ORDERED PRIME SPECTRA OF BOUNDED *DRl*-MONOIDS

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Abstract. Ordered prime spectra of Boolean products of bounded *DRl*-monoids are described by means of their decompositions to the prime spectra of the components.

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R. Cignoli and A. Torrens in [4] described the ordered prime spectrum of an *MV*-algebra which is a weak Boolean product of *MV*-algebras by means of the ordered spectra of those simpler algebras. In [8] and [9] it is shown that *MV*-algebras are in a one-to-one correspondence with *DRl*-monoids from a subclass of the class of bounded *DRl*-monoids. The boundedness of *DRl*-monoids leads to the fact that in any *MV*-algebra the ideals in the sense of *MV*-algebras coincide with those in the sense of *DRl*-monoids, and by [10], Proposition 4, the analogous relationship is also valid for the prime ideals.

In this paper we generalize the result of Cignoli and Torrens in [4] concerning the prime spectra of weak Boolean products of *MV*-algebras to bounded *DRl*-monoids.

Let us recall the notions of an *MV*-algebra and a *DRl*-monoid.

An algebra $A = (A, \oplus, \neg, 0)$ of signature $\langle 2, 1, 0\rangle$ is called an *MV-algebra* if it satisfies the following identities:

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(MV2) $x \oplus y = y \oplus x$;
(MV3) $x \oplus 0 = x$;
(MV4) $\neg\neg x = x$;
(MV5) $x \oplus \neg 0 = \neg 0$;

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(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$.

It is known that MV-algebras were introduced by C. C. Chang in [2] and [3] as an algebraic counterpart of the Łukasiewicz infinite valued propositional logic and that by D. Mundici [6] they can be viewed as intervals of commutative lattice ordered groups (l-groups) with a strong order unit.

If A is an MV-algebra, set $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$ for any $x, y \in A$. Then $(A, \vee, \wedge, 0, \neg 0)$ is a bounded distributive lattice and $(A, \oplus, \vee, \wedge)$ is a lattice ordered monoid (l-monoid).

An algebra $A = (A, +, 0, \vee, \wedge, -)$ of signature $\langle 2, 0, 2, 2, 2\rangle$ is called a *DRl-monoid* if it satisfies the following conditions:

(1) $(A, +, 0)$ is a commutative monoid;

(2) $(A, \vee, \wedge)$ is a lattice;

(3) $(A, +, 0, \vee, \wedge)$ is an l-monoid, i.e. A satisfies the identities

$$x + (y \vee z) = (x + y) \vee (x + z);$$
$$x + (y \wedge z) = (x + y) \wedge (x + z);$$

(4) if $\leqslant$ denotes the order induced by $(A, \vee, \wedge)$ then $x - y$ is the smallest element $z \in A$ such that $y + z \geqslant x$ for each $x, y \in A$;

(5) A satisfies the identity

$$((x - y) \vee 0) + y = x \vee y.$$

DRl-monoids were introduced by K. L. N. Swamy in [11], [12], [13] as a common generalization of, among others, commutative l-groups and Brouwerian and Boolean algebras. By [11], the DRl-monoids form a variety of algebras of signature $\langle 2, 0, 2, 2, 2\rangle$.

If A is a DRl-monoid then by [11], Theorem 2, the lattice $(A, \vee, \wedge)$ is distributive. Moreover, if there exists a greatest element 1 in A then by [5], Theorem 1.2.3, the lattice $(A, \vee, \wedge)$ is bounded also below and 0 is a least element.

Connections between MV-algebras and bounded DRl-monoids were described in [8] and [9]. In the sequel we will consider bounded DRl-monoids as algebras $A = (A, +, 0, \vee, \wedge, -, 1)$ of signature $\langle 2, 0, 2, 2, 2, 0\rangle$ enlarged by one nullary operation 1. Denote by $\mathcal{DR}l_{1(\mathrm{i})}$ the equational category of bounded DRl-monoids satisfying the condition

$$\text{(i)} \qquad 1 - (1 - x) = x$$

and by $\mathcal{MV}$ the equational category of MV-algebras. By [9], Theorem 3, the categories $\mathcal{DR}l_{1(\mathrm{i})}$ and $\mathcal{MV}$ are isomorphic.

If A is a bounded DRl-monoid and $\emptyset \neq I \subseteq A$ then I is called an *ideal* in A if

1. $\forall a, b \in I;\ a + b \in I$,
2. $\forall a \in I, x \in A;\ x \leqslant a \Longrightarrow x \in I$.

For any elements c and d of a DRl-monoid A set $c * d = (c - d) \vee (d - c)$. Then we have

Lemma 1. *Let A be a DRl-monoid and $I \in \mathcal{I}(A)$. If $a, b \in A$, $a * b \in I$ and $a \in I$, then $b \in I$.*

Proof. For any $a, b \in A$ we have

$$\begin{aligned} b &\leqslant ((a - b) + a) \vee b \leqslant ((a - b) + a) \vee ((b - a) + a) \\ &= ((a - b) \vee (b - a)) + a = (a * b) + a. \end{aligned}$$

Hence $a * b \in I$ and $a \in I$ imply $b \in I$. $\square$

The MV-algebra corresponding to a given DRl-monoid $A = (A, +, 0, \vee, \wedge, -)$ from $\mathcal{DR}l_{1(\mathrm{i})}$ is $(A, \oplus, \neg, 0)$, where $x \oplus y = x + y$ and $\neg x = 1 - x$ for any $x, y \in A$. Hence we have ([8]) that ideals in the mutually corresponding MV-algebras and DRl-monoids coincide. By [10] this is also true for prime ideals, and by [10], Propositions 4 and 5, prime ideals are in both types of algebras just finitely meet irreducible elements of the lattices of ideals. That means, if $\mathcal{I}(A)$ denotes the lattice of ideals of a bounded DRl-monoid A then $I \in \mathcal{I}(A)$ is a *prime ideal* in A if it satisfies

$$\forall J, K \in \mathcal{I}(A); J \cap K = I \Longrightarrow J = I \text{ or } K = I.$$

Equivalently, $I \in \mathcal{I}(A)$ is prime if and only if

$$\forall x, y \in A; x \wedge y \in I \Longrightarrow x \in I \text{ or } y \in I.$$

Let us denote by $\operatorname{Spec} A$ the prime spectrum of A, i.e. the set of all proper prime ideals of a DRl-monoid A. $\operatorname{Spec} A$ endowed with the spectral (i.e. hull-kernel) topology is a compact topological space by [7], Corollary 6.

Recall that a *weak Boolean product* (a *Boolean product*) of an indexed family (A_x; $x \in X$) of algebras over a Boolean space X is a subdirect product $A \leqslant \prod\limits_{x \in X} A_x$ such that

(BP1) if $a, b \in A$ then $[[a = b]] = \{x \in X;\ a(x) = b(x)\}$ is open (clopen);

(BP2) if $a, b \in A$ and U is a clopen subset of X, then $a|_U \cup b|_{X \setminus U} \in A$, where $\big(a|_U \cup b|_{X \setminus U}\big)(x) = a$ for $x \in U$ and $\big(a|_U \cup b|_{X \setminus U}\big) = b$ for $x \in X \setminus U$. (See [1] or [4].)

The following theorem makes it possible to compose the ordered prime spectrum of a weak Boolean product of bounded DRl-monoids from the prime spectra of the components of this product and so it is a generalization of Theorem 2.3 in [4] for MV-algebras.

Theorem 2. *Let a DRl-monoid A with a greatest element* 1 *be a weak Boolean product over a Boolean space X of a system $(A_x;\, x \in X)$ of DRl-monoids with greatest elements. Then the ordered prime spectrum* $(\operatorname{Spec} A, \subseteq)$ *is isomorphic to the cardinal sum of the ordered prime spectra* $(\operatorname{Spec} A_x, \subseteq)$, $x \in X$.

Proof. Let us denote $I_x = \{c \in A;\, c(x) = 0\}$ for any $x \in X$. Let $P \in \operatorname{Spec} A$ and let us suppose that $I_x \nsubseteq P$ for each $x \in X$. Then for any $x \in X$ there exists $b_x \in I_x \setminus P$. Obviously $X = \bigcup_{x \in X} [[b_x = 0]]$. Hence by condition (BP1) there is a finite subset $\{x_1, \ldots, x_n\} \subseteq X$ such that $X = \bigcup_{i=1}^{n} [[b_{x_i} = 0]]$. Since $0 \leqslant b_{x_1} \wedge \ldots \wedge b_{x_n} \leqslant b_{x_i}$ for each $i = 1, \ldots, n$, we have $[[b_{x_i} = 0]] \subseteq [[b_{x_1} \wedge \ldots \wedge b_{x_n} = 0]]$, hence $\bigcup_{i=1}^{n} [[b_{x_i} = 0]] \subseteq [[b_{x_1} \wedge \ldots \wedge b_{x_n} = 0]]$, and thus $b_{x_1} \wedge \ldots \wedge b_{x_n} = 0 \in P$. By the assumption P is prime, therefore $b_{x_k} \in P$ for some $k = 1, \ldots, n$, a contradiction. This implies that there exists at least one $x \in X$ such that $I_x \subseteq P$. We will show that such x is unique for P. For this, let $x, y \in X$, $x \neq y$, be such that $I_x \subseteq P$, $I_y \subseteq P$. The space X is Boolean, hence there is a clopen subset $V \subseteq X$ such that $x \in V$ and $y \in X \setminus V$. A is a subalgebra of $\prod_{x \in X} A_x$, thus $0 = (\ldots, 0, \ldots)$, $1 = (\ldots, 1, \ldots) \in A$. Hence by (BP2), $0|_V \cup 1|_{X \setminus V} \in A$, and so $0|_V \cup 1|_{X \setminus V} \in I_x \subseteq P$. Analogously $1|_V \cup 0|_{X \setminus V} \in I_y \subseteq P$. Moreover, $(0|_V \cup 1|_{X \setminus V}) + (1|_V \cup 0|_{X \setminus V}) = 1$, hence $1 \in P$, and therefore $P = A$, a contradiction.

Let us now set $H(I_x) = \{P \in \operatorname{Spec} A;\, I_x \subseteq P\}$ for any $x \in X$. Then from the preceding part it is clear that $(\operatorname{Spec} A, \subseteq)$ is isomorphic to the cardinal sum of the ordered sets $(H(I_x), \subseteq)$, $x \in X$. We will show that the ordered sets $(H(I_x), \subseteq)$ and $(\operatorname{Spec} A_x, \subseteq)$ are isomorphic for any $x \in X$.

Let $P \in H(I_x)$ and $\varphi_x(P) = \{c(x);\, c \in P\}$. We will show that $\varphi_x(P) \in \operatorname{Spec} A_x$. Since $P \in \mathcal{I}(A)$ and A is a subdirect product of A_x, it is obvious that $\varphi_x(P) \in \mathcal{I}(A_x)$.

Suppose $1 \in \varphi_x(P)$. Then there exists $c \in P$ such that $c(x) = 1$. Hence $(c * 1)(x) = 0$, thus $c * 1 \in I_x \subseteq P$. Moreover, $c \in P$, therefore $1 \in P$ by Lemma 1, a contradiction with $P \in \operatorname{Spec} A$. Hence $\varphi_x(P)$ is a proper ideal in A_x.

Let $v, z \in A_x$ and $v \wedge z \in \varphi_x(P)$. Then there exist $c, d \in A$ and $a \in P$ such that $c(x) = v$, $d(x) = z$ and $a(x) = v \wedge z = (c \wedge d)(x)$. Hence $((c \wedge d) * a)(x) = 0$, that means $(c \wedge d) * a \in I_x \subseteq P$, and since $a \in P$, we get $c \wedge d \in P$ by Lemma 1. Therefore $c \in P$ or $d \in P$, and so $v \in \varphi_x(P)$ or $z \in \varphi_x(P)$. That means $\varphi_x(P)$ is a prime ideal in A_x.

Therefore the assignment $\varphi_x\colon P \longmapsto \varphi_x(P)$ is a mapping from $H(I_x)$ into $\operatorname{Spec} A_x$.

Let $Q \in \operatorname{Spec} A_x$. Put $\psi_x(Q) = \{a \in A;\ a(x) \in Q\}$. Clearly $\psi_x(Q) \neq A$ and hence it is obvious that $\psi_x(Q)$ is a proper ideal in A. Moreover, $I_x \subseteq \psi_x(Q)$. Let $c, d \in A$ be such that $c \wedge d \in \psi_x(Q)$. Then $c(x) \in Q$ or $d(x) \in Q$, therefore $c \in \psi_x(Q)$ or $d \in \psi_x(Q)$. That means $\psi_x(Q) \in H(I_x)$.

From this we get that φ_x is a bijection of $H(I_x)$ onto $\operatorname{Spec} A_x$ and that $\psi_x = \varphi_x^{-1}$. Moreover, both the bijections respect set inclusion, hence they are order isomorphisms. $\square$

R e m a r k. It is obvious that the assertion of Theorem 2 can be modified for any subvariety of the variety $\mathcal{DR}l_1$ of all bounded DRl-monoids. For instance, it is valid for MV-algebras (see [4], Theorem 2.3) and Brouwerian algebras.

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