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WEAK BOOLEAN PRODUCTS OF BOUNDED DUALLY RESIDUATED l-MONOIDS

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Abstract. In the paper we deal with weak Boolean products of bounded dually residuated l-monoids (DRl-monoids). Since bounded DRl-monoids are a generalization of pseudo MV-algebras and pseudo BL-algebras, the results can be immediately applied to these algebras.

Keywords: bounded DRl-monoid, weak Boolean product, prime spectrum

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Introduction

Commutative dually residuated lattice-ordered monoids (commutative DRl-monoids) were introduced by K. L. N. Swamy in [26] as a common generalization of Abelian lattice-ordered groups and Brouwerian algebras. Dropping the commutativity assumption, T. Kovář in his thesis [13] defined general DRl-monoids which include all lattice-ordered groups. Recently, it was shown in [20], [21], [23], [24] and [15] that also algebras of logics behind fuzzy reasoning and their non-commutative versions, namely, MV-algebras and pseudo MV-algebras, and BL-algebras and pseudo BL-algebras, can be regarded to be particular cases of bounded DRl-monoids.

Boolean and weak Boolean products of MV-algebras, BL-algebras and bounded commutative DRl-monoids were studied in [4], [7] and [22]. In this paper we concentrate on weak Boolean products of bounded (non-commutative) DRl-monoids. We prove that non-trivial bounded DRl-monoids are representable as weak Boolean products of directly indecomposable bounded DRl-monoids, we characterize weak Boolean products of bounded DRl-chains, and show that the prime spectrum of a weak Boolean product of bounded DRl-monoids is built up from the prime spectra of

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the components of this product. Our results can be immediately applied to pseudo MV-algebras and pseudo BL-algebras.

1. Definitions and basic properties

An algebra $(A; \oplus, 0, \vee, \wedge, \oslash, \odot)$ of type $\langle 2,0,2,2,2,2\rangle$ is called a *dually residuated* l-*monoid* or a DRl-*monoid* if

(i) $(A; \oplus, 0, \vee, \wedge)$ is an l-monoid, i.e., $(A; \oplus, 0)$ is a monoid, $(A; \vee, \wedge)$ is a lattice and $\oplus$ distributes over both $\vee$ and $\wedge$,

(ii) for any $a, b \in A$, $a \oslash b$ is the least $x \in A$ with $x \oplus b \geqslant a$, and $a \odot b$ is the least $y \in A$ such that $b \oplus y \geqslant a$,

(iii) A satisfies the identities

$$((x \oslash y) \vee 0) \oplus y \leqslant x \vee y, \quad y \oplus ((x \odot y) \vee 0) \leqslant x \vee y,$$
$$x \oslash x \geqslant 0, \quad x \odot x \geqslant 0.$$

We note that the condition (ii) is equivalent to the identities

$$(x \oslash y) \oplus y \geqslant x, \quad y \oplus (x \odot y) \geqslant x,$$
$$x \oslash y \leqslant (x \vee z) \oslash y, \quad x \odot y \leqslant (x \vee z) \odot y,$$
$$(x \oplus y) \oslash y \leqslant x, \quad (y \oplus x) \odot y \leqslant x,$$

and hence the class of all DRl-monoids is a variety. T. Kovář proved that this variety is arithmetical and weakly regular.

A DRl-monoid A is said to be *bounded* if there exists an element 1 in A such that $a \leqslant 1$ for all $a \in A$. As a matter of fact, if 1 is the greatest element of A then 0 is the least one.

In what follows, the greatest element 1 of a bounded DRl-monoid A will be considered to be a new nullary operation, and thus bounded DRl-monoids are algebras of the language $\{\oplus, 0, \vee, \wedge, \oslash, \odot, 1\}$.

R e m a r k. Of course, our DRl-monoids are termwise equivalent to a certain class of residuated lattices. These residuated lattices are called *generalized* BL-*algebras* (GBL-*algebras*) in [1], [8] and [12].

E x a m p l e 1.1. Pseudo MV-algebras were independently introduced by the second author in [24] and by G. Georgescu and A. Iorgulescu in [9] as a non-commutative extension of the well-known MV-algebras (see e.g. [3]):

A *pseudo* MV-*algebra* is an algebra $(A;\oplus,\neg,\sim,0,1)$ of type $\langle 2,1,1,0,0\rangle$ satisfying the following axioms:

(A1) $(x\oplus y)\oplus z = x\oplus(y\oplus z)$,

(A2) $x\oplus 0 = 0\oplus x = x$,

(A3) $x\oplus 1 = 1\oplus x = 1$,

(A4) $\neg 1 = \sim 1 = 0$,

(A5) $\neg(\sim x\oplus\sim y) = \sim(\neg x\oplus\neg y)$,

(A6) $x\oplus(y\odot\sim x) = y\oplus(x\odot\sim y) = (\neg x\odot y)\oplus x = (\neg y\odot x)\oplus y$,

(A7) $(\neg x\oplus y)\odot x = y\odot(x\oplus\sim y)$,

(A8) $\sim\neg x = x$,

where the additional operation $\odot$ is defined via

$$x\odot y = \sim(\neg x\oplus\neg y).$$

Obviously, if $\oplus$ is commutative then $\neg$ and $\sim$ coincide and $(A;\oplus,\neg,0,1)$ is an MV-algebra.

Mutual relationships between pseudo MV-algebras and DRl-monoids were described in [24]. If we put $x\leqslant y$ iff $\neg x\oplus y = 1$, then $(A;\leqslant)$ is a bounded distributive lattice (with 0 at the bottom and 1 at the top) in which $x\vee y = x\oplus(y\odot\sim x)$ and $x\wedge y = (\neg x\oplus y)\odot x$ for all $x,y\in A$. Moreover, by defining $x\oslash y = \neg y\odot x$ and $x\otimes y = x\odot\sim y$, the structure $(A;\oplus,0,\vee,\wedge,\oslash,\otimes,1)$ becomes a bounded DRl-monoid satisfying the identities

(i) $1\oslash(1\otimes x) = x = 1\otimes(1\oslash x)$,

(ii) $1\oslash((1\otimes x)\oplus(1\otimes y)) = 1\otimes((1\oslash x)\oplus(1\oslash y))$.

Conversely, if $(A;\oplus,0,\vee,\wedge,\oslash,\otimes,1)$ is a bounded DRl-monoid that fulfils these equations and if we put $\neg x = 1\oslash x$ and $\sim x = 1\otimes x$, then $(A;\oplus,\neg,\sim,0,1)$ is a pseudo MV-algebra.

Example 1.2. Pseudo BL-algebras established in [5] are another special case of bounded DRl-monoids:

An algebra $(A;\vee,\wedge,\odot,\to,\rightsquigarrow,0,1)$ of type $\langle 2,2,2,2,2,0,0\rangle$ is called a *pseudo* BL-*algebra* if $(A;\vee,\wedge,0,1)$ is a bounded lattice, $(A;\odot,1)$ is a monoid and the following conditions hold for all $x,y,z\in A$:

(i) $x\odot y\leqslant z$ iff $x\leqslant y\to z$ iff $y\leqslant x\rightsquigarrow z$,

(ii) $x\wedge y = (x\to y)\odot x = x\odot(x\rightsquigarrow y)$,

(iii) $(x\to y)\vee(y\to x) = (x\rightsquigarrow y)\vee(y\rightsquigarrow x) = 1$.

Pseudo BL-algebras generalize BL-algebras (see e.g. [10]) in the same way in which pseudo MV-algebras generalize MV-algebras: if $\odot$ is commutative then $\to$ and $\rightsquigarrow$

coincide and the algebra $(A; \vee, \wedge, \odot, \to, 0, 1)$ is a BL-algebra. Moreover, pseudo BL-algebras include pseudo MV-algebras: by [5], pseudo BL-algebras satisfying $(x \to 0) \rightsquigarrow 0 = (x \rightsquigarrow 0) \to 0 = x$ are polynomially equivalent to pseudo MV-algebras.

It was proved by the first author in [15] that pseudo BL-algebras correspond one-to-one to bounded DRl-monoids satisfying the identities

$$(*) \qquad \begin{aligned} (x \oslash y) \wedge (y \oslash x) &= 0, \\ (x \obslash y) \wedge (y \obslash x) &= 0; \end{aligned}$$

they are the duals of such DRl-monoids. Let $(A; \vee, \wedge, \odot, \to, \rightsquigarrow, 0, 1)$ be a pseudo BL-algebra and define $x \oplus y = x \odot y$, $x \vee' y = x \wedge y$, $x \wedge' y = x \vee y$, $x \oslash y = y \to x$, $x \obslash y = y \rightsquigarrow x$, $0' = 1$ and $1' = 0$. Then $(A; \oplus, 0', \vee', \wedge', \oslash, \obslash, 1')$ is a bounded DRl-monoid satisfying $(*)$. Also conversely, if $(A; \oplus, 0, \vee, \wedge, \oslash, \obslash, 1)$ is a bounded DRl-monoid which fulfils $(*)$ then $(A; \vee', \wedge', \odot, \to, \rightsquigarrow, 0', 1')$ is a pseudo BL-algebra.

Let us remark that the logical system corresponding to pseudo BL-algebras was recently described by P. Hájek in [11].

When doing calculations, we make use of the following list of basic rules:

Lemma 1.3 [13]**.** *In any* DRl-*monoid we have:*

(1) $x \oslash x = 0 = x \obslash x$;
(2) $((x \oslash y) \vee 0) \oplus y = x \vee y = y \oplus ((x \obslash y) \vee 0)$;
(3) $x \oslash (y \oplus z) = (x \oslash z) \oslash y$, $x \obslash (y \oplus z) = (x \obslash y) \obslash z$;
(4) *if* $x \leqslant y$ *then* $x \oslash z \leqslant y \oslash z$ *and* $z \oslash x \geqslant z \oslash y$, *likewise* $x \obslash z \leqslant y \obslash z$ *and* $z \obslash x \geqslant z \obslash y$;
(5) $x \leqslant y$ *iff* $x \oslash y \leqslant 0$ *iff* $x \obslash y \leqslant 0$;
(6) $x \oslash (y \wedge z) = (x \oslash y) \vee (x \oslash z)$, $x \obslash (y \wedge z) = (x \obslash y) \vee (x \obslash z)$;
(7) $(x \vee y) \oslash z = (x \oslash z) \vee (y \oslash z)$, $(x \vee y) \obslash z = (x \obslash z) \vee (y \obslash z)$;
(8) $(x \oslash y) \oplus (y \oslash z) \geqslant x \oslash z$, $(y \obslash z) \oplus (x \obslash y) \geqslant x \obslash z$.

Now, we briefly recall the necessary facts concerning ideals of DRl-monoids (see [14] and [16]). Let A be any DRl-monoid. We define the *absolute value* of $a \in A$ via $|a| = a \vee (0 \oslash a)$. A non-empty subset I of A is said to be an *ideal* in A if

(i) $a \oplus b \in I$ whenever $a, b \in I$,
(ii) if $|b| \leqslant |a|$ and $a \in I$ then $b \in I$.

In the case that A is bounded we have $|a| = a$ for all $a \in A$, and therefore any ideal in A is an ideal in the lattice $\mathrm{l}(A) = (A; \vee, \wedge)$. By [14], the ideals of any DRl-monoid A form an algebraic distributive lattice $\mathcal{I}(A)$. If $I(X)$ denotes the ideal generated by $\emptyset \neq X \subseteq A$, then

$$I(X) = \{a \in A\colon\ |a| \leqslant |x_1| \oplus \ldots \oplus |x_n| \text{ for some } x_1, \ldots, x_n \in X, n \in \mathbb{N}\}.$$

We call an ideal H *normal* if $(a \oslash b) \vee 0 \in H$ iff $(a \odot b) \vee 0 \in H$ for all $a, b \in A$. There is a one-to-one correspondence between the normal ideals of any DRl-monoid and its congruence relations under which a normal ideal H corresponds to the congruence Θ_H defined by

$$(a, b) \in \Theta_H \quad \text{iff} \quad (a \oslash b) \vee (b \oslash a) \in H.$$

We write a/H instead of $[a]_{\Theta_H}$ and A/H for the quotient DRl-monoid A/Θ_H.

For a bounded DRl-monoid A, denote by $B(A)$ the set of all $a \in A$ such that the complement a' of a in the lattice $\mathrm{l}(A)$ exists. By [17], $B(A)$ is a subalgebra of A in which $a \oplus b = a \vee b$ and $a \oslash b = a \wedge b' = a \odot b$; thus $B(A)$ is a Boolean algebra. Moreover, if $X \subseteq B(A)$ then $(X]$, the lattice ideal in $\mathrm{l}(A)$ generated by X, is a normal ideal of A. Note that in general $(X]$ need not be an ideal in A.

An ideal $I \in \mathcal{I}(A)$ is *prime* if, for all $J, K \in \mathcal{I}(A)$, if $J \cap K \subseteq I$ then $J \subseteq I$ or $K \subseteq I$; equivalently, I is prime iff $|a| \wedge |b| \in I$ implies $a \in I$ or $b \in I$. The set of all proper prime ideals in A is denoted by $\mathrm{Spec}(A)$.

2. Weak Boolean products

Let $\{A_x \colon x \in X\}$ be a non-empty family of DRl-monoids. Recall that a DRl-monoid A is a *subdirect product* of $\{A_x \colon x \in X\}$ if there is an embedding φ of A into the direct product $\prod\{A_x \colon x \in X\}$ such that the homomorphisms $\varphi\pi_x$ map A onto A_x for all $x \in X$, where π_x is the natural projection of $\prod\{A_x \colon x \in X\}$ onto A_x.

A *weak Boolean product* of a collection $\{A_x \colon x \in X\}$ of bounded DRl-monoids is their subdirect product A such that X can be endowed with a Boolean topology (i.e., X is a compact T_2-space in which the clopen subsets form a basis) having the following properties:

(i) for all $a, b \in A$, the set $[[a = b]] = \{x \in X \colon a(x) = b(x)\}$ is open in X,

(ii) if U is a clopen subset of X and $a, b \in A$, then $a\restriction_U \cup b\restriction_{X \setminus U} \in A$, where

$$(a\restriction_U \cup b\restriction_{X\setminus U})(x) = \begin{cases} a(x) & \text{if } x \in U, \\ b(x) & \text{if } x \in X \setminus U. \end{cases}$$

We proved in [14] that $a = b$ iff $(a \oslash b) \vee (b \oslash a) = 0$, and therefore, (i) can be replaced by the condition

(i′) $[[a = 0]]$ is an open subset in X for all $a \in A$.

Since DRl-monoids form a variety, it follows that a weak Boolean product of bounded DRl-monoids is still a bounded DRl-monoid.

Let now B be any Boolean algebra and let $\Omega(B)$ be the Stone space of B, i.e. the set of all maximal (= proper prime) ideals in B equipped with the topology whose basis consists of the sets of the form $\sigma(a)=\{P\in\Omega(B)\colon\ a\notin P\}$. It is well-known that $\Omega(B)$ is a Boolean space which determines B to within isomorphism.

Theorem 2.1. *Let A be a non-trivial bounded* DRl-*monoid and let C be a subalgebra of $B(A)$. Then A is isomorphic to a weak Boolean product of $\{A/I(P)\colon P\in\Omega(C)\}$.*

P r o o f. In order to see that A is a subdirect product of $\{A/I(P)\colon\ P\in\Omega(C)\}$, we have to show that $\bigcap\{I(P)\colon\ P\in\Omega(C)\}=\{0\}$.

Let $a\in A\setminus\{0\}$ and let $a\notin P$ for $P\in\operatorname{Spec}(A)$. Then $P\cap C$ is obviously a proper prime ideal of C. Assume that $a\in I(P\cap C)=(P\cap C]$, i.e. $a\leqslant c$ for some $c\in P\cap C$. Hence $a\wedge c'=0\in P$, which entails $c'\in P$ since $a\notin P$. Then $1=c\vee c'=c\oplus c'\in P$, a contradiction. Thus $a\notin I(P\cap C)$ proving $\bigcap\{I(P)\colon\ P\in\Omega(C)\}=\{0\}$.

In what follows, we will regard A as the corresponding subalgebra of the direct product $\prod\{A/I(P)\colon\ P\in\Omega(C)\}$; so $a\in A$ is a mapping $P\mapsto a(P)=a/I(P)$, $P\in\Omega(C)$.

For (i), we have to prove that, for any $a\in A$, $[[a=0]]$ is an open set in $\Omega(C)$. Let $P\in[[a=0]]$, i.e. $a(P)=a/I(P)=I(P)$, so $a\in I(P)$ and there is $p\in P$ with $a\leqslant p$. Therefore, $P\in\sigma(p')=[[p=0]]\subseteq[[a=0]]$ proving that $[[a=0]]$ is open.

For (ii), let U be a clopen subset of $\Omega(C)$. Then $U=\sigma(c)$ for some $c\in C$ since U is a compact clopen set. If $a,b\in A$ then $a\restriction_U\cup b\restriction_{\Omega(C)\setminus U}=(a\wedge c)\vee(b\wedge c')\in A$. Indeed, if $P\in U$ then $(a\restriction_U\cup b\restriction_{\Omega(C)\setminus U})(P)=a/I(P)=(a/I(P)\wedge c/I(P))\vee(b/I(P)\wedge c'/I(P))$ since $b/I(P)\wedge c'/I(P)=b/I(P)\wedge I(P)=I(P)$ and $a/I(P)\wedge c/I(P)=a/I(P)$ because $a\oslash c\leqslant c'\in I(P)$, i.e. $a/I(P)\leqslant c/I(P)$. Similarly for $P\in\Omega(C)\setminus U$. □

An ideal I of a bounded DRl-monoid A is called *Stonean* if for every $a\in I$ there exists $b\in B(A)\cap I$ such that $a\leqslant b$, i.e. $I=(B(A)\cap I]$. In addition, I is a *maximal Stonean ideal* of A if $B(A)\cap I$ is a maximal (= prime) ideal of $B(A)$.

Lemma 2.2. *Let A be a bounded* DRl-*monoid, $a\in A$ and $b\in B(A)$. Then*

$$1\oslash(a\vee b)=(1\oslash a)\wedge(1\oslash b),\quad 1\otimes(a\vee b)=(1\otimes a)\wedge(1\otimes b).$$

P r o o f. First observe that $(a\oslash b)\wedge(b\oslash a)\leqslant(1\oslash b)\wedge b=0$, so $(a\oslash b)\wedge(b\oslash a)=0$ since $b\in B(A)$. Therefore

$$\begin{aligned}1\oslash(a\vee b)&=(1\oslash(a\vee b))\oplus((a\oslash b)\wedge(b\oslash a))\\&=((1\oslash(a\vee b))\oplus(a\oslash b))\wedge((1\oslash(a\vee b))\oplus(b\oslash a))\\&=((1\oslash(a\vee b))\oplus((a\vee b)\oslash b))\wedge((1\oslash(a\vee b))\oplus((a\vee b)\oslash a))\\&\geqslant(1\oslash b)\wedge(1\oslash a)\end{aligned}$$

by Lemma 1.3 (7) and (8). The other inequality is obvious. □

An ideal $I \in \mathcal{I}(A)$ is called a *direct factor* of A if there is an ideal $J \in \mathcal{I}(A)$ such that the mapping $(a,b) \mapsto a \oplus b$ is an isomorphism of the direct product $I \times J$ onto A, in which case we write $A = I \oplus J$. In other words, $A = I \times J$ and I is identified with $\{(a,0)\colon a \in I\}$ and J with $\{(0,a)\colon a \in J\}$. By [19], Proposition 3.2.3, $I \in \mathcal{I}(A)$ is a direct factor if and only if $I \vee I^{\perp} = A$, where $I^{\perp} = \{x \in A\colon |x| \wedge |a| = 0 \text{ for all } a \in I\}$ is the pseudo-complement of I in the ideal lattice $\mathcal{I}(A)$. Therefore, given a bounded DRl-monoid A, if $a \in B(A)$ then $A = (a] \oplus (a']$. We have obtained

Proposition 2.3. *A bounded* DRl-*monoid A is directly indecomposable if and only if $B(A) = \{0,1\}$.*

Proposition 2.4. *Let A be a bounded* DRl-*monoid. If I is a maximal Stonean ideal of A then A/I is directly indecomposable.*

Proof. Since I is a Stonean ideal of A, it is normal.

Let $a \in A$ be such that $a/I \in B(A/I))$. Then $a/I \wedge (1/I \oslash a/I) = (a \wedge (1 \oslash a))/I = I$, so that $a \wedge (1 \oslash a) \in I$. Hence $a \wedge (1 \oslash a) \leqslant b$ for some $b \in B(A) \cap I$. Let $c = a \vee b$; then

$$\begin{aligned} c \wedge (1 \oslash c) &= (a \vee b) \wedge (1 \oslash (a \vee b)) \\ &= (a \vee b) \wedge (1 \oslash a) \wedge (1 \oslash b) \\ &= (a \wedge (1 \oslash a) \wedge (1 \oslash b)) \vee (b \wedge (1 \oslash a) \wedge (1 \oslash b)) \\ &= 0, \end{aligned}$$

which yields $c \in B(A)$. Since $B(A) \cap I$ is a prime ideal of the Boolean algebra $B(A)$, we have either $c \in B(A) \cap I$ or $c' \in B(A) \cap I$. If $c \in B(A) \cap I$ then $a \in B(A) \cap I$ as $a \leqslant c$. Then clearly $a/I = I$. If $c' \in B(A) \cap I$ then

$$\begin{aligned} (1 \oslash a) \vee b &= ((1 \oslash a) \vee b) \wedge ((1 \oslash b) \vee b) \\ &= ((1 \oslash a) \wedge (1 \oslash b)) \vee b \\ &= (1 \oslash (a \vee b)) \vee b \\ &= (1 \oslash c) \vee b \in B(A) \cap I. \end{aligned}$$

Consequently, $1 \oslash a \in I$, whence $1/I \oslash a/I = (1 \oslash a)/I = I$, so $1/I \leqslant a/I$, i.e. $1/I = a/I$. In either case, $B(A/I) = \{I, 1/I\}$, which entails that A/I is directly indecomposable by the previous proposition. □

Theorem 2.5. *Let A be a weak Boolean product of a non-empty family $\{A_x\colon x \in X\}$ of non-trivial bounded* DRl*-monoids. Define*

$$C = \{a \in A\colon a(x) \in \{0_x, 1_x\} \text{ for all } x \in X\}$$

and

$$P_x = \{a \in C\colon a(x) = 0_x\}, \quad x \in X.$$

Then

(i) *C is a subalgebra of $B(A)$;*
(ii) *the mapping $\varphi\colon x \mapsto P_x$ is a homeomorphism of X onto $\Omega(C)$;*
(iii) *for any $x \in X$, A_x is isomorphic to $A/I(P_x)$;*
(iv) *$C = B(A)$ if and only if all the algebras A_x are directly indecomposable.*

P r o o f. (i) This should be evident.

(ii) First, we prove that $P_x \in \Omega(C)$. It is obvious that P_x is a proper ideal of C since $1 \notin P_x$. Assume that $a \wedge b \in P_x$ for $a, b \in C$. Then $(a \wedge b)(x) = a(x) \wedge b(x) = 0_x$, which yields $a(x) = 0_x$ or $b(x) = 0_x$, and so $a \in P_x$ or $b \in P_x$. Thus P_x is prime.

Let $x, y \in X$, $x \neq y$. Since X is a Boolean space (= a T_2-space with a basis of clopen sets), there exists a clopen subset U of X such that $x \in U$ and $y \notin U$. One readily sees that $a = 0\restriction_U \cup 1\restriction_{X \setminus U} \in A$. Moreover, $a \in C$ as $a(z) \in \{0_z, 1_z\}$ for each $z \in X$. From $x \in U$ it follows that $a(x) = 0_x$, so $a \in P_x$, and from $y \notin U$ we obtain $a(y) = 1_y$, so $a \notin P_y$. Thus $P_x \neq P_y$ and the mapping $\varphi\colon x \mapsto P_x$ is one-to-one.

Assume that φ is not onto, i.e., there exists $P \in \Omega(C)$ with $P \neq P_x$ for any $x \in X$. We have $P_x \nsubseteq P$ since both P_x and P are maximal ideals of $B(A)$. Hence for any $x \in X$, there is $a_x \in P_x$ such that $a_x \notin P$. Then $a_x(x) = 0_x$, so $x \in [[a_x = 0]]$, which entails $X = \bigcup\{[[a_x = 0]]\colon x \in X\}$. Consequently, $X = [[a_{x_1} = 0]] \cup \ldots \cup [[a_{x_n} = 0]]$ for some $x_1, \ldots, x_n \in X$. It is easily seen that $X = [[a_{x_1} = 0]] \cup \ldots \cup [[a_{x_n} = 0]] \subseteq [[a_{x_1} \wedge \ldots \wedge a_{x_n} = 0]]$, whence $X = [[a_{x_1} \wedge \ldots \wedge a_{x_n} = 0]]$, and thus $a_{x_1} \wedge \ldots \wedge a_{x_n} = 0$. But P is a prime ideal of C, and hence $a_{x_i} \in P$ for some $1 \leqslant i \leqslant n$, which contradicts $a_x \notin P$ for any $x \in X$.

We have proved that φ is a bijection of X onto $\Omega(C)$.

Let $c \in C$. Then $x \in \varphi^{-1}(\sigma(c))$ iff $P_x \in \sigma(c)$ iff $c \notin P_x$ iff $c' \in P_x$ iff $x \in [[c' = 0]]$; thus $\varphi^{-1}(\sigma(c)) = [[c' = 0]]$. Since the sets $\sigma(c)$ form a basis for $\Omega(C)$, it follows that φ is continuous. Since both X and $\Omega(C)$ are compact T_2-spaces, φ is a homeomorphism.

(iii) Denote $\operatorname{Ker}(\pi_x) = \{a \in A\colon a(x) = 0_x\}$, where π_x is the natural map of A onto A_x. It is clear that $\operatorname{Ker}(\pi_x)$ is a normal ideal of A and $A/\operatorname{Ker}(\pi_x) \cong A_x$. We will show that $I(P_x) = \operatorname{Ker}(\pi_x)$.

If $a \in I(P_x)$ then $a \leqslant b$ for some $b \in P_x$, whence $a(x) \leqslant b(x) = 0_x$, so $a(x) = 0_x$ proving $I(P_x) \subseteq \operatorname{Ker}(\pi_x)$.

Conversely, let $a \in \mathrm{Ker}(\pi_x)$. Then $x \in [[a = 0]]$ so that $P_x = \varphi(x) \subseteq \varphi([[a = 0]])$, where $\varphi([[a = 0]])$ is an open set in $\Omega(C)$. Therefore, there exists $c \in C$ such that $P_x \in \sigma(c') \subseteq \varphi([[a = 0]])$. To complete the proof of (iii) it suffices to show that $a \leqslant c$, which along with $c \in P_x$ (we have $c' \notin P_x$) entails $a \in I(P_x)$.

Note that if $z \in [[c = 0]]$ then $c \in P_z$, i.e. $P_z \in \sigma(c') \subseteq \varphi([[a = 0]])$, and consequently, $z \in [[a = 0]]$ since φ is a bijection; so $[[c = 0]] \subseteq [[a = 0]]$. Therefore, if $z \notin [[a = 0]]$ then $z \notin [[c = 0]]$, which yields $z \in [[c' = 0]]$ since $c(z) \in \{0_z, 1_z\}$ and $c(z) \neq 0_z$. Hence $X = [[a = 0]] \cup [[c' = 0]] \subseteq [[a \wedge c' = 0]]$, thus $a \wedge c' = 0$ proving $a \leqslant c$.

(iv) If $C = B(A)$ then for any $x \in X$, $I(P_x)$ is a maximal Stonean ideal of A. Indeed, $P_x \in \Omega(B(A))$, so P_x is maximal, whence it follows that $I(P_x)$ is a maximal Stonean ideal of A. Therefore, by Proposition 2.4, $A_x \cong A/I(P_x)$ is directly indecomposable.

Conversely, suppose that each A_x is directly indecomposable, but $C \neq B(A)$. Let $a \in B(A) \setminus C$, i.e., there is $x \in X$ with $a(x) \notin \{0_x, 1_x\}$. However, $a \in B(A)$ entails $a(x) \in B(A_x)$. Hence $B(A_x) \neq \{0_x, 1_x\}$ showing that A_x is not directly indecomposable, the desired contradiction. ☐

Corollary 2.6. *Every non-trivial bounded* DRl-*monoid is isomorphic with a weak Boolean product of directly indecomposable bounded* DRl-*monoids.*

Corollary 2.7. *If a non-trivial bounded* DRl-*monoid A is a weak Boolean product of bounded* DRl-*chains, then each maximal Stonean ideal of A is prime. In addition, if A satisfies the equations* $(*)$ *then A is a weak Boolean product of bounded* DRl-*chains if and only if every maximal Stonean ideal is prime.*

Proof. We have $A_x \cong A/I(P_x)$. By [16], Corollary 2.10, if $A/I(P_x)$ is a DRl-chain then $I(P_x)$ is a prime ideal of A. Moreover, in view of [16], Theorem 2.12, if it fulfils $(*)$ then a normal ideal I of A is prime if and only if A/I is linearly ordered. ☐

3. Prime spectra

Prime spectra of pseudo MV-algebras and DRl-monoids were examined by the authors in [25] and [18], respectively.

Recall that $\mathrm{Spec}(A)$ is the poset of all proper prime ideals of a DRl-monoid A; it is partially ordered by set-inclusion. The *prime spectrum* of A is $\mathrm{Spec}(A)$ endowed with the topology $\{\mathcal{S}(X)\colon X \in \mathcal{I}(A)\}$, where $\mathcal{S}(X) = \{P \in \mathrm{Spec}(A)\colon X \nsubseteq P\}$. We note that $\mathcal{S}(X) = \mathcal{S}(I(X))$ for any $X \subseteq A$. Although $\mathrm{Spec}(A)$ does not characterize A, it does give a great deal of information about A, especially if A fulfils the identities $(*)$ (see [16]).

We wish to generalize [22], Theorem 2, stating that the prime spectrum of a weak Boolean product of commutative bounded DRl-monoids is the cardinal sum of the prime spectra of its components.

Lemma 3.1. *Let A be a lower-bounded* DRl*-monoid and $I \in \mathcal{I}(A)$. If $(a \oslash b) \vee (b \oslash a) \in I$ and $a \in I$, then $b \in I$.*

Proof. For any $a, b \in A$,

$$b \leqslant ((a \oslash b) \oplus a) \vee b \leqslant ((a \oslash b) \oplus a) \vee ((b \oslash a) \oplus a) = ((a \oslash b) \vee (b \oslash a)) \oplus a.$$

Therefore, if both $(a \oslash b) \vee (b \oslash a)$ and a belong to I, then so does b. □

Theorem 3.2. *Let A be a weak Boolean product of a family $\{A_x \colon x \in X\}$ of bounded* DRl*-monoids. Then the ordered prime spectrum of A, $\operatorname{Spec}(A)$, is isomorphic to the cardinal sum of the ordered prime spectra $\{\operatorname{Spec}(A_x) \colon x \in X\}$.*

Proof. Denote $I_x = \operatorname{Ker}(\pi_x) = \{a \in A \colon a(x) = 0_x\}$ for any $x \in X$. Let $P \in \operatorname{Spec}(A)$ and assume that $I_x \nsubseteq P$ for all $x \in X$, i.e., for any $x \in X$ there exists $b_x \in I_x \setminus P$. Then clearly $X = \bigcup\{[[b_x = 0]] \colon x \in X\}$, and consequently, $X = [[b_{x_1} = 0]] \cup \ldots \cup [[b_{x_n} = 0]]$ for some $x_1, \ldots, x_n \in X$. We also have $X = [[b_{x_1} = 0]] \cup \ldots \cup [[b_{x_n} = 0]] \subseteq [[b_{x_1} \wedge \ldots \wedge b_{x_n} = 0]]$, whence $b_{x_1} \wedge \ldots \wedge b_{x_n} = 0 \in P$, which entails $b_{x_i} \in P$ for some $1 \leqslant i \leqslant n$, since P is a prime ideal; a contradiction. Thus given $P \in \operatorname{Spec}(A)$, there exists $x \in X$ with $I_x \subseteq P$. We are going to show that this x is unique. For that purpose, let $x \neq y$, $I_x \subseteq P$ and $I_y \subseteq P$. Since X is a Boolean space, there exists a clopen subset V of X such that $x \in V$ while $y \in X \setminus V$. By the condition (ii), $0\restriction_V \cup 1\restriction_{X\setminus V} \in A$, and in addition, $0\restriction_V \cup 1\restriction_{X\setminus V} \in I_x \subseteq P$ as $(0\restriction_V \cup 1\restriction_{X\setminus V})(x) = 0_x$. Similarly $1\restriction_V \cup 0\restriction_{X\setminus V} \in I_y \subseteq P$. However, it is easily seen that $(0\restriction_V \cup 1\restriction_{X\setminus V}) \oplus (1\restriction_V \cup 0\restriction_{X\setminus V}) = 1$, so $1 \in P$, the desired contradiction.

Let now $\mathcal{H}(I_x) = \{P \in \operatorname{Spec}(A) \colon I_x \subseteq P\}$ for $x \in X$. We have proved that for any $P \in \operatorname{Spec}(A)$, there exists a unique $x \in X$ such that $I_x \subseteq P$. Therefore it is obvious that the ordered prime spectrum $\operatorname{Spec}(A)$ is isomorphic to the cardinal sum of the posets $\mathcal{H}(I_x)$, $x \in X$. In order to complete the proof, we will show that $\operatorname{Spec}(A_x)$ and $\mathcal{H}(I_x)$ are isomorphic.

Let $P \in \mathcal{H}(I_x)$ and $\psi_x(P) = \{c(x) \colon c \in P\}$. One readily sees that $\psi_x(P) \in \mathcal{I}(A_x)$. Moreover, if $1_x \in \psi_x(P)$ then $1_x = c(x)$ for some $c \in P$, so $((c \oslash 1) \vee (1 \oslash c))(x) = 0_x$ and hence $(c \oslash 1) \vee (1 \oslash c) \in I_x \subseteq P$. But by Lemma 3.1 this yields $1 \in P$, which contradicts $P \in \operatorname{Spec}(A)$. Thus $\psi_x(P)$ is a proper ideal of A_x.

Let $u, v \in A_x$ and assume that $u \wedge v \in \psi_x(P)$. Then there exist $a, b \in A$ and $c \in P$ such that $a(x) = u$, $b(x) = v$ and $c(x) = u \wedge v = (a \wedge b)(x)$. Clearly, $((a \wedge b) \oslash c) \vee (c \oslash (a \wedge b))(x) = 0_x$, and so $((a \wedge b) \oslash c) \vee (c \oslash (a \wedge b)) \in I_x \subseteq P$, which yields $a \wedge b \in P$ by Lemma 3.1. Since P is prime, we have $a \in P$ or $b \in P$ so that $u \in \psi_x(P)$ or $v \in \psi_x(P)$. Therefore $\psi_x(P)$ is a proper prime ideal of A_x and $\psi_x\colon P \mapsto \psi_x(P)$ is a (one-to-one) mapping from $\mathcal{H}(I_x)$ into $\mathrm{Spec}_x(A_x)$.

Let $Q \in \mathrm{Spec}(A_x)$ and put $\varrho_x(Q) = \{a \in A\colon a(x) \in Q\}$. It can be easily seen that $\varrho_x(Q)$ is a proper prime ideal of A with $I_x \subseteq \varrho_x(Q)$, that is, $\varrho_x(Q) \in \mathcal{H}(I_x)$. In addition, $\psi_x(\varrho_x(Q)) = Q$ proving that ψ_x is a bijection; obviously, $\psi_x^{-1} = \varrho_x$. Since both ψ_x and ϱ_x preserve set-inclusion, $\psi_x\colon \mathcal{H}(I_x) \to \mathrm{Spec}(A_x)$ is the desired isomorphism. □

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