SOME OSCILLATION CRITERIA FOR THE SECOND-ORDER LINEAR DELAY DIFFERENTIAL EQUATION

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Abstract. Some Wintner and Nehari type oscillation criteria are established for the second-order linear delay differential equation.

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1. Introduction

On the half-line $\mathbb{R}_+ = [0, +\infty[$ we consider the second-order linear delay differential equation

$$u''(t) + p(t)u(\tau(t)) = 0 \tag{1.1}$$

where $p\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a locally integrable function and $\tau\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that

$$\tau(t) \leqslant t \quad \text{for } t \geqslant 0, \qquad \lim_{t\to+\infty} \tau(t) = +\infty. \tag{1.2}$$

Oscillation theory for the linear second-order ordinary differential equation is a widely studied and well-developed topic of the general theory of differential equations. As for the results which are closely related to the results of this paper, we

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should mention, in particular, works of W. B. Fite, E. Hille, Z. Nehari, A. Wintner, and P. Hartman (see, e.g., [1], [3], [4], [10], [19]). These classical results were successfully extended to more general equations such as equations with p-Laplacian, difference equations, or equations on time-scales (see, e.g., [2], [5], [11], [15]–[18] and references therein). In this paper, some Wintner and Nehari type oscillation criteria known for the ordinary differential equations are generalized to the delay equation (1.1). We should also note that similar oscillation criteria for the differential equations with argument deviations and their systems can be found, e.g., in [6], [7], [9], [12], [14].

The following definitions introduce notions of proper oscillatory and non-oscillatory solutions of the equation (1.1) commonly used in literature.

Definition 1.1. Let $t_0 \in \mathbb{R}_+$ and $a_0 = \inf\{\tau(t)\colon\ t \geqslant t_0\}$. A continuous function $u\colon\ [a_0, +\infty[\to \mathbb{R}$ is said to be a proper solution of the equation (1.1) on the interval $[t_0, +\infty[$ if it is absolutely continuous together with its first derivative on every compact interval in $[t_0, +\infty[$, satisfies the equality (1.1) almost everywhere in $[t_0, +\infty[$, and $\sup\{|u(s)|\colon\ s \geqslant t\} > 0$ for $t \geqslant t_0$.

Definition 1.2. A proper solution u of the equation (1.1) is said to be oscillatory if it has a sequence of zeros tending to infinity, and non-oscillatory otherwise.

Oscillation criteria presented in this paper are proved by using the Riccati technique, which is well-developed in the case of ordinary differential equations. Having a proper non-oscillatory solution u of the equation (1.1) and putting $\varrho(t) = u'(t)/u(t)$ for t large enough, we get from the equality (1.1) that

$$\varrho'(t) = -p(t)\frac{u(\tau(t))}{u(t)} - \varrho^2(t) \quad \text{for large } t.$$

Therefore, in order to extend the Riccati technique to differential equations with argument deviations we need to find suitable lower and upper bounds of the quantity $u(\tau(t))/u(t)$, which is equal to 1 in the case of ordinary differential equations. One of such estimates is given in Lemma 3.1 below.

2. Main results

It is known (see, e.g., [13, §3]) that if the integral $\int_0^{+\infty} \tau(s)p(s)\,\mathrm{d}s$ is convergent, then the equation (1.1) has proper non-oscillatory solutions. Therefore, we will assume in the sequel that

$$\int_0^{+\infty} \tau(s)p(s)\,\mathrm{d}s = +\infty. \tag{2.1}$$

Theorem 2.1. *Let the condition* (2.1) *hold and*

$$\limsup_{t\to+\infty}\frac{1}{t}\int_0^t s\tau(s)p(s)\,\mathrm{d}s>1. \tag{2.2}$$

Then every proper solution of the equation (1.1) *is oscillatory.*

Remark 2.1. If the equation (1.1) is the ordinary one, i.e., if

$$\tau(t)=t\quad\text{for } t\geqslant 0, \tag{2.3}$$

then the condition (2.2) is a particular case of the oscillation criterion proved by Z. Nehari (see [10, Theorem III]).

Now let us put

$$G_*=\liminf_{t\to+\infty}\frac{1}{t}\int_0^t s\tau(s)p(s)\,\mathrm{d}s. \tag{2.4}$$

In view of Theorem 2.1, it is natural to suppose in what follows that

$$G_*\leqslant 1. \tag{2.5}$$

A Wintner type criterion is presented in the next theorem.

Theorem 2.2. *Let the conditions* (2.1) *and* (2.5) *be fulfilled, and let*

$$\liminf_{t\to+\infty}\frac{\tau(t)}{t}>0. \tag{2.6}$$

Moreover, let there exist $\lambda<1$ *such that*

$$\int_0^{+\infty}s^{\lambda}\Big(\frac{\tau(s)}{s}\Big)^{1-G_*}p(s)\,\mathrm{d}s=+\infty. \tag{2.7}$$

Then every proper solution of the equation (1.1) *is oscillatory.*

Remark 2.2. It is clear that if the condition (2.3) holds then the condition (2.6) is satisfied and the criterion (2.7) coincides with the well-known results (see E. Hille [4, Lemma 5]; see also A. Wintner [19] and W. B. Fite [1] for $\lambda=0$).

Finally, we give an oscillation criterion which generalizes a result of E. Müller-Pfeiffer proved for ordinary differential equations in the paper [8].

Theorem 2.3. *Let the conditions* (2.1), (2.5), *and* (2.6) *hold, and let*

$$\limsup_{t\to+\infty}\frac{1}{\ln t}\int_0^t s\Big(\frac{\tau(s)}{s}\Big)^{1-G_*}p(s)\,\mathrm{d}s>\frac{1}{4}. \tag{2.8}$$

Then every proper solution of the equation (1.1) *is oscillatory.*

Remark 2.3. The condition (2.6) in Theorems (2.2) and (2.3) is satisfied, in particular, if τ is a proportional delay, i.e., in the case where the equation (1.1) has the form

$$u''(t)+p(t)u(\alpha t)=0$$

with $0<\alpha\leqslant 1$.

3. Auxilliary statements

The next lemma contains a certain a priori estimate of non-oscillatory solutions of the equation (1.1), which plays a crucial role in the proofs of the main results.

Lemma 3.1. *Let* (2.1) *hold and let the equation* (1.1) *have a solution* u *such that*

$$\text{there exists } t_u>0 \text{ such that } u(t)>0 \quad \text{for } t\geqslant t_u. \tag{3.1}$$

Then

$$\limsup_{t\to+\infty}\frac{1}{t}\int_0^t s\tau(s)p(s)\,\mathrm{d}s\leqslant 1. \tag{3.2}$$

If, in addition, the inequality (2.6) *holds then*

$$\liminf_{t\to+\infty}\Big(\frac{t}{\tau(t)}\Big)^{1-G_*}\frac{u(\tau(t))}{u(t)}\geqslant 1, \tag{3.3}$$

where the number G_* *is defined by the relation* (2.4).

Proof. It is not difficult to verify that the inequality $u'(t)\geqslant 0$ holds for sufficiently large t. Since the equation (1.1) is homogeneous, we can assume without loss of generality that $u(t)\geqslant 1$ for sufficiently large t. Consequently, in view of the assumption (1.2), there exists $t_0\geqslant t_u$ such that

$$u'(t)\geqslant 0,\quad u(\tau(t))\geqslant 1 \quad \text{for } t\geqslant t_0. \tag{3.4}$$

It is clear that

$$(tu'(t) - u(t))' = -tp(t)u(\tau(t)) \quad \text{for a.e. } t \geqslant 0.$$

Integration of the latter inequality from t_0 to t yields

$$tu'(t) - u(t) = \delta - \int_{t_0}^{t} sp(s)u(\tau(s))\,\mathrm{d}s \quad \text{for } t \geqslant t_0, \tag{3.5}$$

where $\delta = t_0u'(t_0) - u(t_0)$.

Let $\varepsilon \in]0,1]$ be arbitrary but fixed. Then, in view of the assumption (2.1), there exists $t_1(\varepsilon) \geqslant t_0$ such that

$$\delta \leqslant \frac{\varepsilon}{2}\int_{t_0}^{t} sp(s)u(\tau(s))\,\mathrm{d}s \quad \text{for } t \geqslant t_1(\varepsilon).$$

Hence, it follows from the relation (3.5) that

$$tu'(t) - u(t) \leqslant -\Big(1-\frac{\varepsilon}{2}\Big)\int_{t_0}^{t} sp(s)u(\tau(s))\,\mathrm{d}s \leqslant 0 \quad \text{for } t \geqslant t_1(\varepsilon). \tag{3.6}$$

Therefore,

$$\Big(\frac{u(t)}{t}\Big)' = \frac{1}{t^2}(tu'(t) - u(t)) \leqslant 0 \quad \text{for } t \geqslant t_1(\varepsilon).$$

Using this inequality and the assumption (1.2) in the formula (3.6), we get the existence of $t_2(\varepsilon) \geqslant t_1(\varepsilon)$ such that

$$\begin{aligned} tu'(t) - u(t) &\leqslant -\Big(1-\frac{\varepsilon}{2}\Big)\int_{t_2(\varepsilon)}^{t} s\tau(s)p(s)\frac{u(\tau(s))}{\tau(s)}\,\mathrm{d}s \\ &\leqslant -\Big(1-\frac{\varepsilon}{2}\Big)\frac{u(t)}{t}\int_{t_2(\varepsilon)}^{t} s\tau(s)p(s)\,\mathrm{d}s \quad \text{for } t \geqslant t_2(\varepsilon). \end{aligned}$$

The last inequality implies, in particular, that

$$tu'(t) \leqslant u(t)\Big[1 - \Big(1-\frac{\varepsilon}{2}\Big)\frac{1}{t}\int_{t_2(\varepsilon)}^{t} s\tau(s)p(s)\,\mathrm{d}s\Big] \quad \text{for } t \geqslant t_2(\varepsilon). \tag{3.7}$$

Hence, in view of (3.1) and (3.4), we get

$$\frac{1}{t}\int_{t_2(\varepsilon)}^{t} s\tau(s)p(s)\,\mathrm{d}s \leqslant \frac{2}{2-\varepsilon} \quad \text{for } t \geqslant t_2(\varepsilon)$$

and therefore

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t s\tau(s)p(s)\,\mathrm{d}s \leqslant \frac{2}{2-\varepsilon}.$$

Since $\varepsilon \in \,]0,1]$ was arbitrary, the desired inequality (3.2) holds.

It remains to show the validity of the inequality (3.3). It follows from (2.4) that there exists $t_3(\varepsilon) \geqslant t_2(\varepsilon)$ such that

$$\frac{1}{t}\int_{t_3(\varepsilon)}^t s\tau(s)p(s)\,\mathrm{d}s \geqslant \Bigl(1-\frac{\varepsilon}{2}\Bigr)G_* \quad \text{for } t \geqslant t_3(\varepsilon).$$

By using this relation, from the inequality (3.7) we get

$$tu'(t)-u(t) \leqslant -\Bigl(1-\frac{\varepsilon}{2}\Bigr)u(t)\Bigl(1-\frac{\varepsilon}{2}\Bigr)G_* \leqslant -(1-\varepsilon)u(t)G_* \quad \text{for } t \geqslant t_3(\varepsilon),$$

and thus we have

$$\Bigl(\frac{u(t)}{t}\Bigr)' = \frac{1}{t^2}\bigl(tu'(t)-u(t)\bigr) \leqslant -\frac{(1-\varepsilon)G_*}{t}\frac{u(t)}{t} \quad \text{for } t \geqslant t_3(\varepsilon). \tag{3.8}$$

Notice that, in view of (1.2), there exists $t_4(\varepsilon) \geqslant t_3(\varepsilon)$ such that $\tau(t) \geqslant t_3(\varepsilon)$ for $t \geqslant t_4(\varepsilon)$. Consequently, from the inequality (3.8) we obtain

$$\ln\frac{u(t)/t}{u(\tau(t))/\tau(t)} \leqslant -(1-\varepsilon)G_* \ln\frac{t}{\tau(t)} \quad \text{for } t \geqslant t_4(\varepsilon).$$

On the other hand, by virtue of the assumption (2.6), there exists $t_5(\varepsilon) \geqslant t_4(\varepsilon)$ such that $\tau(t)/t \geqslant \alpha > 0$ for $t \geqslant t_5(\varepsilon)$ and therefore

$$\Bigl(\frac{t}{\tau(t)}\Bigr)^{1-G_*}\frac{u(\tau(t))}{u(t)} \geqslant \alpha^{\varepsilon G_*} \quad \text{for } t \geqslant t_5(\varepsilon).$$

Consequently, we have

$$\liminf_{t\to+\infty}\Bigl(\frac{t}{\tau(t)}\Bigr)^{1-G_*}\frac{u(\tau(t))}{u(t)} \geqslant \alpha^{\varepsilon G_*},$$

which, due to the arbitrariness of $\varepsilon \in \,]0,1]$, yields the validity of the desired inequality (3.3). □

Lemma 3.2. *Let u be a solution of the equation* (1.1) *satisfying* (3.1). *Then there exists a finite limit*

$$\lim_{t\to+\infty}\int_{t_u}^{t} s^{\lambda}\,\frac{u(\tau(s))}{u(s)}\,p(s)\,\mathrm{d}s$$

for all $\lambda<1$. Furthermore,

$$\limsup_{t\to+\infty}\frac{1}{\ln t}\int_{t_u}^{t} s\,\frac{u(\tau(s))}{u(s)}\,p(s)\,\mathrm{d}s\leqslant\frac{1}{4}. \tag{3.9}$$

Proof. Let us choose $\lambda<1$ and put $\varrho(t)=u'(t)/u(t)$ for $t\geqslant t_u$. Then the equality (1.1) yields that

$$\varrho'(t)=-p(t)\frac{u(\tau(t))}{u(t)}-\varrho^2(t)\quad\text{for } t\geqslant t_u.$$

Multiplying both sides of this equality by t^{λ} and integrating it from t_u to t, we get

$$\begin{aligned} t^{\lambda-1}\Big[t\varrho(t)-\frac{\lambda}{2}\Big]&=\delta_1-\frac{\lambda(2-\lambda)}{4(1-\lambda)}\frac{1}{t^{1-\lambda}}-\int_{t_u}^{t}s^{\lambda}\,\frac{u(\tau(s))}{u(s)}\,p(s)\,\mathrm{d}s\\ &\quad-\int_{t_u}^{t}s^{\lambda-2}\Big[s\varrho(s)-\frac{\lambda}{2}\Big]^2\,\mathrm{d}s\quad\text{for } t\geqslant t_u, \end{aligned}\tag{3.10}$$

where $\delta_1=t_u^{\lambda}\varrho(t_u)+\frac{1}{4}\lambda^2(1-\lambda)^{-1}t_u^{\lambda-1}$.

We first show that

$$\int_{t_u}^{+\infty}s^{\lambda-2}\Big[s\varrho(s)-\frac{\lambda}{2}\Big]^2\,\mathrm{d}s<+\infty. \tag{3.11}$$

Assume that, on the contrary, the integral in (3.11) is divergent. Then it follows from the relation (3.11) that, for some $t_1\geqslant t_u$, the inequality

$$t\varrho(t)-\frac{\lambda}{2}\leqslant-\frac{1}{2}t^{1-\lambda}\int_{t_u}^{t}s^{\lambda-2}\Big[s\varrho(s)-\frac{\lambda}{2}\Big]^2\,\mathrm{d}s<0\quad\text{for } t\geqslant t_1 \tag{3.12}$$

holds. Let us denote

$$x(t):=\int_{t_u}^{t}s^{\lambda-2}\Big[s\varrho(s)-\frac{\lambda}{2}\Big]^2\,\mathrm{d}s\quad\text{for } t\geqslant t_1.$$

Then, using the relation (3.12), we get

$$x'(t)=t^{\lambda-2}\Big[t\varrho(t)-\frac{\lambda}{2}\Big]^2\geqslant\frac{1}{4t^{\lambda}}\,x^2(t)\quad\text{for } t\geqslant t_1.$$

Therefore, integration of the last inequality from t_1 to t yields that $4(1-\lambda)/x(t_1)+t_1^{1-\lambda}\geqslant t^{1-\lambda}$ holds for $t\geqslant t_1$, which is a contradiction. The contradiction obtained proves the validity of the inequality (3.11).

Now the equality (3.10) can be rewritten to the form

$$\tag{3.13} \int_{t_u}^{t} s^{\lambda}\,\frac{u(\tau(s))}{u(s)}\,p(s)\,\mathrm{d}s=\delta_2-t^{\lambda}\varrho(t)-\frac{\lambda^2}{4(1-\lambda)}\frac{1}{t^{1-\lambda}} + \int_{t}^{+\infty} s^{\lambda-2}\Big[s\varrho(s)-\frac{\lambda}{2}\Big]^2\,\mathrm{d}s \quad \text{for } t\geqslant t_u,$$

where $\delta_2=\delta_1-\int_{t_u}^{+\infty}s^{\lambda-2}[s\varrho(s)-\lambda/2]^2\,\mathrm{d}s$. Consequently, we get

$$\tag{3.14} -\infty<\lim_{t\to+\infty}\int_{t_u}^{t} s^{\lambda}\,\frac{u(\tau(s))}{u(s)}\,p(s)\,\mathrm{d}s=\delta_2<+\infty$$

because, in view of the condition (3.6), the inequality $\varrho(t)\leqslant 1/t$ holds for large t.

It remains to show the validity of the relation (3.9). Multiplying both sides of the equality (3.13) by $t^{-\lambda}$, integrating it from t_u to t by parts, and using the above proved relation (3.14), we get

$$\int_{t_u}^{t} s\,\frac{u(\tau(s))}{u(s)}\,p(s)\,\mathrm{d}s\leqslant\delta_3+\frac{\lambda(2-\lambda)}{4}\ln t + \int_{t_u}^{t}\frac{1}{s}\Big(s\varrho(s)-\frac{\lambda}{2}\Big)\Big(1-\lambda-\Big[s\varrho(s)-\frac{\lambda}{2}\Big]\Big)\,\mathrm{d}s \quad \text{for } t\geqslant t_u,$$

where δ_3 is a suitable constant. Hence, in view of the relation $4x(1-\lambda-x)\leqslant(1-\lambda)^2$ for all $x\in\mathbb{R}$, it follows that

$$\int_{t_u}^{t} s\,\frac{u(\tau(s)}{u(s)}\,p(s)\,\mathrm{d}s\leqslant\delta_3+\frac{1}{4}\ln t \quad \text{for } t\geqslant t_u,$$

and thus the desired condition (3.9) is satisfied. □

4. Proofs of the main results

P r o o f of Theorem 2.1. Suppose that the assertion of the theorem does not hold. Then there exists a solution u of the equation (1.1) satisfying (3.1). According to Lemma 3.1, the relation (3.2) holds, which contradicts the assumption (2.2). □

P r o o f of Theorem 2.2. Suppose that the assertion of the theorem does not hold. Then there exists a solution u of the equation (1.1) satisfying (3.1). Let $\varepsilon \in \,]0,1[$ be arbitrary but fixed. According to Lemma 3.1, there exists $t_0 \geqslant t_u$ such that

$$\Big(\frac{t}{\tau(t)}\Big)^{1-G_*}\frac{u(\tau(t))}{u(t)} \geqslant 1-\varepsilon \quad \text{for } t \geqslant t_0, \tag{4.1}$$

and thus we have

$$\int_0^t s^\lambda \Big(\frac{\tau(s)}{s}\Big)^{1-G_*} p(s)\,\mathrm{d}s \leqslant \int_0^{t_0} s^\lambda \Big(\frac{\tau(s)}{s}\Big)^{1-G_*} p(s)\,\mathrm{d}s + \frac{1}{1-\varepsilon}\int_{t_u}^t s^\lambda \,\frac{u(\tau(s))}{u(s)}\, p(s)\,\mathrm{d}s \quad \text{for } t \geqslant t_0.$$

Hence, it follows from Lemma 3.2 that

$$\int_0^{+\infty} s^\lambda \Big(\frac{\tau(s)}{s}\Big)^{1-G_*} p(s)\,\mathrm{d}s < +\infty,$$

which contradicts the assumption (2.7). □

P r o o f of Theorem 2.3. Suppose that, on the contrary, the assertion of the theorem does not hold. Then there exists a solution u of the equation (1.1) satisfying (3.1). Let $\varepsilon \in \,]0,1[$ be arbitrary but fixed. According to Lemma 3.1, there exists $t_0 \geqslant t_u$ such that the relation (4.1) holds. It is easy to verify that

$$\frac{1}{\ln t}\int_0^t s\Big(\frac{\tau(s)}{s}\Big)^{1-G_*} p(s)\,\mathrm{d}s \leqslant \frac{1}{\ln t}\int_0^{t_0} s\Big(\frac{\tau(s)}{s}\Big)^{1-G_*} p(s)\,\mathrm{d}s + \frac{1}{(1-\varepsilon)\ln t}\int_{t_u}^t s\,\frac{u(\tau(s))}{u(s)}\, p(s)\,\mathrm{d}s \quad \text{for } t \geqslant t_0.$$

Using the condition (3.9) of Lemma 3.2, we get

$$\limsup_{t\to+\infty} \frac{1}{\ln t}\int_0^t s\Big(\frac{\tau(s)}{s}\Big)^{1-G_*} p(s)\,\mathrm{d}s \leqslant \frac{1}{4(1-\varepsilon)},$$

which, due to the arbitrariness of $\varepsilon \in \,]0,1[$, contradicts the assumption (2.8). □

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