EXPONENTIAL DECAY OF A SOLUTION FOR SOME PARABOLIC EQUATION INVOLVING A TIME NONLOCAL TERM

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Abstract. We consider the large time behavior of a solution of a parabolic type equation involving a nonlocal term depending on the unknown function. This equation is proposed as a mathematical model of carbon dioxide transport in concrete carbonation process, and we proved the existence, uniqueness and large time behavior of a solution of this model. In this paper, we derive the exponential decay estimate of the solution of this model under restricted boundary data and initial data.

Keywords: large time behavior; exponential decay; nonlinear parabolic equation

MSC 2010: 35B40, 35K55

1. Introduction

In this paper, we consider the following initial boundary value problem for a parabolic type equation involving a nonlocal term depending on the unknown function:

$$\frac{\partial}{\partial t}\Big[\phi\Big(1-\mathrm{e}^{-\int_0^t u(\tau)\,\mathrm{d}\tau}\Big)u\Big]-\Delta u=-w_0u\mathrm{e}^{-\int_0^t u(\tau)\,\mathrm{d}\tau}\quad \text{in } Q(T):=(0,T)\times\Omega, \tag{1.1}$$

$$u=u_b\quad \text{on } S(T):=(0,T)\times\Gamma, \tag{1.2}$$

$$u(0)=u_0\quad \text{in } \Omega. \tag{1.3}$$

Here Ω is a bounded domain of $\mathbb{R}^3$ with a smooth boundary $\Gamma=\partial\Omega$, $T>0$ is a fixed finite number, ϕ is a function in $C^1(\mathbb{R})$ satisfying $\phi_0\leqslant\phi(r)\leqslant 1$ for $r\in\mathbb{R}$ where ϕ_0 is a positive constant, u_b is a given function on $Q(T)$, and w_0 and u_0 are given functions on Ω.

The equation (1.1) is a diffusion equation derived from carbon dioxide transport in concrete carbonation process proposed by Aiki and Kumazaki in [1], [2]. The detailed derivation is presented in [5], [6]. Physically, Ω is a domain occupied by concrete, and the unknown function $u = u(t,x)$ represents the concentration of carbon dioxide in water at a time t and a position $x \in \Omega$. Also, $\phi = \phi(z)$ represents the porosity, which is the ratio of the volume of the voids inside the concrete to the volume of the whole concrete and $z = 1 - \mathrm{e}^{-\int_0^t u(\tau)\,\mathrm{d}\tau}$ is the ratio of the volume of consumed calcium hydroxide to the volume of the total calcium hydroxide.

Concerning a mathematical analysis of concrete carbonation, Aiki and Kumazaki [1], [2] proposed a mathematical model of moisture transport which involves the hysteresis operator $\mathcal{S}$, and proved existence and uniqueness of a solution of the model, uniqueness being proved only for the one-dimensional case. Also, in [5] we proved the existence and uniqueness of a global solution of (P) $= \{(1.1), (1.2), (1.3)\}$, and in [6] we showed that the solution converges to a solution u_∞ of the steady state problem

$$-\Delta u_\infty + \phi'(1 - w_\infty) w_\infty u_\infty^2 = -w_0 u_\infty w_\infty \quad \text{in } \Omega, \tag{1.4}$$

$$u_\infty = u_{b\infty} \quad \text{on } \Gamma, \tag{1.5}$$

where $u_{b\infty}$ is a given function in Ω with $u_{b\infty} \geqslant 0$ in Ω which is the limit function of u_b as $t \to \infty$, and $w_\infty = w_\infty(x)$ is the limit of $w(x,t) := \mathrm{e}^{-\int_0^t u(x,\tau)\,\mathrm{d}\tau}$ as $t \to \infty$, namely, $w(x,t) \to w_\infty(x)$ as $t \to \infty$ for each $x \in \Omega$. Moreover, we proved that if u_b does not vanish identically on Γ, the solution u_∞ satisfies the Dirichlet problem $-\Delta u_\infty = 0$ in Ω, $u_\infty = u_b$ on Γ, and $w_\infty = 0$ a.e. on Ω.

The main aim of this paper is to establish the following exponential decay of a solution:

$$|u - u_\infty|^2_{L^2(\Omega)} \leqslant C\mathrm{e}^{-\kappa t} \quad \text{for sufficiently large } t, \tag{1.6}$$

where C and κ are positive constants. As mentioned above, we assume that the solution of (P) converges to the solution of the steady state problem; however, under the assumption that $u_b \geqslant 0$ in Ω and $u_0 \geqslant 0$ in Ω, we could not show the convergence rate as in (1.6). The key lemma for the proof of this decay is to prove that

$$u(t) \geqslant \kappa \quad \text{for sufficiently large } t, \tag{1.7}$$

where κ is a positive constant. Accordingly, in this paper we assume that $u_b = u_b(x) \geqslant \kappa$ in Ω for a positive constant κ. In order to obtain the uniform continuity of a solution, we derive higher regularity of the solution u (Lemma 3.1), and show that the $H^2(\Omega)$ estimate independent of t for the solution u holds (Lemma 3.3). By this result and the fact that $u_\infty \geqslant \kappa$ in Ω, we show that (1.7) holds. Finally, by using (1.7) we prove (1.6).

In this paper we use the following notation. In general, for a Banach space X, we denote by $|\cdot|_X$ its norm. In particular, we denote $H = L^2(\Omega)$, and the norm and the inner product of H are simply denoted by $|\cdot|_H$ and $(\cdot,\cdot)_H$, respectively. Also, $H^1(\Omega)$, $H^1_0(\Omega)$ and $H^2(\Omega)$ are the usual Sobolev spaces.

Throughout this paper we assume the following (A1)–(A5):

(A1) $\Omega \subset \mathbb{R}^3$ is a bounded domain with a smooth boundary Γ.

(A2) ϕ is a non-decreasing function in $C^2(\mathbb{R})$ such that $\phi(0) = \phi_0$ and $\phi'(0) = 0$, $c_0 = \sup_{r\in\mathbb{R}} \phi'(r) + \sup_{r\in\mathbb{R}} |\phi''(r)| < \infty$ and $\phi_0 \leqslant \phi(r) \leqslant 1$ for $r \in \mathbb{R}$ where ϕ_0 is a positive number.

(A3) $u_b \in H^2(\Omega) \cap L^\infty(\Omega)$ with $0 \leqslant u_b \leqslant \kappa_0$ in Ω where κ_0 is a positive constant.

(A4) $u_0 \in H^2(\Omega) \cap L^\infty(\Omega)$, $u_0 \geqslant 0$ in Ω and $u_0 = u_b$ on $\partial\Omega$.

(A5) $w_0 \in L^\infty(\Omega)$ and $w_0 > 0$ in Ω.

Next, we define a solution of (P) on $[0,T]$ in the following way:

Definition 2.1. Let u be a function on $Q(T)$ for $0 < T < \infty$. We call a function u a solution of (P) on $[0,T]$ if the following conditions (S1)–(S4) hold:

(S1) $u \in W^{1,2}(0,T;H) \cap L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$, $u \geqslant 0$ a.e. on $Q(T)$.

(S2) $\left[\phi\left(1 - \mathrm{e}^{-\int_0^t u(\tau)\,\mathrm{d}\tau}\right)u\right]_t - \Delta u = -w_0 u \mathrm{e}^{-\int_0^t u(\tau)\,\mathrm{d}\tau}$ a.e. in $Q(T)$.

(S3) $u = u_b$ a.e. on $S(T)$.

(S4) $u(0) = u_0$ in Ω.

Our first and second results show the existence and uniqueness of a solution, and the large time behavior of the solution, respectively.

Theorem 2.1. *If* (A1)–(A5) *hold, then for any* $T > 0$, (P) *has one and only one solution* u *on* $[0,T]$ *such that* $0 \leqslant u \leqslant u^* := \max\{|u_0|_{L^\infty(\Omega)}, \kappa_0\}$ *a.e. on* $Q(T)$, *where* u_0 *is the initial data and* κ_0 *is the same constant as in* (A3).

Theorem 2.2. *Assume* (A1)–(A5) *hold, and let* u *and* u_∞ *be a solution of* (P) *and* $(\mathrm{P})_\infty := \{(1.4), (1.5)\}$, *respectively. Then*

$$u(t) \to u_\infty \quad \text{strongly in } H \text{ and weakly in } H^1(\Omega) \text{ as } t \to \infty.$$

Moreover, if u_b *does not vanish identically on* Γ, *then* u_∞ *is a solution of the steady state problem* $-\Delta u_\infty = 0$ *a.e. in* Ω, $u_\infty = u_b$ *a.e. on* Γ. *Also,* $w_\infty = 0$ *a.e. on* Ω.

Here we note how much the concrete is carbonated finally. In Theorem 2.2, we showed that $w_\infty = 0$ a.e. on Ω. Therefore, we see that $z = 1 - \mathrm{e}^{-\int_0^t u(\tau)\,\mathrm{d}\tau} \to 1$

as $t \to \infty$ for a.e. $x \in \Omega$. Since z is the ratio of the volume of consumed calcium hydroxide to the volume of the total calcium hydroxide, $z = 1$ a.e. on Ω implies that calcium hydroxide is fully consumed almost everywhere in the concrete. Accordingly, finally, we see that the concrete is carbonated almost everywhere.

Theorems 2.1 and 2.2 are already proved in [5], [6] so that we omit the proof. Now, we state the main result concerning the exponential decay estimate.

Theorem 2.3. *Assume* (A1)–(A5) *hold, and let u and u_∞ be solutions of* (P) *and* $(\mathrm{P})_\infty$*, respectively. In addition, we assume that $u_b \geqslant \kappa$ in Ω for a positive constant κ satisfying $(\phi_0 C_P^2)^{-1} > \kappa$ where C_P is a positive constant in Poincaré's inequality. Then there exists $t^* > 0$ such that*

$$|u(t) - u_\infty|_H^2 \leqslant C\mathrm{e}^{-\kappa t} \quad \textit{for } t > t^*,$$

where C is a positive constant.

3. Proof of Theorem 2.3

In the rest of this paper, we use the following notation: For the solution u of (P),

$$w(t) = \mathrm{e}^{-\int_0^t u(\tau)\,\mathrm{d}\tau}, \quad \alpha(t) = \phi\bigl(1 - \mathrm{e}^{-\int_0^t u(\tau)\,\mathrm{d}\tau}\bigr) = \phi(1 - w(t)) \quad \text{for } t > 0.$$

First, we show the following higher regularity result for the solution u of (P).

Lemma 3.1. *For $0 < T < \infty$,* (P) *has at least one solution u on $[0,T]$ such that*

$$\begin{cases} u_t \in C([0,T];H) \cap L^2(0,T;H_0^1(\Omega)), \\ t^{1/2}u_t \in L^\infty(0,T;H_0^1(\Omega)), \\ t^{1/2}u_{tt} \in L^2(0,T;H). \end{cases}$$

P r o o f. Let u be the solution of (P), and set $l = -w_0 u(t)w(t) - \alpha_t(t)u(t)$. Then by the regularity of u we see that $l_t \in L^2(0,T;H)$. Now, we consider the following problem (AP)

$$(\mathrm{AP}) \begin{cases} \alpha(t)Z_t(t) - \Delta Z(t) + \alpha_t(t)Z(t) = l_t(t) \quad \text{in } Q(T), \\ Z(t) = 0 \quad \text{on } S(T), \\ Z(0) = z_0 := \frac{1}{\phi_0}(\Delta u_0 - w_0 u_0) \quad \text{in } \Omega, \end{cases}$$

where ϕ_0 is the same as in (A2). Here, we remark that $\alpha z_t + \alpha_t z = (\alpha z)_t$ and α, $\alpha_t \in L^\infty(Q(T))$. For $z_0 \in H$ we can take a sequence $\{z_{0,n}\} \subset H_0^1(\Omega)$ such that

$z_{0,n} \to z_0$ in H as $n \to \infty$. Then, for each $n \in \mathbb{N}$, by using a classical result on parabolic equations (for example [7]) we can see that the problem $(\mathrm{AP})_n$ with z_0 in (AP) replaced by $z_{0,n}$ has a unique solution $z_n \in C([0,T];H) \cap L^2(0,T;H_0^1(\Omega))$ with $z_n \in L^\infty(0,T;H_0^1(\Omega))$ and $(z_n)_t \in L^2(0,T;H)$. Now, for $n,m \in \mathbb{N}$, we see that

$$
(3.1)\qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \alpha(t)|z_n(t)-z_m(t)|^2\,\mathrm{d}x + |\nabla(z_n(t)-z_m(t))|_H^2 \\
\leqslant \frac{1}{2}|\alpha_t/\alpha|_{L^\infty(Q(T))}\int_\Omega \alpha(t)|z_n(t)-z_m(t)|^2\,\mathrm{d}x \quad \text{for } 0 \leqslant t \leqslant T.
$$

Hence, Gronwall's lemma implies that $\{z_n\}$ is a Cauchy sequence in $C([0,T];H) \cap L^2(0,T;H_0^1(\Omega))$ so that there exists $z \in C([0,T];H) \cap L^2(0,T;H_0^1(\Omega))$ such that $z_n \to z$ in $C([0,T];H) \cap L^2(0,T;H_0^1(\Omega))$ as $n \to \infty$. Therefore, by [3], Chapter 4, or [4], Chapter 1, it holds that $t^{1/2}z \in L^\infty(0,T;H_0^1(\Omega))$ and $t^{1/2}z_t \in L^2(0,T;H)$. For $\varphi \in C_0^\infty(\Omega)$, by integrating over $[0,t]$ after multiplying by φ the equation of $(\mathrm{AP})_n$ and letting $n \to \infty$, we have

$$
\int_\Omega (\alpha(t)Z(t) - \phi_0 z_0)\varphi\,\mathrm{d}x + \int_\Omega \nabla\Big(\int_0^t Z(\tau)\,\mathrm{d}\tau\Big)\cdot\nabla\varphi\,\mathrm{d}x = \int_\Omega (l(t)-l(0))\varphi\,\mathrm{d}x.
$$

Now, we introduce a new variable

$$
\widetilde{u}(t) := \int_0^t Z(\tau)\,\mathrm{d}\tau + u_0.
$$

Then, due to $z_0 = (1/\phi_0)(\Delta u_0 - w_0u_0)$ and $l(0) = -w_0u_0$, we have

$$
\int_\Omega \alpha(t)\widetilde{u}_t(t)\varphi\,\mathrm{d}x + \int_\Omega \nabla\widetilde{u}(t)\cdot\nabla\varphi\,\mathrm{d}x = \int_\Omega l(t)\varphi\,\mathrm{d}x \quad \text{for } \varphi \in C_0^\infty(\Omega).
$$

Since we can see from (1.1) that the above equality with $\widetilde{u}$ replaced by u holds, we have

$$
\int_\Omega \alpha(t)(\widetilde{u}_t(t) - u_t(t))\varphi\,\mathrm{d}x + \int_\Omega \nabla(\widetilde{u}(t)-u(t))\cdot\nabla\varphi\,\mathrm{d}x = 0 \quad \text{for } \varphi \in C_0^\infty(\Omega).
$$

Similarly to (3.1), by taking $\varphi = u(t) - \widetilde{u}(t)$ in this equation for $t > 0$, Gronwall's inequality implies that $\widetilde{u} = u$ so that Lemma 3.1 holds. $\square$

Now we note the global estimate of a solution u obtained in [6].

Lemma 3.2. *Let u be the solution of* (P). *Then $u_t \in L^2(0,\infty;H)$, $\nabla u \in L^2(0,\infty;H)$ and $\Delta u \in L^2(0,\infty;H)$.*

By Lemmas 3.1 and 3.2 we obtain the estimate for the norm of a solution in $H^2(\Omega)$.

Lemma 3.3. *Let δ_0 be any positive constant. Then there exists a positive constant C depending only on δ such that*

$$\sup_{t\geqslant\delta_0}|u(t)|_{H^2(\Omega)}\leqslant C.$$

P r o o f. Let s and s_1 be any positive numbers with $s<s_1<s+1$. Then, for $t\in[s,s_1]$, multiplying (1.1) by $(t-s)u_t$, we have

$$\frac{\phi_0}{2}(t-s)|u_t(t)|_H^2+\frac{\mathrm{d}}{\mathrm{d}t}(t-s)|\nabla u(t)|_H^2\leqslant C_1(t-s)(u(t),w^2(t))_H+|\nabla u(t)|_H^2,$$

where $C_1=(c_0^2(u^*)^3+|w_0|^2_{L^\infty(\Omega)}u^*)/\phi_0$. Here we note that

$$(t-s)(u(t),w^2(t))_H=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[(t-s)|w(t)|_H^2\Big]+\frac{1}{2}|w(t)|_H^2. \tag{3.2}$$

Therefore, we have

$$\frac{\phi_0}{2}(t-s)|u_t(t)|_H^2+\frac{\mathrm{d}}{\mathrm{d}t}\Big[(t-s)|\nabla u(t)|_H^2+\frac{C_1}{2}(t-s)|w(t)|_H^2\Big]\leqslant|\nabla u(t)|_H^2+\frac{C_1}{2}|w(t)|_H^2.$$

By integrating over $[s,s_1]$ we obtain

$$\frac{\phi_0}{2}\int_s^{s_1}(t-s)|u_t(t)|_H^2\,\mathrm{d}t\leqslant\int_s^{s_1}|\nabla u(t)|_H^2\,\mathrm{d}t+\frac{C_1}{2}|\Omega|\quad\text{for } s<s_1\leqslant s+1. \tag{3.3}$$

Next, we differentiate (1.1) with respect to t and multiply the result by $(t-s)u_t$ to obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[(t-s)\int_\Omega\alpha(t)|u_t(t)|^2\,\mathrm{d}x\Big]+\Big(\alpha_{tt}u(t)+\frac{3}{2}\alpha_t(t)u_t(t),(t-s)u_t(t)\Big)_H\\+(t-s)|\nabla u_t(t)|_H^2\leqslant\big(-w_0u_t(t)w(t)-w_0u(t)w_t(t),(t-s)u_t(t)\big)_H\\+\frac{1}{2}(1-s)\int_\Omega\alpha(t)|u_t(t)|^2\,\mathrm{d}x.\end{aligned} \tag{3.4}$$

Note that $\alpha_{tt}=\phi''(wu)^2-\phi'wu^2+\phi'wu_t$ and that $(\phi'wu_t,(t-s)u_t)_H\geqslant0$, $(\alpha_tu_t,(t-s)u_t)_H\geqslant0$, $(w_0u_tw,(t-s)u_t)\geqslant0$ and

$$(t-s)|((\phi'')^2,(wu)^4)_H|+(t-s)|((\phi')^2,(wu^2)^2)_H|\leqslant C_2(t-s)(u,w^2)_H,$$

where C_2 is a positive constant depending on c_0 and u^*. By virtue of $w_t(t)=-u(t)w(t)$ for $t>0$, (3.2), (3.4) and using Young's inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[(t-s)\int_\Omega\alpha(t)|u_t(t)|^2\,\mathrm{d}x\Big]+C_3\frac{\mathrm{d}}{\mathrm{d}t}\Big[(t-s)|w(t)|_H^2\Big]+(t-s)|\nabla u_t(t)|_H^2\\\leqslant\frac{3}{2}(t-s)|u_t(t)|_H^2+\frac{1}{2}|u_t(t)|_H^2+C_3|w(t)|_H^2,\end{aligned}$$

where $C_3 = C_2/2 + (|w_0|^2_{L^\infty(\Omega)}(u^*)^3)/2$. Integrating this result over $[s,t]$ with $t \in [s,s_1]$ and adding (3.3), we derive

$$\tag{3.5} \begin{aligned} \frac{\phi_0}{2}(t-s)|u_t(t)|^2_H \leqslant \frac{3}{\phi_0}\Big(\int_s^{s_1} |\nabla u(\tau)|^2_H \,\mathrm{d}\tau + \frac{C_1}{2}|\Omega|\Big) \\ + \frac{1}{2}\int_s^{s_1} |u_t(\tau)|^2_H \,\mathrm{d}\tau + C_3|\Omega| \quad \text{for } 0 \leqslant s \leqslant t < s_1 \leqslant s+1. \end{aligned}$$

By Lemma 3.2 and (3.5) we see that there exists a positive constant C such that

$$\tag{3.6} \sup_{s\leqslant t\leqslant s_1} (t-s)|u_t(t)|^2_H \leqslant C \quad \text{for any } s \text{ and } s_1 \text{ with } 0 \leqslant s < s_1 \leqslant s+1.$$

Now, we complete the proof of Lemma 3.3. Multiplying (1.1) by $(t-s)(-\Delta u)$, we have

$$\begin{aligned} \frac{1}{4}(t-s)|\Delta u(t)|^2_H \leqslant (t-s)|u_t(t)|^2_H + |\alpha_t|^2_{L^\infty(Q(T))}(t-s)|u(t)|^2_H \\ + |w_0|^2_{L^\infty(\Omega)}|u(t)w(t)|^2_H \end{aligned}$$

for any s and t with $0 \leqslant s < t \leqslant s_1 \leqslant s+1$. Therefore, from (3.6) we have that $\sup\limits_{s\leqslant t\leqslant s_1} (t-s)|\Delta u|^2_H$ is bounded for any s and s_1 with $0 \leqslant s < s_1 \leqslant s+1$. Finally, by taking $s_1 = s+\tau$ with $0 < \tau \leqslant \min(1,\delta_0)$ for any positive constant δ_0, we have that $\sup_{t\geqslant\delta_0} |\Delta u|_H$ is bounded, which implies that Lemma 3.2 holds. □

P r o o f of Theorem 2.3. By Lemma 3.2 and Theorem 2.2, we see that $u \to u_\infty$ weakly in $H^2(\Omega)$ as $t \to \infty$. Therefore, by the compact embeddings we have that $u \to u_\infty$ in $C(\overline{\Omega})$ as $t \to \infty$. Now we show that $u_\infty \geqslant \kappa$ a.e. on Ω. Since u_∞ satisfies $-\Delta u_\infty = 0$ a.e. in Ω and $u_\infty = u_b$ a.e. on Γ, multiplying the equation by $-[-u_\infty+\kappa]^+$ and using $u_b \geqslant \kappa$, we have $|[-u_\infty+\kappa]^+|^2_H = 0$ so that $u_\infty \geqslant \kappa$ a.e. on Ω. From the convergence in $C(\overline{\Omega})$ and $u_\infty \geqslant \kappa$ in Ω we see that there exists a positive number $t^* \geqslant \delta_0$ such that

$$\tag{3.7} u(t) = u(t)-u_\infty+u_\infty \geqslant -|u(t)-u_\infty|+u_\infty \geqslant -\frac{\kappa}{2}+\kappa = \frac{\kappa}{2} \quad \text{on } \Omega \text{ for } t > t^*.$$

Since u and u_∞ are solutions of (P) and $(\mathrm{P})_\infty$, respectively, we have

$$\alpha u_t(t) + \alpha_t(u(t)-u_\infty) - \Delta(u(t)-u_\infty) = -w_0u(t)w(t) - \alpha_t u_\infty \quad \text{in } \Omega \text{ for } t > 0.$$

Multiplying the equation by $u-u_\infty$, we obtain

$$\tag{3.8} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \alpha(t)|u(t)-u_\infty|^2\,\mathrm{d}x + \frac{1}{2}|\nabla(u(t)-u_\infty)|^2_H \leqslant C_4|w(t)|^2_H,$$

where C_P is the positive constant in Poincaré's inequality and $C_4 = C_P^2\big(|w_0|^2_{L^\infty}(u^*)^2 |\Omega| + 3c_0^2(u^*)^4\big)$. Here, we note that (3.7) yields

$$(3.9)\quad \int_\Omega w^2(t)\,\mathrm{d}x = \int_\Omega \mathrm{e}^{-2\int_0^{t^*} u(\tau)\,\mathrm{d}\tau}\mathrm{e}^{-2\int_{t^*}^t u(\tau)\,\mathrm{d}\tau} \leqslant \int_\Omega \mathrm{e}^{-2\int_{t^*}^t u(\tau)\,\mathrm{d}\tau} \leqslant \mathrm{e}^{-\kappa(t-t^*)}|\Omega|,$$

where $|\Omega|$ is the volume of Ω. By substituting (3.9) in (3.8) and setting $I(t) = \frac{1}{2}\int_\Omega \alpha(t)|u(t)-u_\infty|^2\,\mathrm{d}x$ for $t > 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}I(t) + \frac{1}{\phi_0 C_P^2}I(t) \leqslant C_4\mathrm{e}^{-\kappa(t-t^*)}|\Omega| \quad \text{for } t > t^*.$$

Therefore, by putting $\beta = (\phi_0 C_P^2)^{-1}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(I(t)\mathrm{e}^{\beta t}) \leqslant C_4\mathrm{e}^{(\beta-\kappa)t+\kappa t^*} \quad \text{for } t > t * .$$

By integrating over $[t^*, t]$ and using the fact that

$$\int_{t^*}^t \mathrm{e}^{(\beta-\kappa)s}\,\mathrm{d}s = \frac{1}{\beta-\kappa}\big(\mathrm{e}^{(\beta-\kappa)t} - \mathrm{e}^{(\beta-\kappa)t^*}\big)$$

we obtain

$$I(t) \leqslant \mathrm{e}^{-\beta t}\Big(I(t^*)\mathrm{e}^{\beta t^*} + \Big(\frac{C_4}{\beta-\kappa}\big(\mathrm{e}^{(\beta-\kappa)t} - \mathrm{e}^{(\beta-\kappa)t^*}\big)\Big)\mathrm{e}^{\kappa t^*}\Big) \quad \text{for } t > t^*.$$

Therefore, if $\beta > \kappa$, we have $\mathrm{e}^{-\beta t}\cdot\mathrm{e}^{(\beta-\kappa)t}\cdot\mathrm{e}^{\kappa t^*} \leqslant \mathrm{e}^{-\kappa t}\cdot\mathrm{e}^{\beta t^*}$ so that

$$I(t) \leqslant \mathrm{e}^{-\kappa t}\mathrm{e}^{\beta t^*}\Big(I(t^*) + \frac{C_4}{\beta-\kappa}\Big) \quad \text{for } t > t^*.$$

Since $I(t^*) = (1/2)|u(t^*) - u_\infty|_H^2 \leqslant 2(u^*)^2|\Omega|$, by putting $C_5 = 2\mathrm{e}^{\beta t^*}(2(u^*)^2|\Omega| + C_4/(\beta-\kappa))$, we conclude that

$$\int_\Omega \alpha(t)|u(t)-u_\infty|^2\,\mathrm{d}x \leqslant C_5\mathrm{e}^{-\kappa t} \quad \text{for } t > t^*.$$

Finally, by putting $C = C_5/\phi_0$, Theorem 2.3 is proved. $\square$

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