mathrm{d}x\mathrm{d}t = 0 \tag{4.44}$$ div_phi

thanks to incompressibility condition, $\mathrm{div}_x \mathbf{v} = 0$. Regarding I_2, we have

$$|I_2| \leq \frac{\varepsilon}{2} \int_0^\tau \int_{\mathbb{R}^3} \left(S(\nabla_x \mathbf{U}) - S(\nabla_x \mathbf{u})\right) : \left(\nabla_x \mathbf{U} - \nabla_x \mathbf{u}\right) \mathrm{d}x\mathrm{d}t$$
$$+ c\varepsilon \int_0^\tau \int_{\mathbb{R}^3} |S(\nabla_x \mathbf{U})|^2 \mathrm{d}x\mathrm{d}t, \tag{4.45}$$ diss

where we used Young inequality and the following Korn inequality

$$\int_{\mathbb{R}^3} |\nabla_x \mathbf{U} - \nabla_x \mathbf{u}|^2 \mathrm{d}x \leq c \int_{\mathbb{R}^3} \left(S(\nabla_x \mathbf{U}) - S(\nabla_x \mathbf{u})\right) : \left(\nabla_x \mathbf{U} - \nabla_x \mathbf{u}\right) \mathrm{d}x.$$

The first term on the right-hand side of (4.45) can be absorbed by the second term on the left-hand side in the relation (4.2). For the second term on the right-hand side of (4.45), in view of (2.8) and (3.6), we have

$$c\varepsilon \int_0^\tau \int_{\mathbb{R}^3} |S(\nabla_x \mathbf{U})|^2 \, \mathrm{d}x\mathrm{d}t \leq c(M)\varepsilon. \tag{4.46}$$

diss3

Regarding the terms I_3 and I_4 we deal with the following analysis. First, we have

$$\int_{\mathbb{R}^3} \nabla_x H'(r) \cdot r\mathbf{U}\mathrm{d}x = -\int_{\mathbb{R}^3} p(r)\mathrm{div}_x \mathbf{U}\mathrm{d}x \tag{4.47}$$

grad_H

that it will cancel with its counterpart in I_4. Next,

$$\frac{1}{\varepsilon^2}\int_0^\tau \int_{\mathbb{R}^3} \nabla_x H'(r) \cdot (\varrho\mathbf{u}) \, \mathrm{d}x\mathrm{d}t = \frac{1}{\varepsilon}\int_0^\tau \int_{\mathbb{R}^3} H''(r)\nabla_x s \cdot (\varrho\mathbf{u}) \, \mathrm{d}x\mathrm{d}t$$

$$= \int_0^\tau \int_{\mathbb{R}^3} \frac{H''(1+\varepsilon s) - H''(1)}{\varepsilon} \nabla_x s \cdot (\varrho\mathbf{u}) \, \mathrm{d}x\mathrm{d}t + \frac{1}{\varepsilon}\int_0^\tau \int_{\mathbb{R}^3} p'(1)\nabla_x s \cdot (\varrho\mathbf{u}) \, \mathrm{d}x\mathrm{d}t. \tag{4.48}$$

grad_H_p

Observing that

$$\frac{H''(1+\varepsilon s) - H''(1)}{\varepsilon} = H'''(\xi)s, \quad \xi \in (1, 1+\varepsilon s)\,,$$

$$\left| \frac{H''(1+\varepsilon s) - H''(1)}{\varepsilon} \right| \leq cs,$$

the first term on the right-hand side of (4.48) can be estimated in the following way

$$\int_0^\tau \int_{\mathbb{R}^3} \frac{H''(1+\varepsilon s) - H''(1)}{\varepsilon} \nabla_x s \cdot (\varrho\mathbf{u}) \, \mathrm{d}x\mathrm{d}t$$

$$\leq c\int_0^\tau \|s\|_{L^\infty} \|\nabla_x s\|_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)} \|\varrho\mathbf{u}\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t$$

$$\leq c(M)\left(\varepsilon\gamma\left(\frac{\varepsilon+\tau}{\varepsilon}\right)^{-1/\gamma} - \varepsilon\gamma\right). \tag{4.49}$$

H3

For the second integral on the right-hand side, using the acoustic equation (3.1), we get

$$\frac{1}{\varepsilon}\int_0^\tau \int_{\mathbb{R}^3} p'(1)\nabla_x s \cdot (\varrho\mathbf{u}) \, \mathrm{d}x\mathrm{d}t$$

$$= -\int_0^\tau \int_{\mathbb{R}^3} (\varrho\mathbf{u}) \cdot \partial_t \nabla_x \Psi \mathrm{d}x\mathrm{d}t \tag{4.50}$$

ps

that it will cancel with its counterpart in (4.39). Now, we write

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_0^\tau\int_{\mathbb{R}^3}\left[(r-\varrho)\,\partial_t H'(r) - p(\varrho)\mathrm{div}_x\mathbf{U}\right]\mathrm{d}x\mathrm{d}t \\
&= \frac{1}{\varepsilon}\int_0^\tau\int_{\mathbb{R}^3}(r-\varrho)\,H''(r)\partial_t s\mathrm{d}x\mathrm{d}t \\
&\quad -\frac{1}{\varepsilon^2}\int_0^\tau\int_{\mathbb{R}^3}p(\varrho)\triangle\Psi\mathrm{d}x\mathrm{d}t \\
&= \int_0^\tau\int_{\mathbb{R}^3}\frac{(1-\varrho)}{\varepsilon}H''(r)\partial_t s\mathrm{d}x\mathrm{d}t + \int_0^\tau\int_{\mathbb{R}^3}sH''(r)\partial_t s\mathrm{d}x\mathrm{d}t \\
&\quad -\frac{1}{\varepsilon^2}\int_0^\tau\int_{\mathbb{R}^3}p(\varrho)\triangle\Psi\mathrm{d}x\mathrm{d}t. \qquad (4.51)
\end{aligned}$$

oth

The last term on the right-hand side can be split as follows

$$\begin{aligned}
&-\frac{1}{\varepsilon^2}\int_0^\tau\int_{\mathbb{R}^3}p(\varrho)\triangle\Psi\mathrm{d}x\mathrm{d}t \\
&= -\frac{1}{\varepsilon^2}\int_0^\tau\int_{\mathbb{R}^3}\left[p(\varrho)-p(1)\right]\triangle\Psi\mathrm{d}x\mathrm{d}t \\
&\quad -\frac{1}{\varepsilon^2}\int_0^\tau\int_{\mathbb{R}^3}p(1)\triangle\Psi\mathrm{d}x\mathrm{d}t. \qquad (4.52)
\end{aligned}$$

oth_1

Using integration by parts, we have

$$-\frac{1}{\varepsilon^2}\int_0^\tau\int_{\mathbb{R}^3}\nabla_x p(1)\nabla_x\Psi\mathrm{d}x\mathrm{d}t = 0. \qquad (4.53)$$

oth_2

Now, we have

$$\begin{aligned}
&-\frac{1}{\varepsilon^2}\int_0^\tau\int_{\mathbb{R}^3}\left[p(\varrho)-p(1)\right]\triangle\Psi\mathrm{d}x\mathrm{d}t \\
&= -\int_0^\tau\int_{\mathbb{R}^3}\frac{\left[p(\varrho)-p'(1)(\varrho-1)-p(1)\right]}{\varepsilon^2}\triangle\Psi\mathrm{d}x\mathrm{d}t \\
&\quad -\int_0^\tau\int_{\mathbb{R}^3}\frac{p'(1)(\varrho-1)}{\varepsilon^2}\triangle\Psi\mathrm{d}x\mathrm{d}t. \qquad (4.54)
\end{aligned}$$

oth_3

Then, the following estimates hold

$$\left|\int_0^\tau\int_{\mathbb{R}^3}\frac{\left[p(\varrho)-p'(1)(\varrho-1)-p(1)\right]}{\varepsilon^2}\triangle\Psi\mathrm{d}x\mathrm{d}t\right| \leq c\int_0^\tau\mathcal{E}\left\|\triangle\Psi\right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}\mathrm{d}t. \qquad (4.55)$$

oth_4

Now, we have

$$\frac{1}{2}\int_0^\tau \int_{\mathbb{R}^3} \varrho \partial_t |\nabla_x \Psi|^2 \,\mathrm{d}x\mathrm{d}t$$

$$= \frac{1}{2}\int_0^\tau \int_{\mathbb{R}^3} (\varrho - 1)\, \partial_t |\nabla_x \Psi|^2 \,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^\tau \int_{\mathbb{R}^3} \partial_t |\nabla_x \Psi|^2 \,\mathrm{d}x\mathrm{d}t$$

$$= \frac{1}{2}\int_0^\tau \int_{\mathbb{R}^3} (\varrho - 1)\, \partial_t |\nabla_x \Psi|^2 \,\mathrm{d}x\mathrm{d}t + \left[\frac{1}{2}\int_{\mathbb{R}^3} |\nabla_x \Psi|^2 \,\mathrm{d}x\right]_{t=0}^{t=\tau}, \tag{4.56}$$ phi

where, using (3.1) in the first term on the right-hand side, we have

$$\frac{\varepsilon}{2}\int_0^\tau \int_{\mathbb{R}^3} \frac{(\varrho - 1)}{\varepsilon} \partial_t |\nabla_x \Psi|^2 \,\mathrm{d}x\mathrm{d}t = a\int_0^\tau \int_{\mathbb{R}^3} \frac{(\varrho - 1)}{\varepsilon} \nabla_x \Psi \cdot \nabla_x s \mathrm{d}x\mathrm{d}t \tag{4.57}$$ phi_dec

Now, using (3.7), (4.15) and (4.16) in (4.57), we have

$$\int_0^\tau \int_{\mathbb{R}^3} \frac{(\varrho - 1)}{\varepsilon} \nabla_x \Psi \cdot \nabla_x s \mathrm{d}x\mathrm{d}t$$

$$\leq \int_0^\tau \left\| [\frac{(\varrho - 1)}{\varepsilon}] 1_{\{|\varrho - 1| < 1\}} \right\|_{L^2(\mathbb{R}^3)} \|\nabla_x \Psi\|_{L^2(\mathbb{R}^3)} \|\nabla_x s\|_{L^\infty(\mathbb{R}^3)} \,\mathrm{d}t \leq c(M)\varepsilon \left(\log(\varepsilon + \tau) - \log(\varepsilon)\right) \tag{4.58}$$ rho_phi

and

$$\int_0^\tau \int_{\mathbb{R}^3} \frac{(\varrho - 1)}{\varepsilon} \nabla_x \Psi \cdot \nabla_x s \mathrm{d}x\mathrm{d}t$$

$$\leq \int_0^\tau \left\| [\frac{(\varrho - 1)}{\varepsilon}] 1_{\{|\varrho - 1| \geq 1\}} \right\|_{L^\gamma(\mathbb{R}^3)} \|\nabla_x \Psi\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3)} \|\nabla_x s\|_{L^\infty(\mathbb{R}^3)} \,\mathrm{d}t$$

$$\leq c(M)\varepsilon^{2/\gamma} \left(\gamma \left(\frac{\varepsilon + \tau}{\varepsilon}\right)^{-1/\gamma} - \gamma\right). \tag{4.59}$$ rho_phi_2

where we have used the following interpolation inequality for $\nabla_x \Psi$

$$\|\nabla_x \Psi\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)} \leq \|\nabla_x \Psi\|_{L^1(\mathbb{R}^3;\mathbb{R}^3)}^{\frac{\gamma-1}{\gamma}} \|\nabla_x \Psi\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1-\frac{\gamma-1}{\gamma}}$$

$$\leq \|\nabla_x \Psi\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}^{\frac{\gamma-1}{\gamma}} \|\nabla_x \Psi\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}^{\frac{\gamma-1}{\gamma}} \|\nabla_x \Psi\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma} \leq c(M) \|\nabla_x \Psi\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma}.$$

Now, collecting the remained terms, we write

$$\int_0^\tau \int_{\mathbb{R}^3} \frac{(1-\varrho)}{\varepsilon} H''(r) \partial_t s \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\mathbb{R}^3} s H''(r) \partial_t s \mathrm{d}x\mathrm{d}t$$

$$-\int_0^\tau \int_{\mathbb{R}^3} \frac{p'(1)(\varrho-1)}{\varepsilon^2} \triangle \Psi \mathrm{d}x\mathrm{d}t. \tag{4.60}$$ p1

For the first integrals in (4.60), it is possible to show (see [17]) that,

$$\left| \int_0^\tau \int_{\mathbb{R}^3} \frac{(1-\varrho)}{\varepsilon} H''(r) \partial_t s \mathrm{d}x\mathrm{d}t \right|$$

$$\leq \int_0^\tau \int_{\mathbb{R}^3} \frac{(\varrho-1)}{\varepsilon^2} p'(1) \triangle \Psi \mathrm{d}x\mathrm{d}t + c(M) \int_0^\tau \mathcal{E} \left\| \triangle \Psi \right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t, \tag{4.61}$$ p2

where the first term on the right hand side of the inequality it will cancel with its counterpart in (4.60). While, for the second integral in (4.60) we have

$$\left| \int_0^\tau \int_{\mathbb{R}^3} s H''(r) \partial_t s \mathrm{d}x\mathrm{d}t \right| \leq p'(1) \left[\frac{1}{2} \int_{\mathbb{R}^3} s^2 \mathrm{d}x \right]_{t=0}^{t=\tau} + c(M) \int_0^\tau \mathcal{E} \left\| \triangle \Psi \right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t. \tag{4.62}$$ p4

From (4.55), (4.61), (4.62) we need to estimate the following term

$$\int_0^\tau \mathcal{E} \left\| \triangle \Psi \right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t \leq c(M) \left[\varepsilon \left(\log\left(\varepsilon + \tau\right) - \log\left(\varepsilon\right) \right) \right]. \tag{4.63}$$ p4'

Finally, regarding I_5, we have

$$\int_0^\tau \int_{\mathbb{R}^3} (\varrho \mathbf{u} \times \omega) \cdot (\mathbf{v} - \mathbf{u}) \mathrm{d}x\mathrm{d}t - \int_0^\tau \int_{\mathbb{R}^3} (\varrho \mathbf{v} \times \omega) \cdot (\mathbf{v} - \mathbf{u}) \mathrm{d}x\mathrm{d}t$$

$$= \int_0^\tau \int_{\mathbb{R}^3} (\varrho \mathbf{u} \times \omega) \cdot \mathbf{v} \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\mathbb{R}^3} (\varrho \mathbf{v} \times \omega) \cdot \mathbf{u} \mathrm{d}x\mathrm{d}t$$

$$= \int_0^\tau \int_{\mathbb{R}^3} (\varrho \mathbf{u} \times \omega) \cdot \mathbf{v} \mathrm{d}x\mathrm{d}t - \int_0^\tau \int_{\mathbb{R}^3} (\varrho \mathbf{u} \times \omega) \cdot \mathbf{v} \mathrm{d}x\mathrm{d}t = 0 \tag{4.64}$$ rot1

and, thanks to (2.8), (3.7), (4.8) and (4.12), we have

$$\int_0^\tau \int_{\mathbb{R}^3} (\varrho \mathbf{u} \times \omega) \cdot \nabla_x \Psi \mathrm{d}x\mathrm{d}t$$

$$\leq \int_0^\tau \left\| \varrho \mathbf{u} \right\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^3;\mathbb{R}^3)} \left\| \nabla_x \Psi \right\|_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t$$

$$\leq c(M) \left(\frac{\gamma\left(\varepsilon+\tau\right)\left(\frac{\varepsilon+\tau}{\varepsilon}\right)^{-1/\gamma}}{\gamma-1} - \frac{\gamma\varepsilon}{\gamma-1} \right) \tag{4.65}$$ rot2

and

$$\int_0^\tau \int_{\mathbb{R}^3} (\varrho \mathbf{v} \times \omega) \cdot \nabla_x \Psi \mathrm{d}x\mathrm{d}t$$

$$\leq \int_0^\tau \|\varrho\|_{L^\gamma(\mathbb{R}^3)} \|\mathbf{v}\|^{\frac{\gamma}{\gamma-1}}_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \|\nabla_x \Psi\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t$$

$$\leq c(M)\varepsilon \left(\log(\varepsilon + \tau) - \log(\varepsilon)\right). \tag{4.66}$$

rot3

Combining the previous estimates and letting $\varepsilon \to 0$ we can rewrite (4.2) as

$$[\mathcal{E}(\varrho, \mathbf{u} \mid r, \mathbf{U})](\tau) \leq [\mathcal{E}(\varrho, \mathbf{u} \mid r, \mathbf{U})](0) + c(M) \int_0^\tau \mathcal{E}\mathrm{d}t \tag{4.67}$$

gronwall

In virtue of the integral form of the Gronwall inequality, we have

$$[\mathcal{E}(\varrho, \mathbf{u} \mid r, \mathbf{U})](\tau) \leq ([\mathcal{E}(\varrho, \mathbf{u} \mid r, \mathbf{U})](0)) \left(1 + c(M)\tau e^{c(M)\tau}\right) \quad \text{for} \quad \tau \in [0, T], \tag{4.68}$$

proof

where the quantity $\left(1 + c(M)\tau e^{c(M)\tau}\right)$ is bounded for fixed $\tau \in [0, T]$. Theorem 9 is proved and, consequently, Corollary 10.

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