SOME COHOMOLOGICAL ASPECTS OF THE BANACH FIXED POINT PRINCIPLE

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Abstract. Let $T\colon X \rightarrow X$ be a continuous selfmap of a compact metrizable space X. We prove the equivalence of the following two statements: (1) The mapping T is a Banach contraction relative to some compatible metric on X. (2) There is a countable point separating family $\mathcal{F} \subset \mathcal{C}(X)$ of non-negative functions $f \in \mathcal{C}(X)$ such that for every $f \in \mathcal{F}$ there is $g \in \mathcal{C}(X)$ with $f = g - g \circ T$.

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1. Introduction and notation

The object of our study is a continuous selfmap T of a compact metrizable space X. Let $\mathcal{C}(X)$ and $\mathcal{M}(X)$ denote the set of all continuous real-valued functions and the set of all compatible metrics on X, respectively. Regarding $\mathcal{C}(X)$ as an abelian group, we convert it to an $(\mathbb{N},+)$-module defining the action of $n \in \mathbb{N}$ on $f \in \mathcal{C}(X)$ by

$$n \cdot f = f \circ T^n,$$

where T^n is the nth iteration of T. In most textbooks on homological algebra only group-actions are treated, whereas $(\mathbb{N},+)$ is only a commutative monoid reflecting the fact that T need not have an inverse. But it was shown by S. Maclane, H. Cartan, A. Bakakhanian [1] and others that the homology and cohomology theory of monoids is almost identical with that of groups. In our case only one-cohomology will be relevant. By a *one-chain* we understand a map

$$\varphi\colon \mathbb{N} \rightarrow \mathcal{C}(X)$$

with $\varphi(0) = 0 \in \mathcal{C}(X)$. The cochain φ is a cocycle if it is of the form

$$\varphi(n) = f + 1 \cdot f + 2 \cdot f + \ldots + (n-1) \cdot f, \quad n \in \mathbb{N},$$

where f is a function in $\mathcal{C}(X)$. If f is of the form $g - 1 \cdot g$ with $g \in \mathcal{C}(X)$ then the cocycle is a *coboundary*. In this case we have

$$\sum_{k=0}^{n-1} k \cdot f = g - n \cdot g, \quad n \in \mathbb{N}$$

and by abuse of language we call the function itself a coboundary. If $\mathcal{F} \subset \mathcal{C}(X)$ is a family of functions we say $\mathcal{F}$ is *point separating* if for any $x, y \in X$, $x \neq y$, there is $f \in \mathcal{F}$ with $f(x) \neq f(y)$. A selfmap $T\colon X \to X$ is a *Banach contraction* if for some $c \in [0,1)$ and $d \in M(X)$ we have

$$d(Tx, Ty) \leqslant cd(x, y), \quad x, y \in X.$$

We will prove

Theorem 1. *The selfmap T is a Banach contraction if and only if there is a countable point separating family $\mathcal{F} \subset \mathcal{C}(X)$ of nonnegative coboundaries.*

2. Proof of the theorem

We prove first the easier "only if" part. Thus let T be a c-contraction on a compact metric space (X, d). Let $\{x_k\}_{k \in \mathbb{N}}$ be a countable dense subset of X and for every $k \in \mathbb{N}$ consider the orbit $\mathcal{O}(x_k) = \{T^n x_k \colon n \in \mathbb{N}\}$. We define the function f_k as the distance from $x \in X$ to the orbit $\mathcal{O}(x_k)$, i.e. $f_k(x) = d(x, \mathcal{O}(x_k))$. It is obvious that $f_k \in \mathcal{C}(X)$ and that $\sum_{n=0}^{\infty} f_k(T^n x)$ converges uniformly to some continuous function $g_k \in \mathcal{C}(X)$, so that $f_k(x) = g_k(x) - g_k(Tx)$, $x \in X$, showing that this f_k is a coboundary. It is also easy to see that the family $\mathcal{F} = \{f_k \colon k \in \mathbb{N}\}$ is point separating, which completes the "only if" part of the proof.

To prove the "if" part suppose $\mathcal{F} \subset \mathcal{C}(X)$ is a family with the above mentioned properties. Let $f \in \mathcal{F}$. Since f is a coboundary there is $g \in \mathcal{C}(X)$ with $f(x) = g(x) - g(Tx)$, $x \in X$. The equation

$$\sum_{k=0}^{n-1} f(T^k x) = g(x) - g(T^n x)$$

shows that the infinite sum $\sum_{k=0}^{\infty} f(T^k x)$ exists due to the fact that $f \geqslant 0$, g is continuous and the space X is compact. This implies that

$$\lim_{k\to\infty} f(T^k x) = 0, \ x \in X. \tag{2.1}$$

We will show that the orbit

$$\mathcal{O}(x) = x, Tx, T^2x, \ldots, T^n x, \ldots$$

converges for every $x \in X$. Assume that for some x it is not true. Then there are two distinct points, say $y_1 \neq y_2$ towards which two subsequences of $\mathcal{O}(x)$ converge. But then 1 implies that $f(y_1) = f(y_2) = 0$. Since f is an arbitrary element of $\mathcal{F}$ this implies that $\mathcal{F}$ fails to separate y_1 from y_2, a desired contradiction. This yields that the orbit $\mathcal{O}(x)$ converges and since $\lim[\mathcal{O}(x)] = T\lim[\mathcal{O}(x)]$ the limit is a fixed point, say x^*. From the equality

$$f(x^*) = g(x^*) - g(Tx^*) = 0$$

it follows that T cannot have more fixed points than one since otherwise $\mathcal{F}$ would not separate them. Thus T has a unique fixed point x^* toward which every orbit converges. We will show also that T has no periodic points. Suppose a point $x \in X$ has period $p \geqslant 2$, i.e. $T^p x = x$. Since

$$f(x) + f(Tx) + \ldots + f(T^{p-1}x) = g(x) - g(T^p x) = 0$$

we obtain that

$$f(x) + f(Tx) + \ldots + f(T^{p-1}x) = 0$$

implying that $f(T^k x) = 0$ for $k = 0, 1, \ldots, p-1$ which would again clash with the separation property of $\mathcal{F}$. This also shows how strong the argument of separation is.

In the final stage of our proof the function g appearing in $f(x) = g(x) - g(Tx)$ will play an important role. Note that it is not uniquely determined by f, since $g + c$, with c any constant, can replace g. Thus we can choose the function g which corresponds to $f \in \mathcal{F}$ by setting $g(x^*) = 0$. Since

$$\sum_{k=0}^{n-1} f(T^k x) = g(x) - g(T^n x)$$

and the orbit converges to x^* it follows that the infinite sum $\sum_{k=0}^{\infty} f(T^k x)$ converges to the function g.

One of our previous results (see [1]) is that T is a Banach contraction if and only if the core of T, $\operatorname{core}(T) = \bigcap\{T^n(X)\colon\ n \in \mathbb{N}\}$ is a singleton. It is known that the core is a nonempty compact T-invariant subset of X and that the restriction of T to the core is surjective. Thus, to conclude our proof we will assume that $|\operatorname{core}(T)| \geqslant 2$ and deduce from it a contradiction. Since the fixed point x^* is in the core there is in the core another point, say y, distinct from x^*. Since T maps the core onto itself every point in it has at least one pre-image in the core. Since there are no periodic points we can construct an inverse orbit from y, i.e. a sequence $\{y_k\}_{k=0}^{\infty}$ of distinct points with $Ty_{k+1} = y_k$, $k \in \mathbb{N}$. We observe that there must be at least one $f \in \mathcal{F}$ with $f(y) > 0$ since otherwise $\mathcal{F}$ would not separate y from x^*. Let g be the function corresponding to f and consider the sequence $\{g(y_k)_{k=0}^{\infty}\}$. Since $g(y_{k+1}) \geqslant g(y_k)$, $k \in \mathbb{N}$ and $g(x) \geqslant f(x)$ for every $x \in X$ the limit of this sequence exists and is positive. Let $m > 0$ be the limit. Let Z denote the set of all points in the core obtainable as limits of subsequences from $\{y_k\}_{k=0}^{\infty}$. It follows that for any $z \in Z$ we have $g(z) = m > 0$. We consider the orbit $\mathcal{O}(z)$. Let $\{y_{k(i)}\}$ where $k(1) < k(2) < \ldots \to \infty$ be the subsequence of $\{y_k\}$ the limit of which is z. Then Tz is obtained as the limit of $\{y_{k(i)-1}\}$. From this we see that $\mathcal{O}(z) \subset Z$ and the sequence $\{g(T^n z)\}_{n=0}^{\infty}$ is the sequence of constants $m > 0$. But since $\{T^n z\}$ converges to x^* we obtain the desired contradiction since $g(x^*) = 0$. Thus the core of T is the singleton $\{x^*\}$ which concludes the proof.

References

[1] *A. Bakakhanian*: Cohomological Methods in Group Theory. Marcel Dekker, New York, 1972.

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