

Archimédův výpočet čísla π

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► $\pi \approx 3.14\dots$



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- ▶ obvod kruhu: $o = 2\pi r$

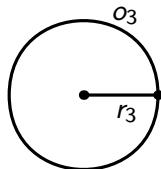
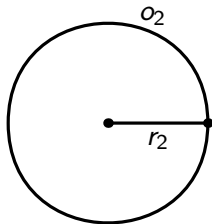
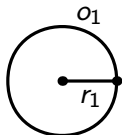


- ▶ $\pi \approx 3.14\dots$
- ▶ obvod kruhu: $o = 2\pi r$
- ▶ plocha kruhu: $S = \pi r^2$

$$o = 2\pi r$$



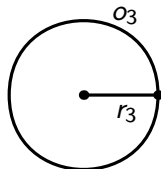
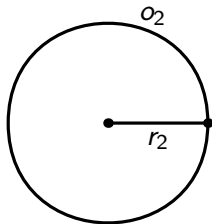
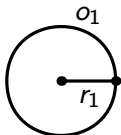
$$\frac{o_1}{2r_1} = \frac{o_2}{2r_2} = \frac{o_3}{2r_3}$$



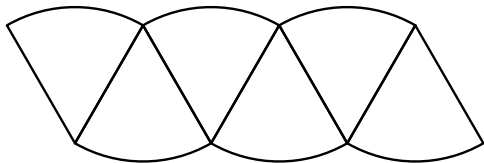
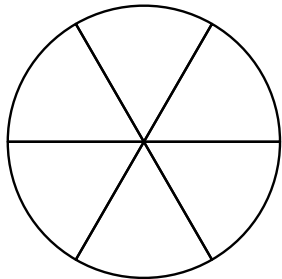


$$o = 2\pi r$$

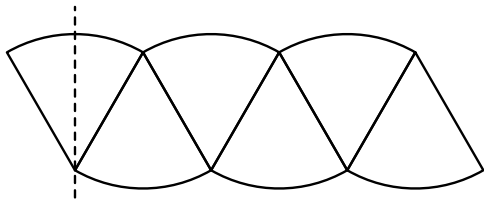
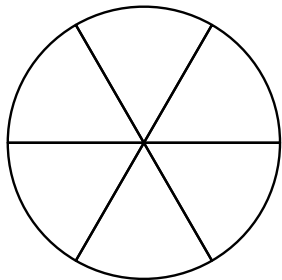
$$\frac{o_1}{2r_1} = \frac{o_2}{2r_2} = \frac{o_3}{2r_3} = \pi$$



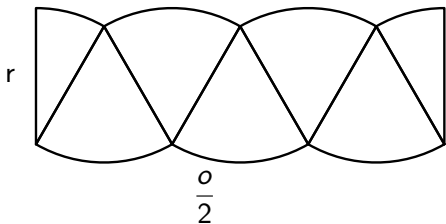
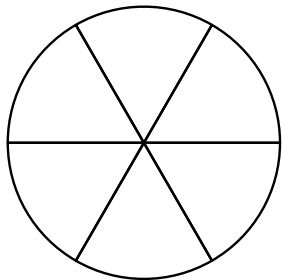
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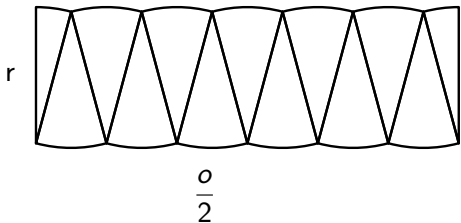
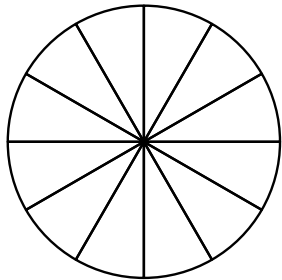


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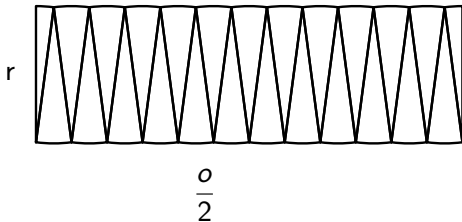
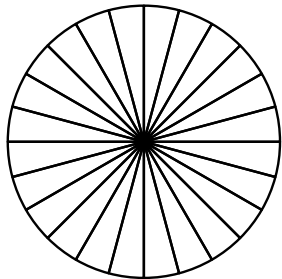
$$S = r \frac{o}{2} = r \frac{2\pi r}{2} = \pi r^2$$

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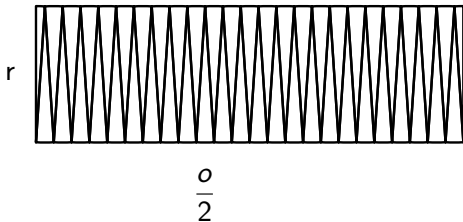
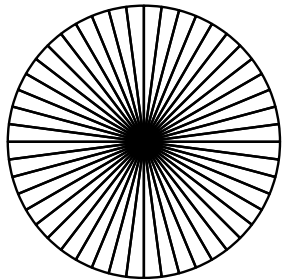
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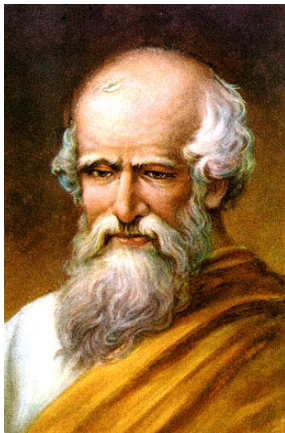


$$S = r \frac{o}{2} = r \frac{2\pi r}{2} = \pi r^2$$

Archimédes (asi 287–212 př.n.l., Syrakusy, Sicílie)



$$\frac{223}{71} < \pi < \frac{22}{7} \quad \Rightarrow \quad \pi \approx 3.14185$$



Archimédes (asi 287–212 př.n.l., Syrakusy, Sicílie)



$$\frac{223}{71} < \pi < \frac{22}{7} \quad \Rightarrow \quad \pi \approx 3.14185$$

Kružnice kolem Prahy s poloměrem 9 km.

Jaká bude chyba, pokud pro obvod použijeme Archimedovo π ?



- (a) asi 500 m
- (b) asi 50 m
- (c) asi 5 m

Archimédes (asi 287–212 př.n.l., Syrakusy, Sicílie)

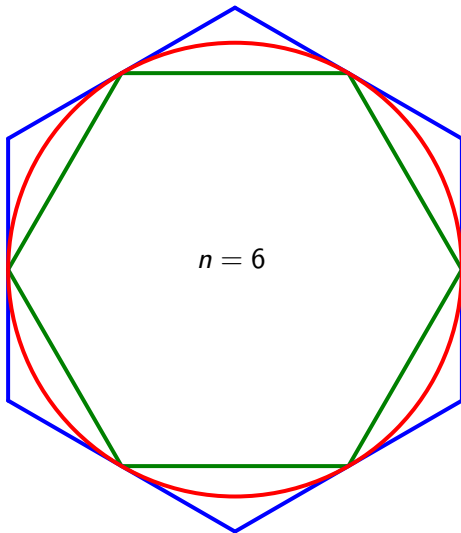


$$\frac{223}{71} < \pi < \frac{22}{7} \quad \Rightarrow \quad \pi \approx 3.14185$$

Archimédes (asi 287–212 př.n.l., Syrakusy, Sicílie)



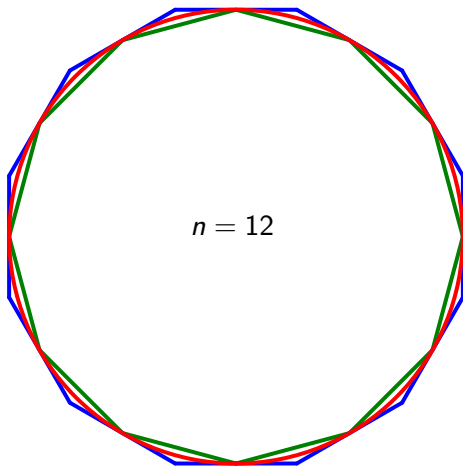
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Archimédes (asi 287–212 př.n.l., Syrakusy, Sicílie)



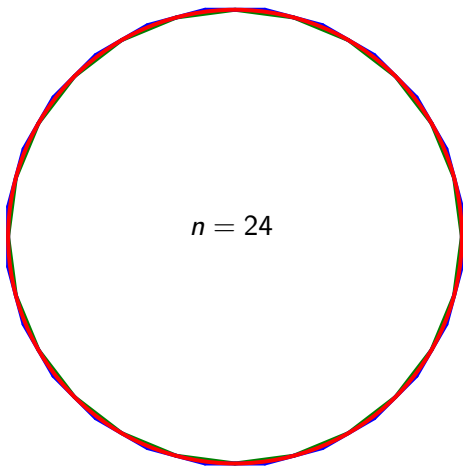
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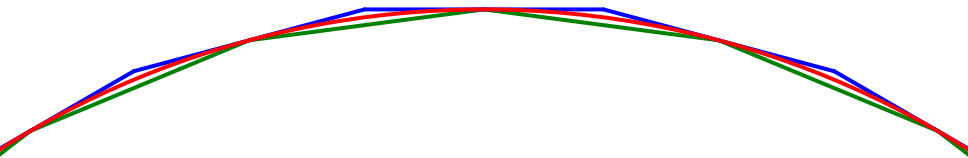
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$n = 24$

Archimédes (asi 287–212 př.n.l., Syrakusy, Sicílie)



$$\frac{223}{71} < \pi < \frac{22}{7} \quad \Rightarrow \quad \pi \approx 3.14185$$

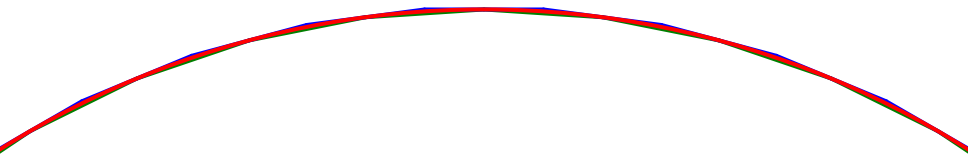


$$n = 24$$

Archimédes (asi 287–212 př.n.l., Syrakusy, Sicílie)



$$\frac{223}{71} < \pi < \frac{22}{7} \quad \Rightarrow \quad \pi \approx 3.14185$$



$$n = 48$$

Archimédes (asi 287–212 př.n.l., Syrakusy, Sicílie)



$$\frac{223}{71} < \pi < \frac{22}{7} \quad \Rightarrow \quad \pi \approx 3.14185$$

$$n = 96$$

Archimédes (asi 287–212 př.n.l., Syrakusy, Sicílie)



$$\frac{223}{71} < \pi < \frac{22}{7} \quad \Rightarrow \quad \pi \approx 3.14185$$

Metoda:

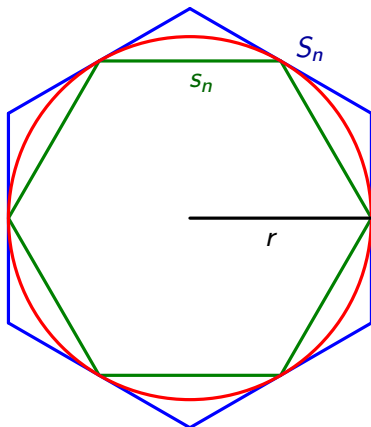
$$o = 2\pi r \quad \Leftrightarrow \quad \pi = \frac{o}{2r}$$

$$ns_n < o < nS_n$$

$$\frac{ns_n}{2r} < \frac{o}{2r} < \frac{nS_n}{2r}$$

$$\underline{\pi}_n < \pi < \bar{\pi}_n$$

$$r = 1$$



Archimédes (asi 287–212 př.n.l., Syrakusy, Sicílie)



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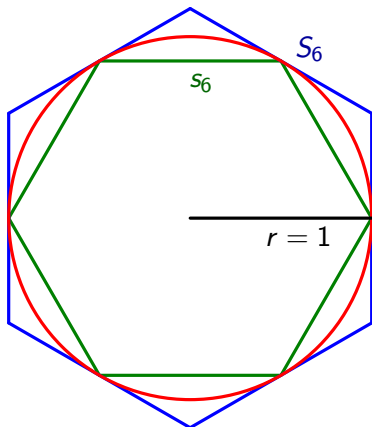
6-ti úhelník: ($n = 6$)

$$s_6 = 1$$

$$\pi_6 = \frac{6 \cdot 1}{2} = 3$$

$$S_6 = \frac{2}{3}\sqrt{3}$$

$$\bar{\pi}_6 = \frac{6 \cdot \frac{2}{3}\sqrt{3}}{2} = 2\sqrt{3} \approx 3.46$$

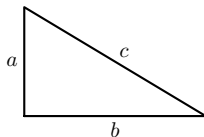


Pomocné věty

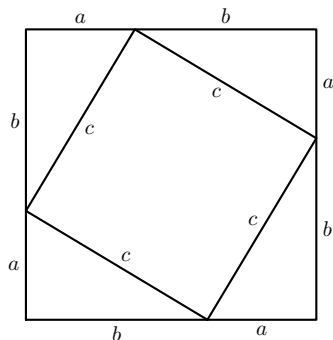
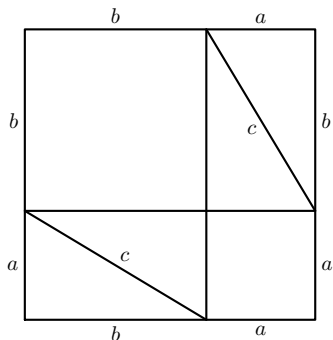


Věta (Pythagorova):

$$a^2 + b^2 = c^2$$



Důkaz:

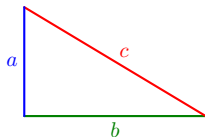


Pomocné věty

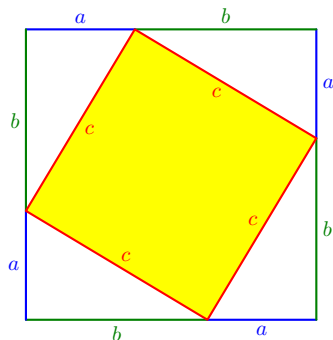
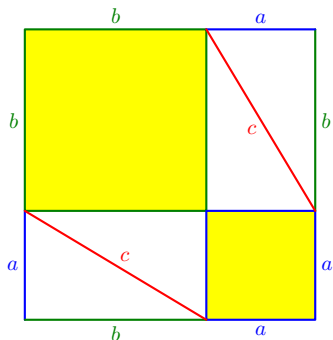


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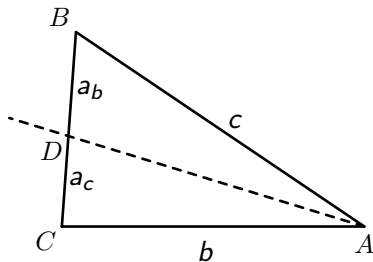


Pomocné věty



Věta: $\frac{a_c}{a_b} = \frac{b}{c}$

D je průsečík osy úhlu CAB s hranou a



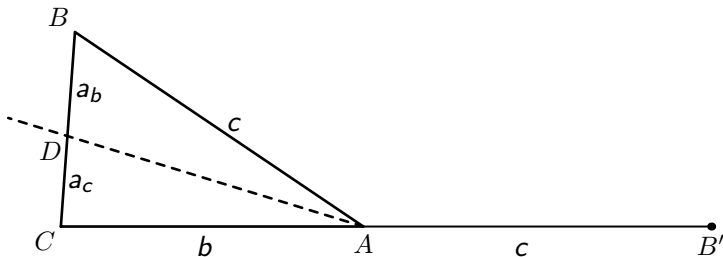
Pomocné věty



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D je průsečík osy úhlu CAB s hranou a

Důkaz:



$$\frac{a_c}{b} = \frac{a_c + a_b}{b + c}$$

$$a_c b + a_c c = a_c b + a_b b$$

$$\frac{a_c}{a_b} = \frac{b}{c}$$

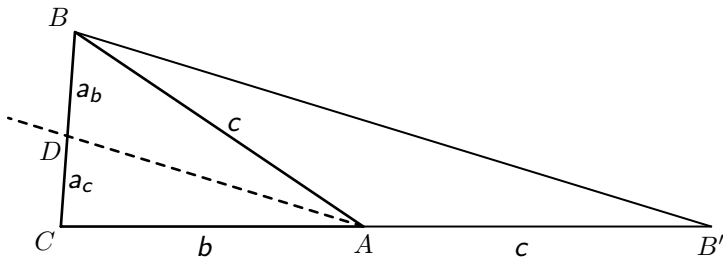
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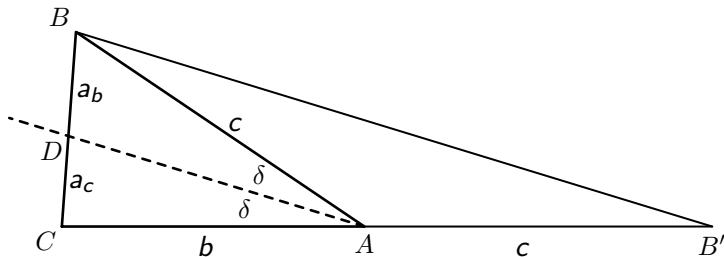
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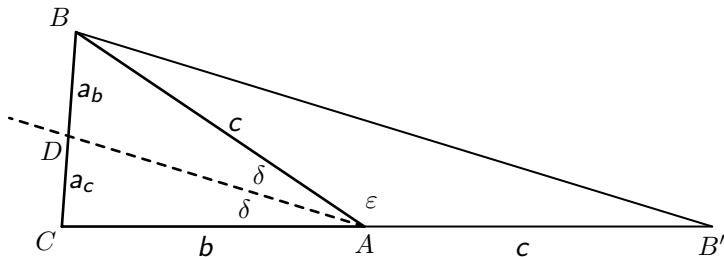
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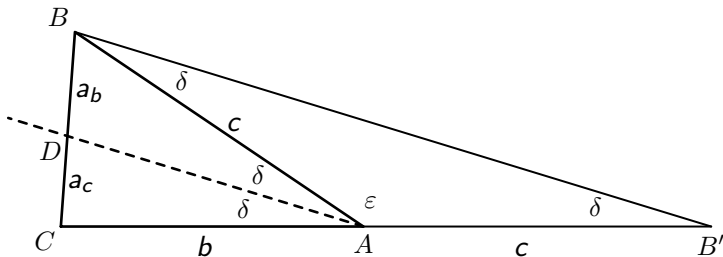
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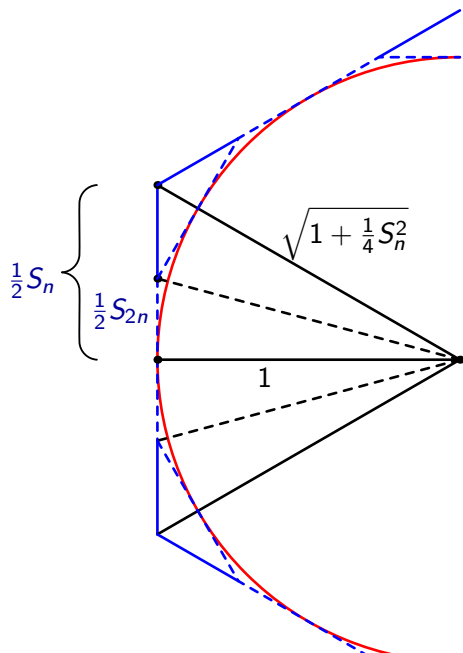


$$\frac{a_c}{b} = \frac{a_c + a_b}{b + c}$$

$$a_c b + a_c c = a_c b + a_b b$$

$$\frac{a_c}{a_b} = \frac{b}{c}$$

Opsané pravidelné n -úhelníky



$$\frac{\frac{1}{2}S_{2n}}{\frac{1}{2}S_n - \frac{1}{2}S_{2n}} = \frac{1}{\sqrt{1 + \frac{1}{4}S_n^2}}$$

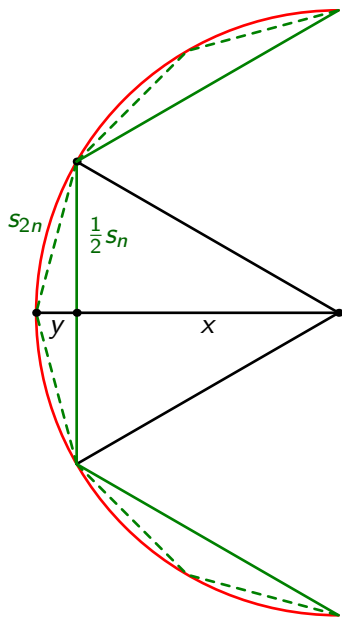
$$\frac{S_{2n}}{S_n - S_{2n}} = \frac{1}{\sqrt{1 + \frac{1}{4}S_n^2}}$$

$$\frac{1}{\frac{S_n}{S_{2n}} - 1} = \frac{1}{\sqrt{1 + \frac{1}{4}S_n^2}}$$

$$\frac{S_n}{S_{2n}} = 1 + \sqrt{1 + \frac{1}{4}S_n^2}$$

$$S_{2n} = \frac{2S_n}{2 + \sqrt{4 + S_n^2}}$$

Vepsané pravidelné n -úhelníky



$$x + y = 1$$

$$x^2 + \frac{s_n^2}{4} = 1$$

$$y^2 + \frac{s_n^2}{4} = s_{2n}^2$$

$$s_{2n}^2 = 1 - 2x + x^2 + \frac{s_n^2}{4}$$

$$s_{2n}^2 = 2 - 2x$$

$$s_{2n}^2 = 2 - 2\sqrt{1 - \frac{s_n^2}{4}}$$

$$s_{2n} = \sqrt{2 - \sqrt{4 - s_n^2}}$$



$$s_{2n} = \sqrt{2 - \sqrt{4 - s_n^2}}$$

$$\underline{\pi}_n = ns_n/2$$

$$S_{2n} = \frac{2S_n}{2 + \sqrt{4 + S_n^2}}$$

$$\bar{\pi}_n = nS_n/2$$

$$s_6 = 1$$

$$\Rightarrow \underline{\pi}_6 = 3$$

$$s_{12} = \sqrt{2 - \sqrt{4 - 1^2}}$$

$$\approx 0.5176$$

$$\Rightarrow \underline{\pi}_{12} = 3.1058$$

$$s_{24} = \dots$$

$$S_6 = 2\sqrt{3}/3 \approx 1.1547$$

$$\Rightarrow \bar{\pi}_6 = 3.4641$$

$$S_{12} = \frac{2 \cdot 1.1547}{2 + \sqrt{4 + 1.1547^2}}$$

$$\approx 0.5359$$

$$\Rightarrow \bar{\pi}_{12} = 3.2154$$

$$S_{24} = \dots$$



Numerický výpočet $\pi = 3.14159265358979$

krok	n	π_n	$\bar{\pi}_n$
1	6	3.000000000000000	3.46410161513775
2	12	3.10582854123025	3.21539030917347
3	24	3.13262861328124	3.15965994209750
4	48	3.13935020304687	3.14608621513144
5	96	3.14103195089053	3.14271459964537
⋮	⋮	⋮	⋮
13	24576	3.14159264532122	3.14159267070200
14	49152	3.14159264532122	3.14159265786785
15	98304	3.14159264532122	3.14159265465931
16	196608	3.14159264532122	3.14159265385717
17	393216	3.14159366984943	3.14159265365664
⋮	⋮	⋮	⋮
26	201326592	3.35410196624968	3.14159265358980
27	402653184	4.24264068711929	3.14159265358980
28	805306368	6.000000000000000	3.14159265358980
29	1610612736	0.000000000000000	3.14159265358980

Numerický výpočet $\pi = 3.14159265358979$



$$s_{2n} = \sqrt{2 - \sqrt{4 - s_n^2}} \quad \Rightarrow \quad \pi_n = s_n/2$$

$$s_{2n}^2 = 2 - \sqrt{4 - s_n^2}$$

$$s_{2n}^2 = \left(2 - \sqrt{4 - s_n^2}\right) \frac{2 + \sqrt{4 - s_n^2}}{2 + \sqrt{4 - s_n^2}}$$

$$s_{2n}^2 = \frac{4 - (4 - s_n^2)}{2 + \sqrt{4 - s_n^2}}$$

$$s_{2n} = \sqrt{\frac{s_n^2}{2 + \sqrt{4 - s_n^2}}} \quad \Rightarrow \quad \pi_n = s_n/2$$



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⋮	⋮	⋮	⋮
26	201326592	3.14159265358979	3.14159265358980
27	402653184	3.14159265358979	3.14159265358980
28	805306368	3.14159265358979	3.14159265358980
29	1610612736	3.14159265358979	3.14159265358980

Děkuji za pozornost

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