α-IDEALS IN 0-DISTRIBUTIVE POSETS

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Abstract. The concept of α-ideals in posets is introduced. Several properties of α-ideals in 0-distributive posets are studied. Characterization of prime ideals to be α-ideals in 0-distributive posets is obtained in terms of minimality of ideals. Further, it is proved that if a prime ideal I of a 0-distributive poset is non-dense, then I is an α-ideal. Moreover, it is shown that the set of all α-ideals $\alpha\,\mathrm{Id}(P)$ of a poset P with 0 forms a complete lattice. A result analogous to separation theorem for finite 0-distributive posets is obtained with respect to prime α-ideals. Some counterexamples are also given.

Keywords: 0-distributive poset; ideal; α-ideal; prime ideal; non-dense ideal; minimal ideal; annihilator ideal

MSC 2010: 06A06, 06A75

1. Introduction

Grillet and Varlet [4] introduced 0-distributive lattices as a generalization of distributive lattices. The theory of 0-distributive lattices was further studied by Balasubramani and Venkatanarasimhan [1] and Jayaram [7]. Cornish [2] introduced and studied the properties of α-ideals in distributive lattices. Generalization of the concept of α-ideals in 0-distributive lattices is carried out by Jayaram [7]. In fact, he proved the separation theorem for prime α-ideals in the case of 0-distributive lattices as follows.

Theorem A (Jayaram [7]). *Let I be an α-ideal of a 0-distributive lattice L and S be a meet subsemilattice of L such that $I \cap S = \emptyset$. Then there exists a prime α-ideal G of L such that $I \subseteq G$ and $G \cap S = \emptyset$.*

Additional properties of α-ideals in 0-distributive lattices were obtained by Pawar and Mane [12] and Pawar and Khopade [11].

In Section 2 of this paper, we show several results concerning α-ideals, which are extensions of the results concerning lattices and semilattices given in Pawar and Mane [12] and Pawar and Khopade [11] to posets, especially to 0-distributive posets. In Section 3, we prove that the set of all α-ideals of a poset with 0 is a complete lattice. Further, we generalize Theorem A for finite 0-distributive posets.

We begin with necessary concepts and terminology. For notation and terminology not mentioned here the reader is referred to Grätzer [3].

Let P be a poset and $A \subseteq P$. The set $A^u = \{x \in P;\ x \geqslant a$ for every $a \in A\}$ is called the *upper cone* of A. Dually, we have the concept of the *lower cone* A^l of A. We shall write A^{ul} instead of $\{A^u\}^l$ and dually. The upper cone $\{a\}^u$ is simply denoted by a^u and $\{a,b\}^u$ is denoted by $(a,b)^u$. Similar notation is used for lower cones. Further, for $A, B \subseteq P, \{A \cup B\}^u$ is denoted by $\{A, B\}^u$ and for $x \in P$, the set $\{A \cup \{x\}\}^u$ is denoted by $\{A, x\}^u$. Similar notation is used for lower cones. We note that $A \subseteq A^{ul}$ and $A \subseteq A^{lu}$. If $A \subseteq B$, then $B^l \subseteq A^l$ and $B^u \subseteq A^u$. Moreover, $A^{lul} = A^l$, $A^{ulu} = A^u$ and $\{a^u\}^l = \{a\}^l = a^l$.

A poset P with 0 is called 0-*distributive*, see Joshi and Waphare [9], if $(x,y)^l = \{0\} = (x,z)^l$ imply $\{x, (y,z)^u\}^l = \{0\}$ for $x, y, z \in P$. Dually, we have the concept of a 1-*distributive* poset.

A nonempty subset I of a poset P is called an *ideal* if $a, b \in I$ implies $(a,b)^{ul} \subseteq I$, see Halaš [5]. A proper ideal I is called *prime*, if $(a,b)^l \subseteq I$ implies that either $a \in I$ or $b \in I$, see Halaš and Rachůnek [6]. Dually, we have the concepts of a *filter* and a *prime filter*. Given $a \in P$, the subset $a^l = \{x \in P;\ x \leqslant a\}$ is an ideal of P generated by a, denoted by $(a]$. We shall call $(a]$ a *principal ideal.* Dually, a filter $[a) = a^u = \{x \in P;\ x \leqslant a\}$ generated by a is called a principal *filter*. It is known that the set of all ideals of a poset P, denoted by $\mathrm{Id}(P)$, forms a complete lattice under set inclusion, see Halaš and Rachůnek [6]. A nonempty subset Q of P is called an *up directed set*, if $Q \cap (x,y)^u \neq \emptyset$ for any $x, y \in Q$. Dually, we have the concept of a *down directed set.* If an ideal I (filter F) is an up (down) directed set of P, then it is called a *u-ideal* (*l-filter*).

For a nonempty subset A of a poset P with 0, define a subset $A^\perp$ of P as follows:

$$A^\perp = \{z \in P;\ (a,z)^l = \{0\}\ \forall\, a \in A\}.$$

If $A = \{x\}$, then we write $a^\perp$ instead of $\{a\}^\perp$. We note that $A \subseteq A^{\perp\perp}$ and $x \in x^{\perp\perp}$. Further, $A^\perp = \bigcap\limits_{a \in A} a^\perp$ and $A \cap A^\perp = \{0\}$. Moreover, if $A \subseteq B$, then $B^\perp \subseteq A^\perp$.

An ideal I of a poset P is said to be an α-*ideal*, if $x^{\perp\perp} \subseteq I$ for all $x \in I$. We denote the set of all α-ideals of P by $\alpha\,\mathrm{Id}(P)$.

2. α-ideals in 0-distributive posets

In this section, we study α-ideals, prime and minimal prime ideals in a 0-distributive poset. We begin by proving a characterization of 0-distributive posets.

Lemma 2.1. *A poset P is 0-distributive if and only if $(x,y)^{ul^{\perp}} = x^{\perp} \cap y^{\perp}$ for all $x, y \in P$.*

Proof. Let P be a 0-distributive poset and let $a \in (x,y)^{ul^{\perp}}$. Since $x, y \in (x,y)^{ul}$, we get $(a,x)^l = \{0\} = (a,y)^l$, which implies $a \in x^{\perp} \cap y^{\perp}$. Hence $(x,y)^{ul^{\perp}} \subseteq x^{\perp} \cap y^{\perp}$.

To show the converse inclusion, suppose that $a \in x^{\perp} \cap y^{\perp}$. We have $(a,x)^l = \{0\} = (a,y)^l$ and by 0-distributivity, we get $\{a, (x,y)^u\}^l = \{0\}$. Let $z \in (x,y)^{ul}$. Then clearly $(a,z)^l = \{0\}$. Thus $a \in (x,y)^{ul^{\perp}}$, which gives $x^{\perp} \cap y^{\perp} \subseteq (x,y)^{ul^{\perp}}$. Therefore $(x,y)^{ul^{\perp}} = x^{\perp} \cap y^{\perp}$.

Conversely, suppose $(x,y)^{ul^{\perp}} = x^{\perp} \cap y^{\perp}$ for all $x, y \in P$. To prove that P is 0-distributive, let us assume that $(a,x)^l = \{0\} = (a,y)^l$ for $a, x, y \in P$. Let $z \in \{a, (x,y)^u\}^l$. Then clearly $(z,x)^l = \{0\} = (z,y)^l$ and $z \in (x,y)^{ul}$. By assumption, $z \in x^{\perp} \cap y^{\perp} = (x,y)^{ul^{\perp}}$ and $z \in (x,y)^{ul}$, which yield $z \in (x,y)^{ul} \cap (x,y)^{ul^{\perp}} = \{0\}$. Therefore $z = 0$ and the 0-distributivity of P follows. □

For an ideal I of a poset P define a subset I' of P as follows:

$$I' = \{x \in P;\ a^{\perp} \subseteq x^{\perp} \text{ for some } a \in I\}.$$

The following is a characterization of an ideal I to be an α-ideal in terms of I' in a 0-distributive poset.

Theorem 2.2. *Let I be a u-ideal of a 0-distributive poset P. Then I' is the smallest α-ideal containing I. Moreover, an ideal I of P is an α-ideal if and only if $I = I'$.*

Proof. First we show that I' is an ideal. For this, assume that $x, y \in I'$ and $z \in (x,y)^{ul}$. We have to show that $z \in I'$. Since $x, y \in I'$, there exist $a, b \in I$ such that $a^{\perp} \subseteq x^{\perp}$ and $b^{\perp} \subseteq y^{\perp}$, and hence $a^{\perp} \cap b^{\perp} \subseteq x^{\perp} \cap y^{\perp}$. Therefore by Lemma 2.1, $a^{\perp} \cap b^{\perp} \subseteq (x,y)^{ul^{\perp}}$. Since I is a u-ideal, there exists an element $c \in (a,b)^u$ and $c \in I$. Now, $c \in (a,b)^u$ implies $c^{\perp} \subseteq a^{\perp} \cap b^{\perp}$, which gives $c^{\perp} \subseteq (x,y)^{ul^{\perp}}$. Since $z \in (x,y)^{ul}$, we have $(x,y)^{ul^{\perp}} \subseteq z^{\perp}$. Hence $c^{\perp} \subseteq z^{\perp}$ and therefore $z \in I'$.

Now, we show that I' is an α-ideal. Let $x \in I'$, i.e., there exists $a \in I$ such that $a^{\perp} \subseteq x^{\perp}$. We show that $x^{\perp\perp} \subseteq I'$. Suppose on the contrary that $x^{\perp\perp} \nsubseteq I'$. Then there exists an element $y \in P$ such that $y \in x^{\perp\perp}$ and $y \notin I'$. Observe that $a^{\perp} \nsubseteq y^{\perp}$, since $a^{\perp} \subseteq y^{\perp}$ and $a \in I$ imply that $y \in I'$, a contradiction to the fact that $y \notin I'$.

Thus $a^{\perp} \nsubseteq y^{\perp}$. So, there exists $b \in a^{\perp}$ and $b \notin y^{\perp}$. Since $a^{\perp} \subseteq x^{\perp}$, we have $b \in x^{\perp}$ and $b \notin y^{\perp}$, which is a contradiction to the fact that $y \in x^{\perp\perp}$. Hence $x^{\perp\perp} \subseteq I'$.

The inclusion $I \subseteq I'$ follows from the fact that $a^{\perp} \subseteq a^{\perp}$ for any element $a \in I$. Now, suppose that there exists an α-ideal J with the property $I \subseteq J$. We have to show that $I' \subseteq J$. Let $x \in I'$, i.e., $a^{\perp} \subseteq x^{\perp}$ for some $a \in I$. Since $I \subseteq J$, we have $a^{\perp} \subseteq x^{\perp}$ and $a \in J$. Using the fact that J is an α-ideal, we get $x^{\perp\perp} \subseteq a^{\perp\perp} \subseteq J$. Since $x \in x^{\perp\perp}$, we get $x \in J$ as required.

Further, let I be an α-ideal. To show that $I = I'$, it is enough to show that $I' \subseteq I$. For this, assume $x \in I'$. Then $a^{\perp} \subseteq x^{\perp}$ for some $a \in I$, which yields $x^{\perp\perp} \subseteq a^{\perp\perp} \subseteq I$. By using the fact that $x \in x^{\perp\perp}$, we get $x \in I$. Hence $I = I'$. ☐

Remark 2.3. The statement of Theorem 2.2 is not necessarily true if we drop the condition of I being a u-ideal. Consider the 0-distributive poset P depicted in Figure 1 and the ideal $I = \{0, a, b\}$, which is not a u-ideal. Observe that $I' = \{0, a, b\} \cup \{x_i\}$, where $i = 1, 2, \ldots$ But I' is not an ideal as $(b, x_1)^{ul} = P \nsubseteq I'$.

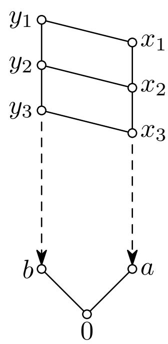

Figure 1.

For a nonempty subset A of a poset P with 0, consider the set $0(A)$ as follows:

$$0(A) = \{x \in P;\ (a, x)^l = \{0\} \text{ for some } a \in A\}.$$

We have the following result.

Theorem 2.4. *Let A be a down directed set of a 0-distributive poset P. Then $0(A)$ is an α-ideal of P.*

Proof. First we prove that $0(A)$ is an ideal. Let $x, y \in 0(A)$ and $z \in (x, y)^{ul}$. We show that $z \in 0(A)$. Since $x, y \in 0(A)$, there exist $a, b \in A$ such that $(a, x)^l = \{0\} = (b, y)^l$. Now, since A is a down directed set, there exists an element $c \in A$

such that $c \in (a, b)^l$, and consequently, $(c, x)^l = \{0\} = (c, y)^l$. By 0-distributivity, we get $\{c, (x, y)^u\}^l = \{0\}$, which gives $(c, z)^l = \{0\}$. Hence $z \in 0(A)$.

Now, we show that $0(A)$ is an α-ideal. Let $x \in 0(A)$, that is, $(a, x)^l = \{0\}$ for some $a \in A$. We claim that $x^{\perp\perp} \subseteq 0(A)$. Suppose that $z \in x^{\perp\perp}$. We obtain $(z, y)^l = \{0\}$ for all $y \in x^{\perp}$. Since $a \in x^{\perp}$, we get $(z, a)^l = \{0\}$, and this yields $z \in 0(A)$. Therefore $0(A)$ is an α-ideal. $\square$

Remark 2.5. The statement of Theorem 2.4 is not true if we remove the condition that A is a down directed set. In the 0-distributive poset P depicted in Figure 2, the set $A = \{1, a, b\}$ is not a down directed set. Observe that $0(A) = \{0, a, b\}$ is not an ideal as $a, b \in 0(A)$, but $(a, b)^{ul} = P \nsubseteq 0(A)$.

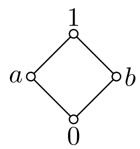

Figure 2.

An immediate consequence of Theorem 2.4 is the following:

Corollary 2.6. *For any l-filter F of a 0-distributive poset P, $0(F)$ is an α-ideal of P.*

However, in the case of meet semilattices we have a theorem of Pawar and Mane [12] following as a corollary.

Corollary 2.7. *For any filter F of a 0-distributive meet semilattice P, $0(F)$ is an α-ideal of P.*

Let I be a proper ideal of a poset P. Then I is said to be a *maximal ideal* of P, if the only ideal properly containing I is P. A *maximal filter*, more usually known as an *ultra filter*, is defined dually. Also, we have the concepts of a *minimal ideal* and a *minimal filter*.

It has to be noticed that Joshi and Mundlik [8], in their two lemmas listed below, have assumed that every maximal l-filter (maximal among all l-filters) is a maximal filter (maximal among all filters).

Lemma 2.8 (Joshi, Mundlik [8]). *Let F be an l-filter of a poset P with 0. Then F is a maximal l-filter if and only if the following condition holds:*

$$\text{for any } x \notin F, \text{ there exists } y \in F \text{ such that } (x, y)^l = \{0\}.$$

Lemma 2.9 (Joshi, Mundlik [8]). *Let P be a finite 0-distributive poset and let I be an ideal of P. Then I is a minimal prime ideal of P if and only if $P - I$ is a maximal l-filter of P.*

The following result is a characterization of prime ideals to be α-ideals in the case of finite 0-distributive posets.

Theorem 2.10. *Every minimal prime ideal of a finite 0-distributive poset P is an α-ideal.*

Proof. Let $x \in I$. To show that I is an α-ideal, we have to show that $x^{\perp\perp} \subseteq I$. Since I is a minimal prime ideal of P, by Lemma 2.9, $P - I$ is a maximal l-filter. Now, as $x \notin P - I$, by Lemma 2.8, there exists $y \in P - I$ such that $(x, y)^l = \{0\}$, that is, $y \notin I$ and $y \in x^{\perp}$. Let $z \in x^{\perp\perp}$. Since $y \in x^{\perp}$, we get, $(z, y)^l = \{0\}$, which gives $(z, y)^l \subseteq I$. Since $y \notin I$, by primeness of I, we have $z \in I$. Hence $x^{\perp\perp} \subseteq I$ as required. □

Let I be an ideal of a poset P with 0. Then I is called *dense* if $I^{\perp} = \{0\}$ and I is said to be an *annihilator* if $I = I^{\perp\perp}$. It is easy to observe that every annihilator ideal of a poset is an α-ideal.

Theorem 2.11. *If a prime ideal I of a 0-distributive poset P is non-dense, then I is an annihilator ideal.*

Proof. By assumption, $I^{\perp} \neq \{0\}$. Hence there exists $x \in I^{\perp}$ such that $x \neq 0$. But then $I^{\perp\perp} \subseteq x^{\perp}$. Since $I \subseteq I^{\perp\perp}$ is always true, we get $I \subseteq x^{\perp}$. Further, if $t \in x^{\perp}$, then $(x, t)^l = \{0\} \subseteq I$. From the fact that $I \cap I^{\perp} = \{0\}$, it is clear that $x \notin I$. Indeed, if $x \in I$, then $x \in I \cap I^{\perp} = \{0\}$, hence $x = 0$ a contradiction to $x \neq 0$. Since $(x, t)^l \subseteq I$ and $x \notin I$, by primeness of I, we get $t \in I$. Therefore $x^{\perp} \subseteq I$. By combining both inclusions, we get $x^{\perp} = I$. Consequently $I = I^{\perp\perp}$, and therefore I is an annihilator. □

As a consequence, we have the following statement, which is another characterization of prime ideals to be α-ideals.

Corollary 2.12. *If a prime ideal I of a 0-distributive poset P is non-dense, then I is an α-ideal.*

3. Prime α-ideal separation theorem in 0-distributive posets

We begin by proving that the set of all α-ideals $\alpha\,\mathrm{Id}(P)$ of a poset P with 0 is closed under the set-theoretical intersection, in fact, it is a complete lattice.

Lemma 3.1. *Let P be a poset with 0 and X be a family of members of $\alpha\,\mathrm{Id}(P)$. Then $\bigcap\limits_{I\in X} I$ is also in $\alpha\,\mathrm{Id}(P)$.*

P r o o f. Let $x \in \bigcap\limits_{I\in X} I$. We have $x \in I$ for all $I \in X$. Since I is an α-ideal, we have $x^{\perp\perp} \subseteq I$ for all $I \in X$, which implies that $x^{\perp\perp} \subseteq \bigcap\limits_{I\in X} I$. Therefore $\bigcap\limits_{I\in X} I \in \alpha\,\mathrm{Id}(P)$. □

Theorem 3.2 follows immediately from Lemma 3.1.

Theorem 3.2. *Let P be a poset with 0. Then $(\alpha\,\mathrm{Id}(P), \subseteq)$ forms a complete lattice in which infima and suprema of a family X of $\alpha\,\mathrm{Id}(P)$ are defined as follows: $\bigwedge\limits_{I\in X} I = \bigcap\limits_{I\in X} I$ and $\bigvee\limits_{I\in X} I = \bigcap\limits_{Y\in\alpha\,\mathrm{Id}(P)} Y$, where $\bigcup\limits_{I\in X} I \subseteq Y$.*

Let P be a given poset. Define the *extension* of an ideal I of P, denoted by I^e, as

$$I^e = \{J \in \mathrm{Id}(P);\ J \subseteq I\}$$

and for an ideal λ of the lattice $(\mathrm{Id}(P), \subseteq)$, define the *contraction* of λ, denoted by λ^c, as

$$\lambda^c = \bigcup\{J;\ J \in \lambda\}.$$

It is obvious that I^e is a principal ideal of $\mathrm{Id}(P)$ for every ideal I of a poset P. More details about these concepts can be found in Kharat and Mokbel [10].

In the following theorem we establish the relation between annihilator ideals of a 0-distributive poset P and the α-ideals of the lattice $\mathrm{Id}(P)$.

Theorem 3.3. *Let P be a poset with 0. If I is an annihilator ideal, then I^e is an α-ideal of $\mathrm{Id}(P)$.*

P r o o f. Suppose $J \in I^e$. Then we have $J \subseteq I$, which yields $J^{\perp\perp} \subseteq I^{\perp\perp}$. Since I is an annihilator, we get $J^{\perp\perp} \subseteq I$. Observe that $J^{\perp\perp} \subseteq I^e$. Indeed, if $J^{\perp\perp} \nsubseteq I^e$, then there exists $J_1 \in \mathrm{Id}(P)$ such that $J_1 \in J^{\perp\perp}$ and $J_1 \notin I^e$, i.e., $J_1 \in J^{\perp\perp}$ and $J_1 \nsubseteq I$. Hence there exists an element $x \in P$ such that $x \in J_1$ and $x \notin I$, which implies $(x] \in J^{\perp\perp} \subseteq I$ and $x \notin I$, a contradiction. Consequently $J^{\perp\perp} \subseteq I^e$. Hence I^e is an α-ideal. □

R e m a r k 3.4. The statement of Theorem 3.3 is not necessarily true if we drop the condition that I is an annihilator. Consider the poset P depicted in Figure 3 and its $\mathrm{Id}(P)$ depicted in Figure 4. Consider the α-ideal $I = \{0, a, b\}$, which is not an annihilator in P. Observe that $I^e = \{(0], (a], (b], \{0, a, b\}\}$ is not an α-ideal in $\mathrm{Id}(P)$, as $\{0, a, b\} \in I^e$, but $\{0, a, b\}^{\perp\perp} = \mathrm{Id}(P) \nsubseteq I^e$.

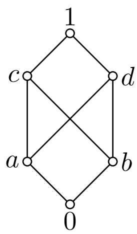

Figure 3.

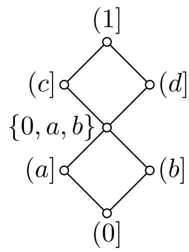

Figure 4.

Theorem 3.5. *Let P be a poset and let λ be an α-ideal of the lattice* $\mathrm{Id}(P)$. *Then λ^c is an α-ideal of P.*

P r o o f. First we prove that λ^c is an ideal. Consider elements $x, y \in \lambda^c$. If x and y belong to some $J \in \lambda$, then the result is obvious. Suppose there exist $J_1, J_2 \in \lambda$ such that $x \in J_1$ and $y \in J_2$, $J_1 \neq J_2$, then we have $(x, y)^{ul} \subseteq J_1 \vee J_2 \in \lambda$, as λ is an ideal. Thus λ^c is an ideal of P.

Now, we show that λ^c is an α-ideal of $\mathrm{Id}(P)$. Let $x \in \lambda^c$. We claim that $x^{\perp\perp} \subseteq \lambda^c$. Observe that $x \in \lambda^c$ implies $(x] \in \lambda$. Since λ is an α-ideal of $\mathrm{Id}(P)$, we have $(x]^{\perp\perp} \subseteq \lambda$. Therefore $x^{\perp\perp} \subseteq \lambda^c$ as required. $\square$

Now, let K be an l-filter of a poset P. Define a subset γ of $\mathrm{Id}(P)$ as follows:

$$\gamma = \{J \in \mathrm{Id}(P);\ J \cap K \neq \emptyset\}. \tag{$*$}$$

We use the following results to prove Theorem 3.9, which is a generalization of Theorem A for finite posets.

Lemma 3.6 (Kharat, Mokbel [10]). *Let P be a poset, K be an l-filter of P and let γ be a subset of* $\mathrm{Id}(P)$ *as defined in* $(*)$. *Then γ is a filter of* $\mathrm{Id}(P)$.

Lemma 3.7 (Kharat, Mokbel [10]). *Let P be a finite poset and λ be a prime ideal of* $\mathrm{Id}(P)$. *Then λ^c is a prime ideal of P.*

Lemma 3.8 (Joshi, Waphare [9]). *A poset P is* 0*-distributive if and only if* $\mathrm{Id}(P)$ *is a* 0*-distributive lattice.*

Theorem 3.9. *Let I be an annihilator ideal and F be an l-filter of a finite* 0*-distributive poset P for which $I \cap F = \emptyset$. Then there exists a prime α-ideal G of P such that $I \subseteq G$ and $I \cap F = \emptyset$.*

P r o o f. Suppose I is an annihilator ideal and F is an l-filter of a finite 0-distributive poset P for which $I \cap F = \emptyset$. By Theorem 3.3, I^e is an α-ideal of $\mathrm{Id}(P)$ and also $\gamma = \{J \in \mathrm{Id}(P);\ J \cap F \neq \emptyset\}$ is a filter of $\mathrm{Id}(P)$ by Lemma 3.6. Observe that $I^e \cap \gamma = \emptyset$. Were this false, then there exists $J_1 \in \mathrm{Id}(P)$ such that $J_1 \in I^e \cap \gamma$. Thus $J_1 \subseteq I$ and $J_1 \cap F \neq \emptyset$. In other words, $I \cap F \neq \emptyset$, which contradicts the hypothesis. By Lemma 3.8, $\mathrm{Id}(P)$ is a 0-distributive lattice. Hence, by Theorem A, there exists a prime α-ideal λ of $\mathrm{Id}(P)$ such that $I^e \subseteq \lambda$ and $\lambda \cap \gamma = \emptyset$. Since λ is a prime α-ideal of $\mathrm{Id}(P)$, by Lemma 3.7 and Theorem 3.5, λ^c is a prime α-ideal of P. Further, $I \subseteq \lambda^c$, since $x \in I$ implies $(x] \in I^e \subseteq \lambda$. Thus $(x] \in \lambda$, and by definition of λ^c, we have $x \in \lambda^c$. Also, we have $\lambda^c \cap F = \emptyset$. Otherwise, if $\lambda^c \cap F \neq \emptyset$, then there exists $x \in P$ such that $x \in \lambda^c \cap F$. Hence $(x] \subseteq J$, where $J \in \lambda$ and $(x] \in \gamma$. In other words, $(x] \in \lambda \cap \gamma$, a contradiction. $\square$

R e m a r k 3.10. (i) The statement of Theorem 3.9 is not necessarily true if we drop the condition that P is finite. Let $\mathbb{N}$ be the set of natural numbers. Consider the set $P = \{\emptyset\} \cup \{X \subseteq \mathbb{N};\ X \text{ is an infinite subset of } \mathbb{N}\} \cup \{X \subseteq \mathbb{N};\ |X| = 1\}$. It is easy to observe that P is an infinite 0-distributive poset under set inclusion and $F = \{X \subseteq \mathbb{N};\ X \text{ is an infinite subset of } \mathbb{N}\}$ is an l-filter of P, see Joshi and Mundlik [8]. Let $I = \{\{1\}, \emptyset\}$. Observe that I is an annihilator ideal for which $I \cap F = \emptyset$. But there does not exist a prime α-ideal G of P for which $I \subseteq G$ and $G \cap F = \emptyset$.

(ii) The condition of F being an l-filter cannot be dropped in the statement of Theorem 3.9. Consider the finite 0-distributive poset P depicted in Figure 5. Consider the annihilator ideal $I = \{0, a, b\}$, which is not prime, and a filter $F = \{a', b', c', d', 1\}$, which is not an l-filter. Observe that $I \cap F = \emptyset$, but there is no prime α-ideal G of P such that $I \subseteq G$ and $G \cap F = \emptyset$.

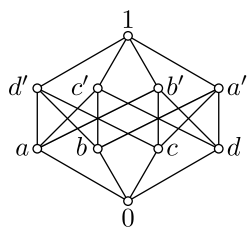

Figure 5.

(iii) Theorem 3.9 is not necessarily true if we drop the condition that I is an annihilator ideal. Consider the finite 0-distributive poset P depicted in Figure 5. Let $I = \{0, a, b, c, d\}$ and $F = \{a', 1\}$. Observe that I is an α-ideal but not prime and F is an l-filter of P for which $I \cap F = \emptyset$, but there is no prime α-ideal G of P such that $I \subseteq G$ and $G \cap F = \emptyset$.

Lemma 3.11 (Kharat, Mokbel [10]). *Let P be a meet semilattice and λ be a prime ideal of* $\operatorname{Id}(P)$. *Then λ^c is a prime ideal of P.*

However, if the poset is a meet semilattice, then by Theorem 3.9 and Lemma 3.11 we have the following:

Corollary 3.12. *Let I be an annihilator ideal and F be a filter of a* 0*-distributive meet semilattice P for which $I \cap F = \emptyset$. Then there exists a prime α-ideal G of P such that $I \subseteq G$ and $I \cap F = \emptyset$.*

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References

[1] *P. Balasubramani, P. V. Venkatanarasimhan*: Characterizations of the 0-distributive lattice. Indian J. Pure Appl. Math. *32* (2001), 315–324. zbl MR
[2] *W. H. Cornish*: Annulets and α-ideals in a distributive lattice. J. Aust. Math. Soc. *15* (1973), 70–77. zbl MR
[3] *G. Grätzer*: General Lattice Theory. New appendices by the author with B. A. Davey et al. Birkhäuser, Basel, 1998. zbl MR
[4] *P. A. Grillet, J. C. Varlet*: Complementedness conditions in lattices. Bull. Soc. R. Sci. Liège (electronic only) *36* (1967), 628–642. zbl MR
[5] *R. Halaš*: Characterization of distributive sets by generalized annihilators. Arch. Math., Brno *30* (1994), 25–27. zbl MR
[6] *R. Halaš, J. Rachůnek*: Polars and prime ideals in ordered sets. Discuss. Math., Algebra Stoch. Methods *15* (1995), 43–59. zbl MR
[7] *C. Jayaram*: Prime α-ideals in an 0-distributive lattice. Indian J. Pure Appl. Math. *17* (1986), 331–337. zbl MR
[8] *V. V. Joshi, N. Mundlik*: Prime ideals in 0-distributive posets. Cent. Eur. J. Math. *11* (2013), 940–955. zbl MR
[9] *V. V. Joshi, B. N. Waphare*: Characterizations of 0-distributive posets. Math. Bohem. *130* (2005), 73–80. zbl MR
[10] *V. S. Kharat, K. A. Mokbel*: Semiprime ideals and separation theorems for posets. Order *25* (2008), 195–210. zbl MR
[11] *Y. S. Pawar, S. S. Khopade*: α-ideals and annihilator ideals in 0-distributive lattices. Acta Univ. Palacki. Olomuc., Fac. Rerum Nat., Math. *49* (2010), 63–74. zbl MR
[12] *Y. S. Pawar, D. N. Mane*: α-ideals in 0-distributive semilattices and 0-distributive lattices. Indian J. Pure Appl. Math. *24* (1993), 435–443. zbl MR

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