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SOME RESULTS ON ORDER WEAKLY COMPACT OPERATORS

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Abstract. We establish some properties of the class of order weakly compact operators on Banach lattices. As consequences, we obtain some characterizations of Banach lattices with order continuous norms or whose topological duals have order continuous norms.

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1. Introduction and notation

Many authors gave necessary and sufficient conditions under which the norm of a Banach lattice or of its topological dual is order continuous. Recall that a norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, the sequence (x_α) converges to 0 for the norm $\|\cdot\|$ where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. Also, the order continuity of norms were used to characterize some interesting classes of Banach lattices. For example, it was proved that a Banach lattice E is reflexive if and only if the norms of its topological dual E' and of its topological bidual E'' are order continuous (Theorem 5.16 of Schaefer [7]).

We desire to give some new characterizations of the order continuity of the norm of a Banach lattice or of its topological dual by using the class of order weakly compact operators defined by Dodds [4]. Let us recall that an operator T from a vector lattice E into a Banach space F is said to be order weakly compact if for each $x \in E^+$, the subset $T([0,x])$ is a relatively weakly compact subset of F where $E^+ = \{x \in E\colon\ 0 \leqslant x\}$. For terminology and results concerning order weakly compact operators we refer the reader to [6].

An interesting problem for order weakly compact operators on Banach lattices is to study the duality problem. In fact, there exist order weakly compact operators whose dual operators are not order weakly compact, and conversely, there exist operators which are not order weakly compact but their dual operators are order weakly compact. The objective of this paper is to give some sufficient and necessary conditions under which a norm of a Banach lattice is order continuous by using duality approach in the class of order weakly compact operators. More precisely, we will prove that for a Banach lattice E, the adjoint operator T' from E' into E' of any order weakly compact operator T from E into E is order weakly compact if and only if E' has an order continuous norm. And conversely, if E is a Dedekind σ-complete Banach lattice, we will establish that each operator T from E into E is order weakly compact whenever its dual operator T' from E' into E' is order weakly compact if and only if E has an order continuous norm.

To state our results, we need to fix some notation and recall some definitions. A vector lattice E is an ordered vector space in which $\sup(x, y)$ exists for every $x, y \in E$. A subspace F of a vector lattice E is said to be a sublattice if for every pair of elements a, b of F the supremum of a and b taken in E belongs to F. A subset B of a vector lattice E is said to be solid if it follows from $|y| \leqslant |x|$ with $x \in B$ and $y \in E$ that $y \in B$ where $|x| = \sup(-x, x)$. An order ideal of E is a solid subspace. Let E be a vector lattice, then for each $x, y \in E$ with $x \leqslant y$, the set $[x, y] = \{z \in E\colon\ x \leqslant z \leqslant y\}$ is called an order interval. A subset of E is said to be order bounded if it is included in some order interval. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm has the following property: for each $x, y \in E$ such that $|x| \leqslant |y|$ we have $\|x\| \leqslant \|y\|$. If E is a Banach lattice, then its topological dual E' endowed with the dual norm and the dual order is also a Banach lattice. A Banach lattice E is called an AM-space if for each $x, y \in E$ such that $\inf(x, y) = 0$, we have $\|x + y\| = \max\{\|x\|, \|y\|\}$. The Banach lattice E is an AL-space if its topological dual E' is an AM-space. For example, the Banach lattice l^1 is an AL-space and the Banach lattice l^∞ is an AM-space. For unexplained terminology on Banach lattice theory we refer the reader to Zaanen [9].

2. Main results

We will use the term an *operator* $T\colon E \longrightarrow F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geqslant 0$ in F whenever $x \geqslant 0$ in E. We denote by $L(E, F)$ the space of all operators from E into F. An operator $T\colon E \longrightarrow F$ is regular if $T = T_1 - T_2$ where T_1, $T_2\colon E \longrightarrow F$ are positive operators. It is well known that each positive linear mapping on a Banach lattice is continuous.

Let us recall that if an operator $T\colon E \longrightarrow F$ between two Banach lattices is positive, then its adjoint $T'\colon F' \longrightarrow E'$ is likewise positive, where T' is defined by

$T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$. For terminology concerning positive operators, we refer the reader to [9].

Note that each weakly compact operator is order weakly compact, but an order weakly compact operator is not necessarily weakly compact. However, contrary to weakly compact operators [2], [8], the class of order weakly compact operators satisfies the domination problem as we prove in the following proposition:

Proposition 2.1. (i) *The subspace of order weakly compact operators is a two sided ideal in the space of all order bounded operators on Banach lattices.*

(ii) *Let E and F be two Banach lattices and let S and T be operators from E to F such that $0 \leqslant S \leqslant T$. If T is order weakly compact, then S is order weakly compact. In particular, if the modulus of T exists and is order weakly compact, then T is order weakly compact.*

Proof. (i) a. If T is a regular operator and S is an order weakly compact operator, the composed operator $S \circ T\colon E \longrightarrow F$ is order weakly compact. In fact, for each $x \in E^+$, the subset $T[0, x]$ is order bounded. Then $S \circ T[0, x]$ is weakly relatively compact in G. And hence $S \circ T$ is order weakly compact.

(i) b. If $T\colon E \longrightarrow F$ is an order weakly compact operator and $S\colon G \longrightarrow E$ is an order bounded operator then the product $T \circ S\colon G \longrightarrow F$ is order weakly compact. In fact, as $S[0, x]$ is order bounded for each $x \in G^+$, the subset $T \circ S[0, x]$ is weakly relatively compact in F, and hence $T \circ S$ is order weakly compact.

(ii) Let $x \in E^+$. Since T is order weakly compact, $T([0, x])$ is relatively compact in the topology $\sigma(F, F')$. It follows from [3], Theorem 1.2 that for each disjoint sequence (x_n) of $[0, x]$ we have $\|T(x_n)\| \longrightarrow 0$. Since $S(x_n) \leqslant T(x_n)$ for each n, then $\|S(x_n)\| \leqslant \|T(x_n)\|$ for each n, and hence $\|S(x_n)\| \longrightarrow 0$. This proves that S is order weakly compact.

In particular, since $-|T| \leqslant T \leqslant |T|$ we have $0 \leqslant T + |T| \leqslant 2|T|$. As $|T|$ is order weakly compact, it follows that $T + |T|$ is order weakly compact, and hence T is order weakly compact.

Remarks. (1) There exist Banach lattices E and F and an operator $T\colon E \longrightarrow F$ which is order weakly compact such that its modulus $|T|$ does not exist. In fact, let $T\colon L^1[0, 1] \longrightarrow c_0$ be the operator defined by

$$T(f) = \left(\int_0^1 f(t) r_n(t)\,\mathrm{d}t \right)_{n=1}^{\infty} \quad \text{for each } f \in L^1[0, 1]$$

where $\{r_n, n \in \mathbb{N}\}$ is the sequence of Rademacher functions on $[0, 1]$. It is clear that T is order weakly compact, but it fails to be regular (Exercise 2.8.E 2 of Meyer-Neiberg [6], p. 147) and hence the modulus of T does not exist.

(2) There exist Banach lattices E and F and a regular operator $T\colon E \longrightarrow F$ which is order weakly compact such that its modulus $|T|$ exists but it is not necessarily order weakly compact. In fact, we may take $E_n = (l^\infty, \|\cdot\|_n)$ that we supply with the norm

$$\|(\lambda_1, \lambda_2, \ldots)|_n = \|(\lambda_1, \lambda_2, \ldots)\|_\infty + n \cdot \limsup |\lambda_k| \quad \text{for each} \quad (\lambda_n) \in l^\infty.$$

Next, we consider the operator T from [5], Example 3.4 defined by

$$T\colon C[0,1] \longrightarrow E,\ f \longmapsto T(f) = \Big(\frac{1}{n} T_n(f)\Big)_{n \geqslant 1}$$

where $E = \{(x_n)\colon x_n \in E_n$ for each n and the sequence $(\|x_n\|_n)$ is bounded$\}$ that we equip with the norm $\|(x_n)\| = \sup\{\|x_n\|_n\colon n \in \mathbb{N}\}$, the operator $T_n\colon C[0,1] \longrightarrow E_n$ is defined by $(T_n(f))(p) = 2^n \int_{B_n} f(t) \cdot \sin(2\pi pt)\,\mathrm{d}t$ ($p \in \mathbb{N}$; $f \in C[0,1]$) and $B_n = [2^{-n}, 2^{-n+1}] \subset [0,1]$ for each $n \in \mathbb{N}$.

Since $\|T_n(f)\|_n = \|T_n(f)\|_\infty$ for all $f \in C[0,1]$, each operator T_n is order weakly compact and regular and its modulus $|T_n|$ exists and is given by

$$(|T_n|(f))(p) = 2^n \int_{B_n} f(t) \cdot |\sin(2\pi pt)|\,\mathrm{d}t \quad (p \in \mathbb{N};\ f \in C[0,1]).$$

Also, as E is a Dedekind complete AM-space, the operator T is order weakly compact and its modulus $|T|$ exists and is given by $|T|(f) = (\frac{1}{n}|T_n|(f))_{n \geqslant 1}$ (for $f \in C[0,1]$). But $|T|$ is not order weakly compact. Indeed, if $|T|$ were order weakly compact, since $C[0,1]$ is an AM-space with unit, $|T|$ would be weakly compact but this is false by [5], Example 3.4.

Recall from Meyer-Neiberg ([6], p. 47) that a Banach lattice E is said to have the property (P) if there exists a positive contractive projection $P\colon E'' \longrightarrow E$ where E is identified with a sublattice of its topological bidual E''.

A linear mapping $T\colon E \longrightarrow F$ between two vector lattices is called disjointness preserving if $|T(x)| \wedge |T(y)| = 0$ for all $x, y \in E$ satisfying $|x| \wedge |y| = 0$.

Now, we give some sufficient conditions under which each order weakly compact operator has a modulus which is order weakly compact.

Theorem 2.2. *Let E, F be Banach lattices and let $T\colon E \longrightarrow F$ be an order weakly compact operator. Then the modulus of T is order weakly compact if one of the following assertions is valid:*

(i) *E is an AL-space and F has the property* (P).

(ii) *T is an order bounded disjointness preserving operator.*

P r o o f. (i) If the Banach lattice E is an AL-space and the Banach lattice F has the property (P), then ([7], Theorem 1.5, Chapter IV) implies that $L^r(E,F) = L(E,F)$. On the other hand, since the norm of E is order continuous, the modulus $|T|\colon E \longrightarrow F$ is order weakly compact.

(ii) If $T\colon E \longrightarrow F$ is an order bounded disjointness preserving operator, then a theorem of Meyer-Neiberg ([6], Theorem 3.1.4) implies that $|T|$ exists and that $|T|(x) = |T(x)|$ for all $x \in E^+$. Now, let (x_n) be an order bounded sequence of E such that $x_n \longrightarrow 0$ in the weak topology $\sigma(E,E')$. Since T is order weakly compact, it follows from [6], Corollary 3.4.9 that $\|T(x_n)\| \longrightarrow 0$ as $n \longrightarrow \infty$. Finally, the equality $|T|(x) = |T(x)|$ for each $x \in E$ implies that $\||T|(x_n)\| = \||T(x_n)|\| = \|T(x_n)\| \longrightarrow 0$ as $n \longrightarrow \infty$. Hence, it results from [6], Corollary 3.4.9 that $|T|$ is order weakly compact.

Now, we give a necessary and sufficient condition under which each operator is order weakly compact.

Theorem 2.3. *Let E be a Banach lattice, then the following assertions are equivalent:*

(i) *each bounded linear operator $T\colon E \longrightarrow E$ is order weakly compact;*
(ii) *the identity operator of E is order weakly compact;*
(iii) *the norm of E is order continuous.*

P r o o f. (i) $\Longrightarrow$ (ii) is evident.

(ii) $\Longrightarrow$ (iii) Let $x \in E^+$. Since the identity operator $\mathrm{Id}_E\colon E \longrightarrow E$ is order weakly compact, $\mathrm{Id}_E([0,x]) = [0,x]$ is relatively compact in the topology $\sigma(E,E')$. But the order interval $[0,x]$ is weakly closed, thus $[0,x]$ is compact in the topology $\sigma(E,E')$. Finally, [1], Theorem 22.1 implies that the norm of E is order continuous.

(iii) $\Longrightarrow$ (i) Assume that the norm of E is order continuous. It follows from [1], Theorem 22.1 that for each $x \in E^+$, the subset $[0,x]$ is weakly relatively compact. Hence $T[0,x]$ is weakly relatively compact.

R e m a r k s 2.4. (1) It follows from Theorem 2.3 that the norm of the topological dual E' is order continuous if and only if its identity operator is order weakly compact.

(2) Let E and F be two Banach lattices. If each regular operator T from E into F is order weakly compact, then:

(i) The topological dual E' is not necessarily discrete. In fact, for $E = l^\infty$ and $F = c_0$, each operator from l^∞ to c_0 is order weakly compact but $E' = (l^\infty)'$ is not discrete.

(ii) The Banach lattice F is not necessarily discrete. In fact, for $E = F = L^1[0,1]$, each operator from $L^1[0,1]$ to $L^1[0,1]$ is order weakly compact but $E' = L^\infty[0,1]$ is not discrete.

(iii) The norm of E' is not necessarily order continuous. In fact, for $E = l^1$, each operator from l^1 to F is order weakly compact but the norm of $E' = l^\infty$ is not order continuous.

(iv) The norm of E is not necessarily order continuous. In fact, for $E = l^\infty$ and F finite-dimensional, each operator $T\colon E \longrightarrow F$ is compact and hence order weakly compact but the norm of l^∞ is not order continuous.

Recall that a vector lattice E is Dedekind σ-complete if every majorized countable nonempty subset of E has a supremum. In the sequel we give a sufficient condition under which the norm of a topological dual is order continuous.

As we established by examples, the order weakly compactness property of an operator is not inherited by its dual operator and conversely.

Our first result concerning the duality problem gives also a characterization of order continuity of the dual norm. In fact:

Theorem 2.5. *Let E be a Banach lattice. The following assertions are equivalent:*

(1) *the adjoint operator T' from E' into E' of an order weakly compact operator T from E into E is order weakly compact;*

(2) *each operator T from l^1 into E has an adjoint operator T' from E' into l^∞ which is order weakly compact;*

(3) *E' has an order continuous norm.*

Proof. (1) $\Longrightarrow$ (3) Assume that the norm of E' is not order continuous. It follows from [6], Theorem 2.4.14 and Proposition 2.3.11 that E contains a sublattice isomorphic to l^1 and there exists a positive projection $P\colon E \longrightarrow l^1$.

We consider the operator product

$$i \circ P\colon E \longrightarrow l^1 \longrightarrow E,$$

where $i\colon l^1 \longrightarrow E$ is the canonical injection. Since the identity operator $\mathrm{Id}_{l^1}\colon l^1 \longrightarrow l^1$ is order weakly compact, Proposition 2.1 implies that the operator

$$i \circ P = i \circ \mathrm{Id}_{l^1} \circ P\colon E \longrightarrow l^1 \longrightarrow l^1 \longrightarrow E$$

is also order weakly compact. However, its adjoint operator

$$P' \circ i'\colon E' \longrightarrow l^\infty \longrightarrow E'$$

is not order weakly compact. Indeed, if $P' \circ i'$ is order weakly compact, then the operator

$$P' \circ i' \circ P'\colon l^\infty \longrightarrow E' \longrightarrow l^\infty \longrightarrow E'$$

is also order weakly compact. And since the Banach lattice l^∞ is an AM-space with unit, the operator $P' \circ i' \circ P'\colon\ l^\infty \longrightarrow E'$ is weakly compact. Now, an application of Gantmacher Theorem proves that the operator product $P \circ i \circ P\colon\ E \longrightarrow l^1$ is weakly compact. And hence its restriction to the Banach lattice l^1, which is the identity operator Id_{l^1}, is weakly compact. This presents a contradiction, and hence the norm of E' is order continuous.

(2) $\Longrightarrow$ (3) If the norm of E' is not order continuous, it follows from [6], Theorem 2.4.14 and Proposition 2.3.11 that E contains a sublattice which is isomorphic to l^1 and there exists a positive projection $P\colon\ E \longrightarrow l^1$. Let us denote by $i\colon\ l^1 \longrightarrow E$ the canonical injection of l^1 into E. This operator i is order weakly compact but its adjoint operator $i'\colon\ E' \longrightarrow l^\infty$ is not order weakly compact. Indeed, if $i'\colon\ E' \longrightarrow l^\infty$ is order weakly compact, then the operator product $i' \circ P'\colon\ l^\infty \longrightarrow F' \longrightarrow l^\infty$, which is the identity operator of l^∞, is order weakly compact. And hence it follows from Theorem 2.3 that l^∞ has an order continuous norm. This gives a contradiction, and hence the norm of E' is order continuous.

(3) $\Longrightarrow$ (1) If E' has an order continuous norm, then by Theorem 2.3 each operator from E' into E' is order weakly compact, and hence the result follows.

(3) $\Longrightarrow$ (2) If E' has an order continuous norm, then by the same Theorem 2.3 the identity operator of E' is order weakly compact, and since $T' = \mathrm{Id}_{E'} \circ T'$, the result follows from Proposition 2.1.

R e m a r k 2.6. At the beginning of this paper we said that the class of order weakly compact operators has not the duality property. In fact, since the norm of l^1 is order continuous, it follows from Theorem 2.3 that the identity operator of the Banach lattice l^1 is order weakly compact. However, the identity operator of the Banach lattice l^∞ is not order weakly compact. If it were, the norm of l^∞ would be order continuous, and this is false. Also, as the norm of the topological dual $(l^\infty)'$ is order continuous, the identity operator of the Banach lattice $(l^\infty)'$ is order weakly compact, however, the identity operator of the Banach lattice l^∞ is not order weakly compact.

Our second result concerning duality gives also a characterization of the order continuity of the norm. In fact:

Theorem 2.7. *Let E be a Dedekind σ-complete Banach lattice. The following assertions are equivalent:*

(1) *Each operator T from E into E is order weakly compact whenever its dual operator T' from E' into E' is order weakly compact.*
(2) *Each operator from E into l^∞ is order weakly compact.*
(3) *E has an order continuous norm.*

P r o o f. (1) $\Longrightarrow$ (3) Assume that the norm of E is not order continuous. Since E is Dedekind σ-complete, it follows from [6], Corollary 2.4.3 that E contains a sublattice isomorphic to l^∞ and there exists a positive projection $P\colon E \longrightarrow l^\infty$. We denote by $i\colon l^\infty \longrightarrow E$ the canonical injection of l^∞ into E.

Consider the operator

$$T = i \circ P\colon E \longrightarrow l^\infty \longrightarrow E.$$

Since the identity operator $\mathrm{Id}_{(l^\infty)'}$ is order weakly compact, it follows from Proposition 2.1 (i) that the adjoint operator

$$T' = P' \circ i' = P' \circ \mathrm{Id}_{(l^\infty)'} \circ i'\colon E' \longrightarrow (l^\infty)' \longrightarrow (l^\infty)' \longrightarrow E'$$

is order weakly compact. However, the operator

$$T = i \circ P\colon E \longrightarrow l^\infty \longrightarrow E$$

is not order weakly compact. Indeed, i.e. if $i \circ P$ is order weakly compact, then the operator product

$$P \circ i \circ P\colon E \longrightarrow l^\infty \longrightarrow E \longrightarrow l^\infty$$

is order weakly compact. Hence its restriction to l^∞, which is the identity operator of l^∞, is also order weakly compact. But this is false. This shows that the norm of E is order continuous.

(2) $\Longrightarrow$ (3) Assume that the norm of E is not order continuous. Since E is Dedekind σ-complete, it follows from [[6], Corollary 2.4.3] that E contains a sublattice isomorphic to l^∞ and there exists a positive projection $P\colon E \longrightarrow l^\infty$.

The operator $P\colon E \longrightarrow l^\infty$ is not order weakly compact. If it were, its restriction to l^∞, which is the identity operator Id_{l^∞} of l^∞, would be order weakly compact. But this is false.

(3) $\Longrightarrow$ (1) If E has an order continuous norm, then by Theorem 2.3 each operator from E into E is order weakly compact, and hence the result follows.

(3) $\Longrightarrow$ (2) If E has an order continuous norm, then by the same Theorem 2.3 the identity operator of E is order weakly compact, and since $T = \mathrm{Id}_E \circ T$, the result follows from Proposition 2.1 (i).

Finally, we have

Theorem 2.8. *Let F be a Banach lattice. Then the following assertions are equivalent:*

(1) *Each order bounded operator from E into F is order weakly compact for each Banach lattice E.*

(2) *F has an order continuous norm.*

Proof. (2) $\Longrightarrow$ (1) It is just a consequence of Proposition 2.1 (i) and Theorem 2.3.

(1) $\Longrightarrow$ (2) If each order bounded operator from E into F is order weakly compact for each Banach lattice E, then the identity operator of the Banach lattice F is order weakly compact. Finally, it follows from Theorem 2.3 that F has an order continuous norm.

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