A NEVANLINNA THEOREM FOR SUPERHARMONIC FUNCTIONS ON DIRICHLET REGULAR GREENIAN SETS

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Abstract. A generalization of Nevanlinna's First Fundamental Theorem to superharmonic functions on Green balls is proved. This enables us to generalize many other theorems, on the behaviour of mean values of superharmonic functions over Green spheres, on the Hausdorff measures of certain sets, on the Riesz measures of superharmonic functions, and on differences of positive superharmonic functions.

Keywords: Nevanlinna theorem, superharmonic function, δ-subharmonic function, Riesz measure, mean value

MSC 2000: 31B05, 31B10

1. Introduction

Nevanlinna's First Fundamental Theorem is concerned with superharmonic functions on balls, and has applications to superharmonic functions on $\mathbb{R}^n$ and δ-subharmonic functions on balls [5], [6]. Here we prove a generalization to superharmonic functions on Green balls, which are sets of the form

$$B_D(x_0,r) = \{y \in D\colon\ G_D(x_0,y) > \tau(r)\},$$

where D is a Dirichlet regular Greenian open set, G_D is its Green function, $\tau(r) = -\log r$ if $n=2, \tau(r) = r^{2-n}$ if $n \geqslant 3$, and $0<r<1$ if $n=2, 0<r<\infty$ if $n \geqslant 3$. Any Green ball $B_D(x_0,r)$ is a bounded domain with its closure in D [11], and is Dirichlet regular. Our result involves the mean values of superharmonic functions over Green spheres, introduced in [11]. Easy corollaries generalize results of Armitage [2], Kuran [7], and Parker [8].

The theorem leads to generalizations of several other results, including some on the behaviour of quotients of differences of mean values of δ-subharmonic functions in [14], on the size of the sets where certain singularities occur in [13], on conditions for a positive measure to be the Riesz measure for a superharmonic function with a harmonic minorant in [6] p. 128, and on conditions for a δ-subharmonic function to be expressible as a difference of two positive superharmonic functions in [5] p. 510. Our results are analogues of theorems on supertemperatures given in [15].

We note that Armitage [1] has given a Nevanlinna theorem for superharmonic functions on half-spaces, but his approach is not related to ours.

For all $x, y \in \mathbb{R}^n$, we put $G(x,y) = \tau(\|x-y\|)$ and

$$B(x,r) = \{y\colon\ G(x,y) > \tau(r)\} = \{y\colon\ \|x-y\| < r\}$$

for all $r > 0$. We also put $p_n = \max\{1, n-2\}$, and note that $\tau'(r) = -p_n r^{1-n}$ for all $n \geqslant 2$.

For almost every r such that $\tau(r) > 0$, the set $\{y \in D\colon\ G_D(x_0,y) = \tau(r)\}$ is a smooth regular $(n-1)$-dimensional manifold. Such a value of r is called *regular*. If r is a regular value, then the set is the Green sphere $\partial B_D(x_0,r)$, and the surface mean value L_D of a function u is defined by

$$L_D(u,x_0,r) = \frac{1}{p_n\sigma_n}\int_{\partial B_D(x_0,r)} \|\nabla G_D(x_0,\cdot)\| u \,\mathrm{d}\sigma$$

whenever the integral exists. Here σ_n denotes the surface area of the unit ball in $\mathbb{R}^n$, and σ denotes surface area measure. If G_D is replaced by G, then the formula for $L_D(u,x_0,r)$ reduces to the standard formula for the mean value of u over the sphere $\partial B(x_0,r)$, which we denote by $L(u,x_0,r)$.

2. The generalization of Nevanlinna's first fundamental theorem

In this section we present our generalization of formula (3.9.6) in [6]. We also present some immediate consequences, which generalize, and to some extent unify, results of Armitage [2], Kuran [7] and Parker [8].

Theorem 1. *Let E be an open set, let D be a Dirichlet regular Greenian open superset of E, let $x_0 \in E$, and let r and s be regular values such that $0 < r < s$ and $\overline{B}_D(x_0,s) \subseteq E$. If u is superharmonic on E with Riesz measure μ, then*

$$L_D(u,x_0,r) = L_D(u,x_0,s) + p_n\int_r^s t^{1-n}\mu(\overline{B}_D(x_0,t))\,\mathrm{d}t \tag{1}$$

and

$$u(x_0) = L_D(u,x_0,s) + p_n\int_0^s t^{1-n}\mu(\overline{B}_D(x_0,t))\,\mathrm{d}t. \tag{2}$$

P r o o f. Let V be a bounded open set such that $\overline{B}_D(x_0,s) \subseteq V$ and $\overline{V} \subseteq E$. Then there is a harmonic function h such that

$$u = G_D\mu_V + h$$

on V. Recall that, by [11] Theorem 1, the means L_D are finite-valued and $L_D(h,x_0,r) = h(x_0)$. It follows that

$$\begin{aligned} L_D(u,x_0,r) - L_D(u,x_0,s) &= L_D(G_D\mu_V,x_0,r) - L_D(G_D\mu_V,x_0,s) \\ &= \int_V (L_D(G_D(\cdot,y),x_0,r) - L_D(G_D(\cdot,y),x_0,s))\,\mathrm{d}\mu(y) \\ &= \int_V ((G_D(x_0,y)\wedge\tau(r)) - (G_D(x_0,y)\wedge\tau(s)))\,\mathrm{d}\mu(y) \end{aligned}$$

by [12] Theorem 2. By definition of $B_D(x_0,r)$, we have $G_D(x_0,y)\wedge\tau(r) = \tau(r)$ if and only if $y \in \overline{B}_D(x_0,r)$. Therefore

$$\begin{aligned} &(G_D(x_0,y)\wedge\tau(r)) - (G_D(x_0,y)\wedge\tau(s)) \\ &\qquad = \begin{cases} \tau(r)-\tau(s) & \text{if } y \in \overline{B}_D(x_0,r), \\ G_D(x_0,y)-\tau(s) & \text{if } y \in \overline{B}_D(x_0,s)\setminus\overline{B}_D(x_0,r), \\ 0 & \text{if } y \notin \overline{B}_D(x_0,s). \end{cases} \end{aligned}$$

Hence

$$L_D(u,x_0,r) - L_D(u,x_0,s) = \int_{\overline{B}_D(x_0,s)} ((G_D(x_0,y)\wedge\tau(r)) - \tau(s))\,\mathrm{d}\mu(y).$$

If we now put $\lambda(t) = \mu(\overline{B}_D(x_0,t))$ whenever $0 \leqslant t \leqslant s$, we obtain

$$\begin{aligned} L_D(u,x_0,r) - L_D(u,x_0,s) &= \int_{[0,s]} ((\tau(t)\wedge\tau(r)) - \tau(s))\,\mathrm{d}\lambda(t) \\ &= (\tau(r)-\tau(s))\lambda(0) + ((\tau(t)\wedge\tau(r)) - \tau(s))\lambda(t)|_{0+}^{s} \\ &\quad - \int_r^s \tau'(t)\lambda(t)\,\mathrm{d}t \\ &= p_n \int_r^s t^{1-n}\mu(\overline{B}_D(x_0,t))\,\mathrm{d}t. \end{aligned}$$

This proves (1). Making $r \to 0$ in (1), we obtain (2). □

R e m a r k s. The formula (2) is a direct extension of Nevanlinna's first fundamental theorem ([6] p. 127). If $n = 2$, consider the case $E = D = B(0,r_0)$ and $x_0 = 0$. Then, whenever $0 < r < 1$, we have

$$B_D(x_0,r) = \{x \in B(0,r_0)\colon\ G_{B(0,r_0)}(0,x) > \tau(r)\} = B(0,rr_0)$$

and

$$L_D(u,x_0,r) = \kappa_2^{-1}\int_{\partial B_D(0,r)} \|\nabla G_{B(0,r_0)}(0,\cdot)\| u \,\mathrm{d}\sigma$$
$$= \kappa_2^{-1}\int_{\partial B(0,rr_0)} \frac{1}{\|x\|} u \,\mathrm{d}\sigma = L(u,0,rr_0),$$

so that (2) becomes

$$u(0) = L(u,0,sr_0)+p_n\int_0^s t^{-1}\mu(\overline{B}(0,tr_0))\,\mathrm{d}t = L(u,0,sr_0)+p_n\int_0^{sr_0} t^{-1}\mu(\overline{B}(0,t))\,\mathrm{d}t$$

for $0 < s < 1$. On the other hand, if $n \geqslant 3$ we can take $E = B(0,r_0)$, $x_0 = 0$, and $D = \mathbb{R}^n$. Then, whenever $0 < r < \infty$, we have $B_D(x_0,r) = B(0,r)$ and

$$L_D(u,x_0,r) = \kappa_n^{-1}\int_{\partial B(0,r)} (2-n)\|x\|^{1-n} u \,\mathrm{d}\sigma = L(u,0,r),$$

so that we can obtain the classical formula from (2) by removing the subscripts D.

Similarly, formula (1) extends a variant of the classical result given, for example, in [2] Lemma 3.

If we put

$$N_D(x_0,s) = p_n\int_0^s t^{1-n}\mu(\overline{B}_D(x_0,t))\,\mathrm{d}t,$$

then $N_D(x_0,\cdot)$ is obviously increasing, and a standard argument ([6] p. 127) shows that there is a convex function φ such that $N_D(x_0,\cdot) = \varphi\circ\tau$.

We now give three corollaries of Theorem 1, all of which are extensions of known results. Theorem 3 (iv), (v) of [11] imply that the surface means can be replaced by the corresponding volume means in the first two corollaries.

Corollary 1. *Let E be an open set, let D be a Dirichlet regular Greenian open superset of E, and let $x_0 \in E$. If u is superharmonic on E with Riesz measure μ, then as $r \to 0$ through regular values*

$$\lim \frac{L_D(u,x_0,r)}{\tau(r)} = \mu(\{x_0\}). \tag{3}$$

Proof. Given $\varepsilon > 0$, choose $\delta > 0$ such that $|\mu(\overline{B}_D(x_0,t)) - \mu(\{x_0\})| < \varepsilon$ for all $t < \delta$. Fix a regular value of $s < \delta$. Then, whenever r is a regular value and $r < s$, we have

$$L_D(u,x_0,r) = L_D(u,x_0,s) + p_n\int_r^s t^{1-n}\mu(\overline{B}_D(x_0,t))\,\mathrm{d}t \tag{4}$$

by (1). Since $s < \delta$, as $r \to 0$ we have

$$\frac{p_n}{\tau(r)} \int_r^s t^{1-n} \mu(\overline{B}_D(x_0, t))\, \mathrm{d}t < \frac{(\mu(\{x_0\}) + \varepsilon)(\tau(r) - \tau(s))}{\tau(r)} \to \mu(\{x_0\}) + \varepsilon,$$

so that

$$\limsup \frac{p_n}{\tau(r)} \int_r^s t^{1-n} \mu(\overline{B}_D(x_0, t))\, \mathrm{d}t \leqslant \mu(\{x_0\}) + \varepsilon.$$

Similarly

$$\liminf \frac{p_n}{\tau(r)} \int_r^s t^{1-n} \mu(\overline{B}_D(x_0, t))\, \mathrm{d}t \geqslant \mu(\{x_0\}) - \varepsilon,$$

so that the corresponding limit exists and is $\mu(\{x_0\})$. The result (3) now follows from (4). □

The cases $D = \mathbb{R}^n$ with $n \geqslant 3$, and D a ball centred at x_0 with $n = 2$, of Corollary 1 were proved by Parker ([8] Lemma). Earlier, Armitage ([2] Lemma 3 Corollary 1) had proved (for the same cases) that if either side of (3) is zero then so is the other, and Kuran ([7] Theorem 2) had proved that if $u(x_0) = 0$ then $\mu(\{x_0\}) = 0$.

Corollary 2. *Let D be a Dirichlet regular Greenian open set, let $x_0 \in D$, and let u be superharmonic with Riesz measure μ on D. If $u \geqslant 0$, then*

$$\tau(r)\mu(\overline{B}_D(x_0, r)) \leqslant L_D(u, x_0, r) \tag{5}$$

for all regular values of r.

P r o o f. Let r and s be regular values such that $r < s$. Put $R = 1$ if $n = 2$, and $R = \infty$ if $n \geqslant 3$. Then $0 < r < s < R$, so that by (1)

$$\begin{aligned} L_D(u, x_0, r) &\geqslant p_n \int_r^s t^{1-n} \mu(\overline{B}_D(x_0, t))\, \mathrm{d}t \\ &\geqslant \mu(\overline{B}_D(x_0, r)) p_n \int_r^R t^{1-n}\, \mathrm{d}t = \mu(\overline{B}_D(x_0, r))\tau(r). \end{aligned}$$

□

R e m a r k s. The special case of Corollary 2 in which $D = \mathbb{R}^n$ and $n \geqslant 3$, was proved earlier by Kuran ([7] Theorem 4) and Armitage ([2] Lemma 3 Corollary 2). The proof given above follows that of Armitage.

The case where $D = B(0, \varrho_0)$ and $n = 2$ of Corollary 2, implies the second inequality of [7] Theorem 4. For then, taking $x_0 = 0$, we have $B_D(x_0, r) = B(0, r\varrho_0)$, and $L_D(u, x_0, r) = L(u, 0, r\varrho_0)$, so that (5) becomes

$$\Big(\log \frac{1}{r}\Big)\mu(\overline{B}(0, r\varrho_0)) \leqslant L(u, 0, r\varrho_0)$$

whenever $0 < r < 1$. If we now put $\varrho = r\varrho_0$, and confine r to $]0, \theta[$ for some $\theta < 1$, we get

$$\mu(\overline{B}(0, \varrho)) \leqslant \Big(\log \frac{1}{r}\Big)^{-1} L(u, 0, \varrho) \leqslant \Big(\log \frac{1}{\theta}\Big)^{-1} L(u, 0, \varrho),$$

which is (13) of [7].

Corollary 3. *Let D be a Dirichlet regular Greenian open set, and let u be superharmonic with Riesz measure μ on D. Put $R = 1$ if $n = 2$, and $R = \infty$ if $n \geqslant 3$. If $u \geqslant 0$, then*

$$\lim_{r \to R} \tau(r)\mu(\overline{B}_D(x, r)) = 0$$

for all $x \in D$.

Proof. The greatest harmonic minorant h of u is given by

$$h(x) = \lim_{r \to R} L_D(u, x, r)$$

for all $x \in D$, in view of [11] Theorem 1 and [3] p. 123, (11.1). Since μ is the Riesz measure for $u - h$, it therefore follows from Corollary 2 above and [11] Theorem 1 that

$$\tau(r)\mu(\overline{B}_D(x, r)) \leqslant L_D(u - h, x, r) = L_D(u, x, r) - h(x) \to 0$$

as $r \to R$ through regular values. Therefore, given $\varepsilon > 0$ we can find K such that $\mu(\overline{B}_D(x, r)) \leqslant \varepsilon/\tau(r)$ for all regular values of $r > K$, and hence for every $r > K$ because $1/\tau$ is continuous and $\mu(\overline{B}_D(x, \cdot))$ is an increasing function. □

The case where $D = \mathbb{R}^n$ and $n \geqslant 3$ of Corollary 3 was first proved by Kuran [7] Theorem 1.

3. The behaviour of the means for small regular values

Theorem 2 below generalizes part of [14] Theorem 2, which dealt only with the classical spherical means.

We need some notation. Let E be an open set, and let D be a Dirichlet regular Greenian open superset of E. If $\overline{B}_D(x_0,s) \subseteq E, \nu$ is a positive measure on E, and $0 \leqslant r < s$, we put

$$I_{\nu,D}(x_0;r,s) = p_n \int_r^s t^{1-n} \nu(\overline{B}_D(x_0,t))\,\mathrm{d}t.$$

Theorem 2. *Let E be an open set, let D be a Dirichlet regular Greenian open superset of E, let u be δ-subharmonic on E with Riesz measure μ, and let ν be a positive measure on E. Then*

$$\limsup_{0<r<s\to 0} \frac{L_D(u,x,r) - L_D(u,x,s)}{I_{\nu,D}(x;r,s)} \leqslant \limsup_{t\to 0} \frac{\mu(\overline{B}_D(x,t))}{\nu(\overline{B}_D(x,t))}$$

whenever the latter exists. Furthermore, if $u(x)$ is defined and finite, and $I_{\nu,D}(x;0,s) < \infty$ for all sufficiently small values of s, then

$$\limsup_{s\to 0} \frac{u(x) - L_D(u,x,s)}{I_{\nu,D}(x;0,s)} \leqslant \limsup_{t\to 0} \frac{\mu(\overline{B}_D(x,t))}{\nu(\overline{B}_D(x,t))}.$$

Proof. The proof of the first inequality is similar to the proof of the first part of [14] Theorem 2. The proof of the second part is similar again, using (2) instead of (1). □

Theorem 2 can easily be rewritten in a form that generalizes [14] Theorem 6.

Theorem 3. *Let E be an open set, and let D be a Dirichlet regular Greenian open superset of E. Let u be δ-subharmonic with Riesz measure μ, and let v be superharmonic with Riesz measure ν, on E. Then*

$$\limsup_{0<r<s\to 0} \frac{L_D(u,x,r) - L_D(u,x,s)}{L_D(v,x,r) - L_D(v,x,s)} \leqslant \limsup_{t\to 0} \frac{\mu(\overline{B}_D(x,t))}{\nu(\overline{B}_D(x,t))}$$

whenever the latter exists. Furthermore, if $u(x)$ is defined and finite, and $v(x) < \infty$, then

$$\limsup_{s\to 0} \frac{u(x) - L_D(u,x,s)}{v(x) - L_D(v,x,s)} \leqslant \limsup_{t\to 0} \frac{\mu(\overline{B}_D(x,t))}{\nu(\overline{B}_D(x,t))}.$$

Proof. In view of Theorem 1 and the finiteness of the means ([11] Theorem 1), the result follows from Theorem 2. □

We can also generalize [14] Theorem 5, as follows.

Theorem 4. *Let E be an open set, let D be a Dirichlet regular Greenian open superset of E, and let u be δ-subharmonic with Riesz measure μ on E. Let $\alpha > 0$, let f be a positive, increasing, absolutely continuous function on $[0, \alpha]$, and let*

$$\hat{f}(r, s) = p_n \int_r^s t^{1-n} f(t)\,\mathrm{d}t$$

whenever $0 \leqslant r < s \leqslant \alpha$. Then

$$\limsup_{0<r<s\to 0} \frac{L_D(u, x, r) - L_D(u, x, s)}{\hat{f}(r, s)} \leqslant \limsup_{t\to 0} \frac{\mu(\overline{B}_D(x, t))}{f(t)}$$

for all x in E. Furthermore, if $u(x)$ is defined and finite, and $\hat{f}(0, s) < \infty$ for all sufficiently small values of s, then

$$\limsup_{s\to 0} \frac{u(x) - L_D(u, x, s)}{\hat{f}(0, s)} \leqslant \limsup_{t\to 0} \frac{\mu(\overline{B}_D(x, t))}{f(t)}.$$

P r o o f. Given x, we choose $\varrho \leqslant \alpha$ such that $\overline{B}_D(x, \varrho) \subseteq E$, and define a positive measure ν on E by putting

$$\mathrm{d}\nu = -\kappa_n^{-1} \|\nabla G_D(x, \cdot)\|^2 \Big(\frac{f'}{\tau'}\Big)(\tau^{-1}(G_D(x, \cdot)))\chi_{B_D(x,\varrho)}\,\mathrm{d}\lambda + f(0)\,\mathrm{d}\delta_x,$$

where τ^{-1} denotes the inverse function of τ, χ_A denotes the characteristic function of a set A, λ denotes n-dimensional Lebesgue measure, and δ_x denotes the unit mass at x. If $0 < t < \varrho$, it follows from results in [11] pp. 309–310 that

$$\begin{aligned}
\nu(\overline{B}_D(x, t)) &= -\kappa_n^{-1} \int_{B_D(x,t)} \|\nabla G_D(x, \cdot)\|^2 \Big(\frac{f'}{\tau'}\Big)(\tau^{-1}(G_D(x, \cdot)))\,\mathrm{d}\lambda + f(0)\\
&= \kappa_n^{-1} \int_0^t \bigg(\int_{\partial B_D(x,r)} \|\nabla G_D(x, \cdot)\| f'(r)\,\mathrm{d}\sigma \bigg)\,\mathrm{d}r + f(0)\\
&= \int_0^t L_D(1, x, r) f'(r)\,\mathrm{d}r + f(0) = \int_0^t f'(r)\,\mathrm{d}r + f(0) = f(t).
\end{aligned}$$

Therefore, whenever $0 \leqslant r < s \leqslant \varrho$,

$$I_{\nu,D}(x; r, s) = p_n \int_r^s t^{1-n} f(t)\,\mathrm{d}t = \hat{f}(r, s).$$

The results now follow from Theorem 2. □

The corollaries of [14] Theorem 5 can now easily be generalized. We leave this to the reader.

4. The Hausdorff measure of certain sets

We use Theorem 4 to study the size of the set of points x where

$$\limsup_{0<r<s\to 0} \frac{L_D(u,x,r) - L_D(u,x,s)}{\hat{f}(r,s)}$$

is unbounded, or is positive, for a given function f and superharmonic function u. The size is estimated in terms of Hausdorff measures [9]. Our results generalize theorems of Armitage [2] and Watson [13] in two directions, namely the mean values considered and the Hausdorff measures used.

Theorem 5. *Let $n \geqslant 3$, let E be an open set, let D be a Dirichlet regular Greenian open superset of E, and let u be superharmonic on E. Let h be an increasing, absolutely continuous function on $[0,\infty[$ such that $h(0)=0$ and $h(2s) \leqslant Kh(s)$ for all $s>0$, where K is a constant. Put*

$$\hat{h}(r,s) = p_n \int_r^s t^{1-n} h(t)\,\mathrm{d}t.$$

Then the set

$$\left\{x \in E\colon \limsup_{0<r<s\to 0} \frac{L_D(u,x,r) - L_D(u,x,s)}{\hat{h}(r,s)} = \infty\right\} \tag{6}$$

has h-measure zero, and

$$\left\{x \in E\colon \limsup_{0<r<s\to 0} \frac{L_D(u,x,r) - L_D(u,x,s)}{\hat{h}(r,s)} > 0\right\} \tag{7}$$

is σ-finite with respect to h-measure.

P r o o f. Let μ denote the Riesz measure for u. It suffices to prove the results locally, and we may therefore suppose that E is bounded and μ is finite. By Theorem 4, the set (6) is a subset of

$$\left\{x \in E\colon \limsup_{t\to 0} \frac{\mu(\overline{B}_D(x,t))}{h(t)} = \infty\right\}, \tag{8}$$

and the set (7) is contained in

$$\left\{x \in E\colon \limsup_{t\to 0} \frac{\mu(\overline{B}_D(x,t))}{h(t)} > 0\right\}. \tag{9}$$

Since $n \geqslant 3$, $G_D(x,y) \leqslant \|x-y\|^{2-n}$ for all $x, y \in D$, so that $B_D(x,t) \subseteq B(x,t)$ for all $t > 0$. Therefore $\overline{B}_D(x,t)$ is contained in a closed interval $I(x,s)$ of centre x and edge length $s = 2r$, which is contained in E if r is sufficiently small. It follows that the set (8) is a subset of

$$\left\{x \in E\colon \limsup_{s\to 0} \frac{\mu(I(x,s))}{h(s/2)} = \infty\right\}, \tag{10}$$

and that the set (9) is a subset of

$$\left\{x \in E\colon \limsup_{s\to 0} \frac{\mu(I(x,s))}{h(s/2)} > 0\right\}. \tag{11}$$

If i is chosen so that $2^{i-1} > \sqrt{n}$, then

$$h(s/2) \geqslant K^{-i} h(2^{i-1}s) \geqslant K^{-i} h(s\sqrt{n}) = K^{-i} h(\operatorname{diam} I(x,s)).$$

Therefore the sets (10) and (11) are contained in the sets

$$S = \left\{x \in E\colon \limsup_{s\to 0} \frac{\mu(I(x,s))}{h(s\sqrt{n})} = \infty\right\}$$

and

$$T = \left\{x \in E\colon \limsup_{s\to 0} \frac{\mu(I(x,s))}{h(s\sqrt{n})} > 0\right\}$$

respectively. By a result of Rogers and Taylor ([10], Lemma 2), for each $k > 0$ the set

$$S_k = \left\{x \in E\colon \limsup_{s\to 0} \frac{\mu(I(x,s))}{h(s\sqrt{n})} > k\right\}$$

has h-measure $h - m(S_k) \leqslant M\mu(E)/k$ for some constant M. It follows that $h - m(S) = 0$, and that T is σ-finite with respect to h-measure. This implies the results of the theorem. □

Corollary 1. *Let $n \geqslant 3$, and let E, D, h and $\hat{h}$ be as in Theorem 5. If u is δ-subharmonic on E, and $\hat{h}(0,\alpha) = \infty$ for some α, then the set*

$$\left\{x \in E\colon \limsup_{s\to 0} \frac{L_D(u,x,s)}{\hat{h}(s,\alpha)} = \infty\right\}$$

has h-measure zero, and

$$\left\{x \in E\colon \limsup_{s\to 0} \frac{L_D(u,x,s)}{\hat{h}(s,\alpha)} > 0\right\}$$

is σ-finite with respect to h-measure.

P r o o f. Let μ denote the Riesz measure of u. It is enough to prove the result locally, and so we may suppose that E is bounded (and hence Greenian) and that μ has finite total variation. Since $G_E\mu \leqslant G_E|\mu|$, we may also suppose that μ is positive (so that u is superharmonic). Using the method of proof of [14] Theorem 5 Corollary 1, we can now show that

$$\limsup_{s\to 0} \frac{L_D(u,x,s)}{\hat{h}(s,\alpha)} \leqslant \limsup_{0<r<s\to 0} \frac{L_D(u,x,r) - L_D(u,x,s),}{\hat{h}(r,s)}$$

and so the result follows from Theorem 5. □

In the next result, we denote by m_β the h-measure constructed from the function $h(s) = s^\beta$, where $\beta > 0$.

Corollary 2. *Let $n \geqslant 3$, and let E, D and u be as in Corollary 1. Then the set S_β, defined by*

$$S_\beta = \Big\{x \in E\colon \limsup_{s\to 0} s^{n-\beta-2} L_D(u,x,s) = \infty\Big\}$$

if $0 < \beta < n-2$, and by

$$S_\beta = \Big\{x \in E\colon \limsup_{s\to 0} \Big(\log\frac{1}{s}\Big)^{-1} L_D(u,x,s) = \infty\Big\}$$

if $\beta = n-2$, has m_β-measure zero. Furthermore, the set T_β given by

$$T_\beta = \Big\{x \in E\colon \limsup_{s\to 0} s^{n-\beta-2} L_D(u,x,s) > 0\Big\}$$

if $0 < \beta < n-2$, and by

$$T_\beta = \Big\{x \in E\colon \limsup_{s\to 0} \Big(\log\frac{1}{s}\Big)^{-1} L_D(u,x,s) > 0\Big\}$$

if $\beta = n-2$, is σ-finite with respect to m_β.

P r o o f. If we take $h(s) = s^\beta$, $0 < \beta \leqslant n-2$, in Corollary 1, then $\hat{h}(0,\alpha) = \infty$ so that the corollary is applicable. Furthermore,

$$\hat{h}(s,\alpha) = \begin{cases} \dfrac{p_n}{n-\beta-2}(s^{\beta+2-n} - \alpha^{\beta+2-n}) & \text{if } 0<\beta<n-2, \\ p_n \log(\alpha/s) & \text{if } \beta = n-2. \end{cases}$$

The result follows. □

The case of Corollary 2 in which $D = \mathbb{R}^n$, u is superharmonic, and $0 < \beta < n-2$, was proved by Armitage in [2] Theorem 3. The case $D = \mathbb{R}^n$ was subsequently proved by Watson in [13] Theorem 16 (in the statement of which $|u|$ should be replaced by u).

In Theorem 5 Corollary 1, the condition on $\hat{h}$ ensures that $L_D(u,x,s) \to \infty$ as $s \to 0$, for every x in either of the sets in question. Therefore the sets are polar. In the theorem itself, polarity is not so readily determined, and in fact depends on h. We demonstrate this in the context of the next corollary.

Corollary 3. *Let $n \geqslant 3$, and let E, D and u be as in Theorem 5.*

(i) *If $0 < \beta < n-2$, then the set*

$$\left\{x \in E\colon \limsup_{0<r<s\to 0} \frac{L_D(u,x,r) - L_D(u,x,s)}{r^{-(n-2-\beta)} - s^{-(n-2-\beta)}} = \infty\right\}$$

has m_β-measure zero, and

$$\left\{x \in E\colon \limsup_{0<r<s\to 0} \frac{L_D(u,x,r) - L_D(u,x,s)}{r^{-(n-2-\beta)} - s^{-(n-2-\beta)}} > 0\right\}$$

is σ-finite with respect to m_β.

(ii) *The set*

$$\left\{x \in E\colon \limsup_{0<r<s\to 0} \frac{L_D(u,x,r) - L_D(u,x,s)}{\log(s/r)} = \infty\right\}$$

has m_{n-2}-measure zero, and

$$\left\{x \in E\colon \limsup_{0<r<s\to 0} \frac{L_D(u,x,r) - L_D(u,x,s)}{\log(s/r)} > 0\right\}$$

is σ-finite with respect to m_{n-2}.

(iii) *If $n-2 < \beta \leqslant n$, then the set*

$$\left\{x \in E\colon \limsup_{0<r<s\to 0} \frac{L_D(u,x,r) - L_D(u,x,s)}{s^{\beta+2-n} - r^{\beta+2-n}} = \infty\right\}$$

has m_β-measure zero, and

$$\left\{x \in E\colon \limsup_{0<r<s\to 0} \frac{L_D(u,x,r) - L_D(u,x,s)}{s^{\beta+2-n} - r^{\beta+2-n}} > 0\right\}$$

is σ-finite with respect to m_β.

Proof. Take $h(s) = s^\beta, 0 < \beta \leqslant n$, in Theorem 5. □

Sets of finite m_{n-2}-measure are polar ([4] p. 78, or [6] p. 228). Therefore the sets in Corollary 3 (i) and (ii) are all polar. The sets in (iii), however, need not be. Given β such that $n-2 < \beta \leqslant n$, choose γ such that $n-2 < \gamma < \beta$. If S is an m_γ-measurable set for which $0 < m_\gamma(S) < \infty$, and μ is the restriction to S of m_γ, then by [4] p. 25 we have

$$\limsup_{t \to 0} \frac{\mu(B(x,t))}{t^\gamma} \geqslant 1$$

μ-a.e. on S. Therefore

$$\limsup_{t \to 0} \frac{\mu(B(x,t))}{t^\beta} = \infty$$

for all $x \in S_0$, say, where $\mu(S \setminus S_0) = 0$. It follows from Theorem 4 that

$$\limsup_{0<r<s \to 0} \frac{L(u,x,r) - L(u,x,s)}{s^{\beta+2-n} - r^{\beta+2-n}} = \infty$$

for all $x \in S_0$. Since $m_\gamma(S_0) > 0$ and $\gamma > n-2$, the set S_0 is not polar ([4] p. 78, or [6] p. 225), and so the sets in Corollary 3 (iii) are not polar.

5. The Riesz measures of superharmonic functions on Dirichlet regular Greenian sets

In this section we generalize [6] Theorem 3.20 from the case where $D = \mathbb{R}^n$ for some $n \geqslant 3$, to that where D is an arbitrary Dirichlet regular Greenian domain in $\mathbb{R}^n$ for any $n \geqslant 2$.

Theorem 6. *Let D be a Dirichlet regular Greenian domain, and let μ be a positive measure on D. Put $R = 1$ if $n = 2$, and $R = \infty$ if $n \geqslant 3$.*

(i) *If μ is the Riesz measure of a superharmonic function that has a harmonic minorant on D, then*

$$\int_{\frac{1}{2}}^{R} t^{1-n} \mu(\overline{B}_D(x,t))\, \mathrm{d}t < \infty \tag{12}$$

for all $x \in D$.

(ii) *Conversely, if there is a point $x \in D$ such that* (12) *holds, then $G_D\mu$ is superharmonic on D. If, in addition, $\mu(\{x\}) = 0$ and*

$$\int_0^{\frac{1}{2}} t^{1-n} \mu(\overline{B}_D(x,t))\, \mathrm{d}t < \infty,$$

then $G_D\mu(x) < \infty$.

P r o o f. (i) Let w be a superharmonic function which has a harmonic minorant u on D, and whose Riesz measure is μ. Then μ is also the Riesz measure for $w-u$. Therefore, if $x \in D$ and r, s are regular values such that $r < s$, Theorem 1 shows that

$$\begin{aligned} L_D(w-u,x,r) &= L_D(w-u,x,s) + p_n \int_r^s t^{1-n}\mu(\overline{B}_D(x,t))\,\mathrm{d}t \\ &\geqslant p_n \int_r^s t^{1-n}\mu(\overline{B}_D(x,t))\,\mathrm{d}t. \end{aligned}$$

Since $L_D(w-u,x,r) < \infty$ by [11] Theorem 1, if we fix r and make $s \to R$ we obtain (12).

(ii) Now suppose that (12) holds for some $x = x_0 \in D$. Let $\{k_j\}$ be an increasing sequence of regular values such that $k_j \to R$ as $j \to \infty$, and put

$$\begin{aligned} &A_D(x_0;k_1,R) = D \setminus \overline{B}_D(x_0,k_1), \\ &A_D(x_0;k_1,k_j) = B_D(x_0,k_j) \setminus \overline{B}_D(x_0,k_1) \qquad \text{for all } j > 1. \end{aligned}$$

If $u = G_D\mu$, then for all $x \in D$ we put

$$u(x) = \int_{\overline{B}_D(x_0,k_1)} G_D(x,y)\,\mathrm{d}\mu(y) + \int_{A_D(x_0;k_1,R)} G_D(x,y)\,\mathrm{d}\mu(y) = v_1(x) + v_2(x),$$

say, and

$$u_j(x) = \int_{A_D(x_0;k_1,k_j)} G_D(x,y)\,\mathrm{d}\mu(y)$$

for all $j \geqslant 1$. Since μ is locally finite, v_1 and every u_j is superharmonic on D. Since $\{u_j\}$ is increasing to the limit v_2, if $v_2(x_0) < \infty$ then v_2 will be superharmonic on D. Writing $\lambda(t) = \mu(\overline{B}_D(x_0,t))$ for all $t > 0$, we have

$$u_j(x_0) = \int_{k_1}^{k_j} \tau(t)\,\mathrm{d}\lambda(t) = [\tau(t)\lambda(t)]_{k_1}^{k_j} - \int_{k_1}^{k_j} \tau'(t)\lambda(t)\,\mathrm{d}t.$$

Since (12) holds when $x = x_0$, we have

$$\lambda(s)\tau(s) = \lambda(s)\int_s^R -\tau'(t)\,\mathrm{d}t \leqslant -\int_s^R \tau'(t)\lambda(t)\,\mathrm{d}t \to 0$$

as $s \to R$. Therefore

$$v_2(x_0) = \lim_{j\to\infty} u_j(x_0) = -\tau(k_1)\lambda(k_1) - \int_{k_1}^R \tau'(t)\lambda(t)\,\mathrm{d}t < \infty,$$

so that v_2, and hence u, is superharmonic on D.

For the last part, let $\{r_j\}$ be a decreasing null sequence of regular values (relative to x_0). Then, if $\mu(\{x_0\}) = 0$,

$$u(x_0) = \lim_{j\to\infty} \int_{r_j}^{R} \tau(t)\,\mathrm{d}\lambda(t) = \lim_{j\to\infty} \left(-\tau(r_j)\lambda(r_j) - \int_{r_j}^{R} \tau'(t)\lambda(t)\,\mathrm{d}t \right)$$
$$\leqslant - \int_{0}^{R} \tau'(t)\lambda(t)\,\mathrm{d}t < \infty.$$

□

6. Differences of positive superharmonic functions

Let D be a Dirichlet regular, Greenian open set, and let u be δ-subharmonic on D. If μ is the Riesz measure for u, then μ can be written minimally as a difference $\mu^+ - \mu^-$ of two positive measures on D. For all $r \in]0, R[$ (where $R = 1$ if $n = 2, R = \infty$ if $n \geqslant 3$), we put

$$\lambda_D^+(x, r) = \mu^+(\overline{B}_D(x, r)), \qquad N_D^+(x, r) = p_n \int_0^r t^{1-n} \lambda_D^+(x, t)\,\mathrm{d}t,$$

and similarly for μ^-. We say that $u(x_0)$ is *finite* if $N_D^+(x_0, \cdot)$ and $N_D^-(x_0, \cdot)$ are both finite-valued, in which case it follows from (2) that u is the difference of two superharmonic functions which are finite at x_0. If $u(x_0)$ is finite, we define the *characteristic* T_D of u at x_0 by

$$T_D(u, x_0, r) = L_D(u^+, x_0, r) + N_D^+(x_0, r) - u(x_0)$$

for each regular value of r.

We use T_D to characterize those δ-subharmonic functions on D that can be written as a difference of two positive superharmonic functions, and thus generalize [5] Theorem 7.42, which deals with the case where D is a ball.

Theorem 7. *Let D be a Dirichlet regular Greenian domain, and let u be δ-subharmonic on D.*

(i) *If $u = u_1 - u_2$ is the difference of two positive superharmonic functions on D, and $u(x_0)$ is finite, then $T_D(u, x_0, \cdot)$ is an increasing function such that $0 \leqslant T_D(u, x_0, r) \leqslant u_2(x_0)$ for all regular values of r, and there is a convex function φ such that $T_D(u, x_0, \cdot) = \varphi \circ \tau$.*

(ii) *Conversely, if $u(x_0)$ is finite and $T_D(u, x_0, \cdot)$ is bounded above, then u is the difference of two positive superharmonic functions on D.*

P r o o f. (i) For $i \in \{1,2\}$, let μ_i be the Riesz measure for u_i, and put

$$\lambda_D^i(x_0,r) = \mu_i(\overline{B}_D(x_0,r)), \qquad N_D^i(x_0,r) = p_n \int_0^r t^{1-n} \lambda_D^i(x_0,t)\,\mathrm{d}t$$

for all $r \in\,]0,R[$. Since $u_1 \geqslant 0$, it follows from (2) that

$$0 = L_D(u_1,x_0,r) + N_D^1(x_0,r) - u_1(x_0).$$

Since μ_1 and μ_2 are positive and $\mu = \mu_1 - \mu_2$, we have $\mu^+ \leqslant \mu_1$ and $\mu^- \leqslant \mu_2$, so that

$$N_D^+(x_0,r) \leqslant N_D^1(x_0,r) = u_1(x_0) - L_D(u_1,x_0,r).$$

Furthermore $u_1 \geqslant u^+$, so that $L_D(u^+,x_0,r) \leqslant L_D(u_1,x_0,r)$. Hence

$$T_D(u,x_0,r) \leqslant L_D(u_1,x_0,r) + (u_1(x_0) - L_D(u_1,x_0,r)) - u(x_0) = u_2(x_0),$$

which establishes the upper bound for $T_D(u,x_0,r)$.

Now put $v_2 = G_D\mu^-$ and $v_1 = u + v_2$. Then both v_1 and v_2 are superharmonic, so that we can apply (2) to both of them and subtract. Thus we obtain

$$u(x_0) = L_D(u,x_0,r) + N_D^+(x_0,r) - N_D^-(x_0,r).$$

It follows that

$$\begin{aligned} T_D(u,x_0,r) &= L_D(u^+,x_0,r) + N_D^-(x_0,r) - L_D(u,x_0,r) = L_D(u^-,x_0,r) + N_D^-(x_0,r) \\ &= L_D(u^-,x_0,r) + v_2(x_0) - L_D(v_2,x_0,r) = v_2(x_0) - L_D(v_2 - u^-,x_0,r). \end{aligned}$$

Let $x \in D$. If $u(x) \geqslant 0$, then $v_1(x) \geqslant v_2(x)$ and $v_2(x) - u^-(x) = v_2(x) = (v_1 \wedge v_2)(x)$. On the other hand, if $u(x) \leqslant 0$ then $v_1(x) \leqslant v_2(x)$ and $v_2(x) - u^-(x) = v_1(x) = (v_1 \wedge v_2)(x)$. Hence

$$T_D(u,x_0,r) = v_2(x_0) - L_D(v_1 \wedge v_2,x_0,r).$$

Since $v_1 \wedge v_2$ is superharmonic, the characteristic $T_D(u,x_0,\cdot)$ is increasing on the set of all regular values (by [11] Theorem 1), there is a convex function φ such that $T_D(u,x_0,\cdot) = \varphi \circ \tau$ (by [11] Theorem 2), and $T_D(u,x_0,r) \geqslant v_2(x_0) - (v_1 \wedge v_2)(x_0) \geqslant 0$ (by [11] Theorem 1).

(ii) Let w_1, w_2 be superharmonic functions such that $u = w_1 - w_2$ on D. Applying (2) to each w_j and subtracting, we obtain

(13) $$T_D(u,x_0,r) = L_D(u^-,x_0,r) + N_D^-(x_0,r)$$

for all regular values of r. Therefore $N_D^-(x_0,\cdot) \leqslant T_D(u,x_0,\cdot)$, and so $N_D^-(x_0,\cdot)$ is bounded. Hence

$$\int_0^R t^{1-n}\lambda_D^-(x_0,t)\,\mathrm{d}t < \infty,$$

so that the function $v_2 = G_D\mu^-$ is superharmonic on D, by Theorem 6. Furthermore,

$$N_D^+(x_0,r) = T_D(u,x_0,r) - L_D(u^+,x_0,r) + u(x_0) \leqslant T_D(u,x_0,r) + u(x_0)$$

for all regular values of r, so that $N_D^+(x_0,\cdot)$ is bounded, and hence the function $v_1 = G_D\mu^+$ is superharmonic on D. It follows that the function h, defined q.e. on D by $h = u + v_2 - v_1$, can be extended to a harmonic function h on D. Furthermore, because v_1 and v_2 are positive,

$$L_D(h^-,x_0,\cdot) \leqslant L_D(u^-,x_0,\cdot) + L_D(v_1,x_0,\cdot) = T_D(u,x_0,\cdot) - N_D^-(x_0,\cdot) + L_D(v_1,x_0,\cdot)$$

by (13), so that

$$L_D(h^-,x_0,\cdot) \leqslant T_D(u,x_0,\cdot) + v_1(x_0)$$

by [11] Theorem 1. Therefore $L_D(h^-,x_0,\cdot)$ is bounded, so that h^- has a harmonic majorant v on D, by [11] Theorem 1. Hence $h = (h+v) - v$ is a difference of two positive harmonic functions on D, so that

$$u = h + v_1 - v_2 = (h + v + v_1) - (v + v_2)$$

is a difference of two positive superharmonic functions on D.

Remark. A representation formula for the difference of two positive superharmonic functions on D, follows from the Riesz decomposition theorem and the Martin representation theorem for differences of positive harmonic functions on Greenian domains given in [3] p. 204.

References

[1] *D. H. Armitage*: A Nevanlinna theorem for superharmonic functions in half-spaces, with applications. J. London Math. Soc. *23* (1981), 137–157.
[3] *D. H. Armitage*: Mean values and associated measures of superharmonic functions. Hiroshima Math. J. *13* (1983), 53–63.
[3] *J. L. Doob*: Classical Potential Theory and its Probabilistic Counterpart. Springer, New York, 1984.
[4] *K. J. Falconer*: The Geometry of Fractal Sets. Cambridge University Press, Cambridge, 1985.
[5] *W. K. Hayman*: Subharmonic Functions, Vol. 2. Academic Press, London, 1989.
[6] *W. K. Hayman, P. B. Kennedy*: Subharmonic Functions, Vol. 1. Academic Press, London, 1976.

[7] *Ü. Kuran*: On measures associated to superharmonic functions. Proc. Amer. Math. Soc. *36* (1972), 179–186.
[8] *M. J. Parker*: The fundamental function of the distance. Bull. London Math. Soc. *19* (1987), 337–342.
[9] *C. A. Rogers*: Hausdorff Measures. Cambridge University Press, Cambridge, 1970.
[10] *C. A. Rogers, S. J. Taylor*: Functions continuous and singular with respect to a Hausdorff measure. Mathematika *8* (1961), 1–31.
[11] *N. A. Watson*: Mean values of subharmonic functions over Green spheres. Math. Scand. *69* (1991), 307–319.
[12] *N. A. Watson*: Generalizations of the spherical mean convexity theorem on subharmonic functions. Ann. Acad. Sci. Fenn. Ser. A I Math. *17* (1992), 241–255.
[13] *N. A. Watson*: Superharmonic extensions, mean values and Riesz measures. Potential Anal. *2* (1993), 269–294.
[14] *N. A. Watson*: Mean values and associated measures of δ-subharmonic functions. Math. Bohem. *127* (2002), 83–102.
[15] *N. A. Watson*: A generalized Nevanlinna theorem for supertemperatures. Ann. Acad. Sci. Fenn. Math. *28* (2003), 35–54.

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