TWO CLASSES OF GRAPHS RELATED TO EXTREMAL ECCENTRICITIES

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Abstract. A graph G is called an S-graph, if its periphery $\mathrm{Peri}(G)$ is equal to its center eccentric vertices $\mathrm{Cep}(G)$. Further, a graph G is called a D-graph, if $\mathrm{Peri}(G) \cap \mathrm{Cep}(G) = \emptyset$.

We describe S-graphs and D-graphs for small radius. Then, for a given graph H and natural numbers $r \geqslant 2$, $n \geqslant 2$, we construct an S-graph of radius r having n central vertices and containing H as an induced subgraph. We prove an analogous existence theorem for D-graphs, too. At the end, we give some properties of S-graphs and D-graphs.

Keywords: eccentricity, central vertex, peripheral vertex

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1. Introduction

We consider here nonempty and finite graphs. Basic notions are as in [1] and [5] and we recall some of them. Let $d(u, v)$ denote the *distance* between vertices u, v of G and let $e(v)$ denote the *eccentricity* of a vertex v of G.

The *radius* of G, $r(G)$, is the minimum eccentricity, whereas the *diameter* of G, $d(G)$, is the maximum eccentricity. The *center* of G, $C(G)$, is the set of all vertices of minimum eccentricity and the *periphery* of G, $\mathrm{Peri}(G)$, is the set of all vertices of maximum eccentricity. The symbol $\langle C(G)\rangle$ will denote the subgraph induced by all central vertices of G. The vertex u is an *eccentric vertex* for v, if $d(v, u) = e(v)$. The vertex u is a *center eccentric vertex* of G if u is an eccentric vertex of some center vertex of G. The symbol $\mathrm{Cep}(G)$ denotes the set of all center eccentric vertices of G. (For every $x \in \mathrm{Cep}(G)$ there exists $c \in C(G)$ such that $d(x, c) = r(G)$.)

The short paper [5] posed the problem of further study of the following five classes of graphs:

a) A graph G is an *F-graph* if its center $C(G)$ contains at least two vertices and the distance between any two vertices of $C(G)$ equals $r(G)$ (i.e. maximum).

b) A graph G is an *L-graph* if its all paths of length $d(G)$ contain a center vertex; a graph G is an *L′-graph* if none of its paths of length $d(G)$ contains a center vertex.

c) A graph G is an *S-graph* if $\mathrm{Peri}(G) = \mathrm{Cep}(G)$; a graph G is a *D-graph* if $\mathrm{Peri}(G) \cap \mathrm{Cep}(G) = \emptyset$.

Paper [2] defines F-graphs, L-graphs and L'-graphs, discusses the possibilities of their applications and gives their basic properties. The paper [3] defines S-graphs and D-graphs, gives estimation of their diameter, studies embedding problems for these graphs which minimize either the diameter or the order of the host graph, and some other.

In this paper we describe S-graphs of radius one and S-graphs od radius $r \geqslant 2$ with one central vertex. Then we prove that for natural numbers $r \geqslant 2$, $n \geqslant 2$, $p \geqslant 2$ and any graph G with n vertices there exists an S-graph Q of radius r having n central vertices, p peripheral vertices, and containing G as an induced subgraph. From paper [3] it follows that no D-graph of radius one, two or three exists. We prove that for any graph G and natural numbers $r \geqslant 4$, $n \geqslant 1$ and $p \geqslant 2r - 6$ there exists a D-graph H of radius r having n central vertices, p peripheral vertices and containing G as an induced subgraph.

At the end we study some properties of S-graphs and D-graphs. E.g. we show that there exist a lot of graphs belonging to the intersection of either S-graphs and a unique eccentric vertex (u.e.v.) graphs or a D-graph and u.e.v. graphs.

(A graph G is a u.e.v. graph if every vertex u of G has exactly one eccentric vertex i.e. a vertex of distance $e(u)$). U.e.v. graphs were defined by Nandakumar and Parathasarathy, see [1], p. 37)

2. The Existence of S-graphs

Now we describe S-graphs of radius one and S-graphs with one central vertex by using Construction 1.

Construction 1. *Let $r \geqslant 1$, $n \geqslant 2$ be natural numbers. Let G_i, $i = 1, 2, \ldots, n$ be vertex disjoint graphs having at least one vertex v_i of eccentricity r. Let the graph H arise from graphs G_i by identification of all vertices v_i with one common vertex w.*

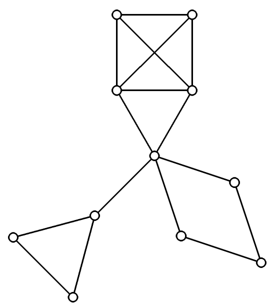

Fig. 1

Example of Construction 1 is in Fig. 1.

Theorem 2. *Let Q be a graph. Then Q is an S-graph of radius one if and only if Q is either a complete graph K_n, $n \geqslant 2$ or can be constructed according to Construction* 1, *for* $r = 1$.

P r o o f. If Q is either a complete graph K_n, $n \geqslant 2$ or can be constructed according to Construction 1, for $r = 1$, then it is clear that Q is an S-graph of radius one.

Let Q be an S-graph of radius one. Then Q contains at least one vertex w of eccentricity one. If Q is a complete graph K_n, $n \geqslant 2$ then the theorem holds.

Let Q be not a complete graph and let $G_1, G_2, \ldots, G_n$ be a maximal (with respect to the number of subgraphs) decomposition of Q into subgraphs G_i such that each G_i, $i = 1, 2, \ldots, n$ contains the vertex w and any two subgraphs G_r, G_s have only the vertex w in common.

Since Q is not complete, then $n \geqslant 2$, the graph Q fulfils the conditions of Construction 1 for $r = 1$, and the theorem holds. $\square$

Theorem 3. *Let $r \geqslant 2$ be a natural number and let Q be a graph. Then Q is an S-graph of radius r with one central vertex if and only if Q can be constructed according to Construction* 1.

P r o o f. If Q is constructed according to Construction 1, then it directly follows that $r(Q) = r, C(Q) = \{w\}$ and that Q is an S-graph.

Let Q be any S-graph of radius r having exactly one central vertex w. Let $G_1, G_2, \ldots, G_n$ be a maximal decomposition of Q into subgraphs G_i such that each G_i, $i = 1, 2, \ldots, n$ contains the vertex w and any two subgraphs G_r, G_s have only the vertex w in common.

It is clear that the eccentricity $e_{G_i}(w) \leqslant r$ for all i. Eccentricity of w in at least two subgraphs G_j, G_k is equal to r, because otherwise the vertex w would not be exactly one central vertex of Q.

If $e_{G_i}(w) < r$ for some i, then we unify the whole subgraph G_i with the subgraph G_j. In this way we decrease the number n, but in each new subgraph G_i we have $e_{G_i}(w) = r$. According to the previous, there are at least two new subgraphs G'_j, G'_k of Q and then Q has the required properties. The theorem follows. $\square$

The following Construction 4 is a generalization of Construction 1 and gives S-graphs, too.

Construction 4. *Let $r \geqslant 2$, $n \geqslant 2$ be natural numbers. Let K_n be a complete graph of n vertices $u_1, u_2, \ldots, u_n$. Let G_i, $i = 1, 2, \ldots, n$ be any graph that has one vertex v_i of eccentricity $r - 1$. Let all graphs K_n, $G_1, G_2, \ldots, G_n$ be mutually disjoint. Finally, let the graph H arise from the above mentioned graphs K_n and $G_1, G_2, \ldots, G_n$ by identifying only the vertices u_i and v_i for $i = 1, 2, \ldots, n$.*

Lemma 5. *Let $r \geqslant 2$, $n \geqslant 2$ be natural numbers. Let H be a graph constructed by Construction 4. Then the graph H is an S-graph of radius r, having exactly n central vertices.*

P r o o f. It follows directly from the Construction 4 that $r(H) = r$ and $\langle C(H) \rangle = K_n$. The graph H is also an S-graph because $\mathrm{Peri}(H) = \mathrm{Cep}(H)$. $\square$

The above described Construction 4 does not give all S-graphs as one can see from the following theorem.

Theorem 6. *Let $r \geqslant 2, n \geqslant 2, p \geqslant 2$ be natural numbers. Let G be any graph with n vertices. Then there exists an S-graph Q of radius r, containing n central vertices, p peripheral vertices and such that the central graph $G = \langle C(Q) \rangle$.*

P r o o f. As usual, the sequential join $G_1 + G_2 + \ldots + G_t$ of graphs $G_1, G_2, \ldots, G_t$ is formed from $G_1 \cup G_2 \cup \ldots \cup G_t$ by adding the additional edges joining each vertex of G_k with each vertex of G_{k+1}, $1 \leqslant k \leqslant t - 1$. Now the graph Q will be $Q = H_r + H_{r-1} + \ldots + H_1 + G + L_1 + L_2 + \ldots + L_r$, where

1) G is any graph with n vertices;
2) $H_1, \ldots, H_r$ and $L_1, \ldots, L_r$ are any (non-empty) graphs such that the union $H_r \cup L_r$ has exactly p vertices.

Directly from the construction it follows that $r(Q) = r$, $C(Q) = V(G)$, $\mathrm{Peri}(Q) = \mathrm{Cep}(Q) = H_r \cup L_r$ and $G = \langle C(Q) \rangle$. The theorem follows. $\square$

Paper [3] (Theorem 6) has shown that for every pair of positive integers a and b such that $a \leqslant b \leqslant 2a$, there exists an S-graph with radius a and diameter b. We generalize this result by similar methods. First we give two useful notions:

a) A path P_n, $n \geqslant 2$, contains n vertices and $n-1$ edges;

b) Let G, Q be disjoint graphs and $u \in V(Q)$. Then the graph H is a *substitution of G into Q instead of u*, if the vertex set $V(H) = (V(Q) - \{u\}) \cup V(G)$ and the edge set $E(H)$ consists of all edges of G, of all edges of Q which are not adjacent to u and of all edges which join a vertex of G with a vertex belonging to the neighbourhood of u in Q.

Theorem 7. *Let G be a graph. Let a, b be positive integers such that $a \leqslant b \leqslant 2a$ and $b \neq 1$. Then there exists an S-graph Q of radius a and diameter b, containing G as an induced subgraph.*

Proof. If $a = 1$ then $b = 2$. If $r(G) = 1$ then we insert the graph G in the path P_3 instead of its central vertex. If $r(G) > 1$ then we insert the graph G in the circuit C_3 instead of any of its vertex.

If $a \geqslant 2$ and $b = a$ then we insert the graph G in the cycle C_{2a} instead of any of its central vertices. If $a \geqslant 2$ and $b = 2a - 1$ then we insert the graph G in the path P_{2a} instead of any of its central vertices. If $a \geqslant 2$ and $b = 2a$, then we insert G in the path P_{2a+1} instead of its central vertex.

If $a \geqslant 2$ and $a < b < 2a - 1$ then the basic graph B consists of the cycle $C_{2(2a-b)}$ and the paths P_{b-a+1} such that from every vertex of the cycle there starts one path P_{b-a+1}. Then we insert the graph G in the graph B instead of any of its central vertices (i.e. any vertex from $C_{2(2a-b)}$).

One can easily see that in each case $r(Q) = a$ and $d(Q) = b$ and that Q is an S-graph. The theorem follows. □

Now we will study the existence of D-graphs.

3. The Existence of D-graphs

In paper [4] it was proved that for any D-graph G we have

$$r(G) + 2 \leqslant d(G) \leqslant 2r(G) - 2.$$

These inequalities imply that a D-graph of radius one, two or three does not exist. But there exist a lot of D-graphs of radius $r \geqslant 4$. We prove that for any graph G and natural numbers $r \geqslant 4$, $c \leqslant 1$, $p \geqslant 2r - 6$ there exists a D-graph H of radius r having c central vertices, p peripheral vertices and containing G as an induced subgraph.

We begin with a construction.

Construction 8. *Let $r \geqslant 4$ be a natural number. Let the graph H contain*

a) *a $(2r+1)$-gon formed by vertices $\{0,1,2,\dots,2r\}$;*

b) *vertices $\{0',2',\dots,(r-1)',(r+2)',\dots,(2r-1)'\}$;*

c) *edges (i,i') for every $i = 0,2,\dots,r-1,r+2,\dots,2r-1$ and two edges $((r-1)',r)$ and $(r+1,(r+2)')$.*

The graph H does not contain other vertices and edges.

An example of such a graph with $r = 6$ is in Fig. 2.

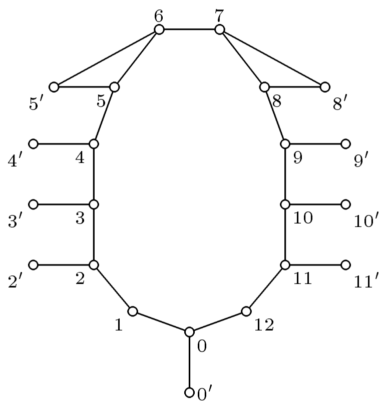

Fig. 2

Lemma 9. *Let $r \geqslant 4$ be a natural number. Then the graph H constructed by Construction 8 is a D-graph of radius r.*

P r o o f. As a useful illustration one can verify from Fig. 2 that $e(0) = 6$ and it is attained at the vertices $5', 6, 7, 8'$; the eccentricities of the vertices $2', 3', 4', 9', 10', 11'$ are equal to 8, and the eccentricities of the other vertices are equal to 7. So, for this case, $r(G) = 6$, $C(H) = \{0\}$, $\mathrm{Peri}(H) = \{2', 3', 4', 9', 10', 11'\}$, $\mathrm{Cep}(H) = \{5', 6, 7, 8',\}$ and Lemma 9 holds for $r = 6$.

One can see by verification of the general construction that the eccentricity $e(0) = r$ and it is attained at the vertices $(r-1)', r, r+1, (r+2)'$. The eccentricities of the vertices $x = 2', \dots, (r-2)', (r+3)', \dots, (2r-1)'$ are equal to $r+2$. The eccentricities of all other vertices are equal to $r+1$. Thus we have $r(H) = r, C(H) = \{0\}$, $\mathrm{Peri}(H) = \{2', \dots, (r-2)', (r+3)', \dots, (2r-1)'\}$, $\mathrm{Cep}(H) = \{(r-1)', r, r+1, (r+2)'\}$. So $\mathrm{Peri}(H) \cap \mathrm{Cep}(H) = \emptyset$ and Lemma 9 is proved. $\square$

Using the above described construction we prove the following existence theorem.

Theorem 10. *Let $r \geqslant 4$, $c \geqslant 1$, $p \geqslant 2r-6$, $k \geqslant 4$ be natural numbers. Let G be a graph with at least one vertex. Then there exists a D-graph Q of radius r having c central vertices, p peripheral vertices, k central peripheral vertices, and containing the graph G as an induced subgraph.*

Proof. Let H be a D-graph of radius r constructed by Construction 8. Then $|C(H)| = 1$, $|\operatorname{Cep}(H)| = 4$, and $|\operatorname{Peri}(H)| = (r-3)+(r-3) = 2r-6$. Now we will construct the graph Q from the graph H by some substitutions.

Let C, P and K be graphs with $c_1 \geqslant 1, p_1 \geqslant 1$ and $k_1 \geqslant 1$ vertices, respectively. Let us substitute succesively:

1) the graph G instead of the vertex $0'$ (i.e. we join every vertex of G with every neighbour of the vertex $0'$ (now the vertex 0));
2) the graph C instead of the vertex 0 (i.e. we join every vertex of C with vertices 1, 12 and vertices $V(G)$);
3) the graph P instead of the vertex $2'$,
4) the graph K instead of the vertex $(r-1)'$.

Let the graph Q contain no other vertices and edges.

One can verify directly from the construction of Q that:

1) the eccentricities of vertices $V(C)$ in the graph Q are the same as $e(0)$ in the graph H;
2) the eccentricities of $V(G)$ are the same as $e(0')$ in H;
3) the eccentricities of $V(P)$ are the same as $e(2')$ in H;
4) the eccentricities of $V(K)$ are the same as $e(r-1)'$ in H;
5) the eccentricities of the other vertices of Q are the same as in H.

So we have

1) G is an induced subgraph of G;
2) $C(Q) = C$;
3) $\operatorname{Peri}(Q) = \{V(P), 3', \ldots, (r-2)', (r+3)', \ldots, (2r-1)'\}$ and $|\operatorname{Peri}(Q)| \geqslant 2r-6$;
4) $\operatorname{Cep}(Q) = \{V(K), r, r+1, (r+2)'\}$ and $|\operatorname{Cep}(Q)| \geqslant 4$.

This completes the proof. □

Other examples of D-graphs are constructed in the paper [3], Theorem 3. The basic graph D_x, $x \geqslant 2$, is shown in Fig. 3, where the dashed paths P_x and P_{x+1}

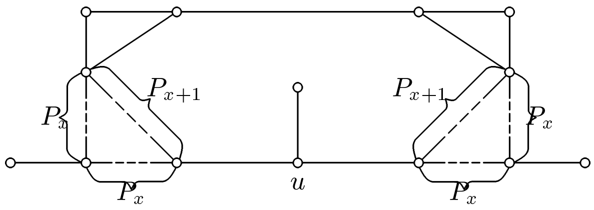

Fig. 3

contain x and $x+1$ vertices, respectively (so they have lengths $x-1$ and x). One can see that the vertex u is a central vertex of D_x. The diameter $d(D_x) = 2x+2$, but the radius $r(D_x)$ is not equal to $x+2$. (Hence the proof of Theorem 3 in [3] that for any positive integers a, b such that $a+2 \leqslant b \leqslant 2a-2$ there exists a D-graph of radius a and diameter b is not correct).

Now for a given D-graph G of radius $r \geqslant 2$ and for a graph Q we will construct a new D-graph H of radius r, containing Q as an induced subgraph.

Theorem 11. *Let G be a D-graph of radius $r \geqslant 2$ and $u \in C(G)$. Let Q be a graph. Then the substitution H of graph Q into G instead of u is a D-graph of radius r containing Q as an induced subgraph.*

Proof. Let G be a D-graph of radius $r \geqslant 2$ and $u \in C(G)$. Let Q be a graph. It is clear that the substitution H of Q into G instead of u contains G as an induced subgraph.

Let $a \in V(G) - \{u\}$. If a shortest path $P_H(a,b)$ does not contain a vertex $x \in V(Q)$, then this path is a shortest path in G, too. If a shortest path $P_H(a,b)$ contains a vertex $x \in V(Q)$, than this path contains exactly one vertex $\bar{x} \in V(Q)$ (because it is shortest) and then we can obtain (by substituting u instead of $\bar{x}$) a shortest path $P_H(a,b)$. So $e_H(a) = e_G(a)$.

If $P_H(x,y)$, where $x \in V(Q)$ is a shortest path in H, then this path contains exactly one vertex from Q, because P_H is a shortest path. Then from the path $P_H(x,y)$ we can obtain a new path $P_G(u,y)$ (by substituting u instead of x) that will be a shortest path in G. So $e_G(u) = e_H(x)$ for every $x \in V(Q)$. Then we have $r(H) = r(G)$ and $\mathrm{Peri}(G) = \mathrm{Peri}(H)$.

We can show that $\mathrm{Cep}(G) = \mathrm{Cep}(H)$ by analogous methods. So if G is a D-graph, then H is also a D-graph. The theorem follows. $\square$

4. Some properties of S-graphs and D-graphs

We will show that a self-centered graph is an S-graph as well. We will also show that there exist a lot of graphs arising as intersections of either unique eccentric vertex (u.e.v.) graphs and S-graphs or u.e.v. graphs and D-graphs.

A graph G is self-centered if all eccentricities of G have the same value (i.e. $r(G) = d(G)$). The self-centered graphs were studied in many papers, see [1]. These graphs are S-graphs as well.

Theorem 12. *Let G be a self-centered graph. Then G is an S-graph as well.*

P r o o f. If G is a self-centered graph, then $V(G) = C(G) = \mathrm{Peri}(G) = \mathrm{Cep}(G)$. So the theorem follows. $\square$

It is clear that S-graphs and D-graphs are disjoint. But the intersections of u.e.v. graphs with S-graphs and D-graphs are non-empty. Now we give two theorems about the intersections.

Theorem 13. *Let G be a graph with $n \geqslant 3$ vertices. Let $r \geqslant 3$, $k \geqslant 2n+2$ be given integers. Then there exists a regular graph Q of radius r, degree k, containing G as an induced subgraph and such that Q is a u.e.v. graph as well as an S-graph.*

P r o o f. The construction of the graph Q is identical with the construction described in [4], Theorem 4. Now we describe it in detail. As usual, $N_G(u)$ will denote the set of neighbour vertices (adjacent vertices) of u in G.

1) Let $r = 3$. We construct an overgraph G_1 of G in which for every vertex $u \in V(G_1)$ we have

$(*)$ for every $a \in N_{G_1}(u)$ there exists $b \in N_{G_1}(u)$ such that $(a,b) \notin E(G_1)$.

The graph G_1 is constructed by successively verifying the property $(*)$ and by joining one new vertex to every vertex not fulfilling the property $(*)$. It can be verified that $|V(G_1)| \leqslant 2n$. Let a graph G_2 arise from G_1 by adding $k - |V(G_1)|$ isolated vertices. Let a graph G_3 arise from G_2 by adding one new vertex joined with every vertex of G_2. One can see that $d(G_3) = 2$, $|V(G_3)| = k+1$ and the condition $(*)$ is valid for every vertex u of G_3. Let G_3' be a copy of G_3 and let u' be the vertex of G_3' corresponding to a vertex u of G_3. Finally, the graph Q is constructed as follows: $V(Q) = V(G_3) \cup V(G_3')$. The edge set of Q consists of $E(G_3)$, $E(G_3')$, and moreover, every vertex $x \in V(G_3)$ is adjacent to any vertex from $V(G_3') - (N_{G_3'}(x) \cup \{x'\})$.

It follows directly from the construction that Q is a regular graph of degree k and contains G as an induced subgraph.

Let $a \in V(G_3)$, $a' \in V(G_3')$. The inequality $d_Q(a,x) \leqslant 2$ is valid for every $x \in V(Q) - \{a'\}$ because either $x \in V(G_3)$ and then $d(a,x) \leqslant d(G_3) = 2$, or

$x \in V(G'_3) - N_{G'_3}(a')$ and then $(a, x) \in E(Q)$, or $x \in N_{G'_3}(a')$ and then we can verify that $d_Q(a, x) = 2$ by using the condition $(*)$. Analogously one can verify that $d_Q(a', x) \leqslant 2$ for every $x \in V(Q) - \{a\}$. Since $d_Q(a, a') = 3$, Q is a u.e.v. graph of radius 3. Moreover, Q is an S-graph of radius 3, because $V(Q) = C(Q) = \mathrm{Peri}(Q) = \mathrm{Cep}(Q)$.

2) Let $r \geqslant 4$, $s = r - 3$. Let us put $i = 1$ for $r = 4$ and $i = 2$ for $r \geqslant 5$. According to part 1 of this proof we construct the graph G_1 from G; then the graph G_2 from G_1 by adding $k - 1 - |V(G_1)|$ isolated vertices, and then the graph G_3 from G_2 by adding one new vertex adjacent to every vertex adjacent to every vertex of G_2. So $|V(G_3)| = k + 1 - i$. Finally, we construct the graph Q_i analogously as the graph Q in part 1 of this proof. So the graph Q_i is a regular graph of degree $k - 1$ that is a u.e.v. graph as well as an S-graph of radius 3.

Let $Q(4) = K_2 \times Q_1$ and let $Q(r) = C_{2s} \times Q_2$ for $r \geqslant 5$, where the symbol $\times$ denotes the Cartesian product, K_2 is the complete graph with two vertices and C_{2s}, $s \geqslant 2$ is the circuit with $2s$ vertices.

Then, according to the definition of the Cartesian product and to Lemma 1 of [4], the graph $Q(r)$, $r \geqslant 4$, is a regular graph of degree k, radius r, containing G as an induced subgraph and such that $Q(r)$ is a u.e.v. graph as well as an S-graph. The theorem follows. $\square$

Remark 14. The graph Q constructed in the previous theorem is a u.e.v. graph and an S-graph, but also a vertex-critical graph by radius (see [4]), and a maximal graph by radius (see Gliviak et al.: On radially maximal graphs, Australas. J. Combin. *9*, 1995, 275–284). Moreover, this graph Q is a base for the construction of D-graphs that are u.e.v. graphs.

Theorem 15. *Let G be a graph with $n \geqslant 3$ vertices. Let $r \geqslant 4, k \geqslant 2n + 2$ be given integers. Then there exists a graph H of radius $r + 1$ containing G as an induced subgraph and such that H is a u.e.v. graph as well as a D-graph and the degrees of its vertices are either $k + 1$ or 1.*

Proof. Let Q be a u.e.v. graph constructed to a given graph G and let $r - 1, k$ be numbers given according to Theorem 14. Let the graph H arise from the graph Q by adding one new vertex to every vertex of Q. (The illustration of this construction of H from Q is in Fig. 4.) It is clear that the eccentricity of every vertex $x \in V(Q)$ is r, the eccentricity of every $x \in V(H) - V(Q)$ is $r + 1$. Directly from the construction of Q it follows that its degrees are either $k + 1$ or 1.

All eccentricities of Q are attained at exactly one vertex of degree one. So H is a u.e.v. graph. Moreover, $C(H) = V(Q), \mathrm{Peri}(H) = \mathrm{Cep}(H) = V(H) - V(Q)$. So H is a D-graph. The theorem follows. $\square$

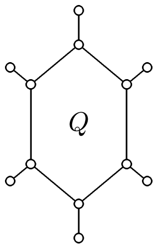

Fig. 4

In the end we give two problems for further study:

a) The description of intersection of either S-graphs and u.e.v. graphs or D-graphs and u.e.v. graphs, respectively. A similar problem was solved for the intersection of self-centered graphs and u.e.v. graphs by R. Nandakumar and K. R. Parathasarathy, see [1], Theorem 2.9, p. 40.

b) The lower and upper estimate of the number of edges of either S-graphs or D-graphs. Analogous lower and upper estimate of the number of edges of connected self-centered graphs was given by F. Buckley, see [1], Theorem 2.6, p. 38. Another example is the upper estimate of the number of edges of graphs with radius $r \geqslant 1$ and p vertices by V. G. Vizing, see [1], Theorem 5.1, p. 96.

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