NOTE ON A DISCRETIZATION OF A LINEAR FRACTIONAL DIFFERENTIAL EQUATION

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Abstract. The paper discusses basics of calculus of backward fractional differences and sums. We state their definitions, basic properties and consider a special two-term linear fractional difference equation. We construct a family of functions to obtain its solution.

Keywords: fractional difference, fractional sum, discrete Mittag-Leffler function

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1. Introduction

The theory of fractional calculus (derivatives and integrals of non-integer orders) has a long tradition originating already from the correspondence between Leibniz and L'Hospital in 1695. Since that time, several alternative approaches have appeared discussing the question what meaning could be attributed to the m-th order derivative of a function f and m-th multiple integral of f provided m is not a positive integer. We mention here at least the Riemann-Louville fractional integral of a function f of order ν in the form

$$D^{-\nu}f(t)=\frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}f(s)\,\mathrm{d}s, \qquad t\in\mathbb{R}^+,$$

where ν is a positive real and f is piecewise continuous on $(0,\infty)$ and integrable on any bounded subinterval of $[0,\infty)$. This key (and broadly accepted) definition originates from the Cauchy formula for n-multiple integrals and can be employed also in the introduction of fractional derivatives. The two notions are the starting point for the theory of differential equations of non-integer orders (called fractional differential equations). General references to this topic are the books [7] and [10].

Contrary to the continuous case, the theory of fractional difference calculus (involving differences and sums of non-integer orders) is much less developed. The basic notions and properties of fractional differences can be found in [5], [6] and [9], while essentials of the theory of discrete fractional equations are the subject matter of recent papers [1], [2].

Our principal interest in this paper is to discuss and solve a special two-term linear fractional difference equation involving, together with an unknown function, also its backward h-difference of a (generally) non-integer order. The reason for considering backward (nabla) fractional differences instead of forward (delta) ones, as is done in [1] and [2], is two-fold. First, the resulting discrete equation involves an unknown dependent variable and its non-integer differences at the same values of arguments (for more details on domains of fractional delta differences we refer to [1] and [4]). Secondly, keeping in mind possible applications to numerical analysis of fractional differential equations, our scheme is motivated by the backward Euler method for numerical solution of classical ordinary differential equations, which is preferred to the forward Euler method especially because of its stability property.

The structure of this paper is as follows. In Section 2, we mention some necessary mathematical background concerning the calculus of backward fractional differences and sums. Section 3 presents some auxiliary results (especially the power rule) which turn out to be useful in the subsequent considerations. In the last section, we find eigenfunctions of a non-integer order difference operator, which enables us to solve a two-term linear nabla fractional difference equation.

2. Preliminaries

To make this paper self-contained, we give a brief survey of some related mathematical tools. Considering the h-calculus instead of the conventional difference calculus, we need to recall or introduce the corresponding modifications of some relevant functions which are of the key importance in the study of both the continuous and discrete fractional calculus. In particular, the well-known Euler Gamma function Γ satisfying the factorial equation $\Gamma(t+1) = t\Gamma(t)$ with the normalizing condition $\Gamma(1) = 1$ should be modified to satisfy

$$\Gamma_h(t+h) = t\Gamma_h(t), \qquad \Gamma_h(h) = 1. \tag{2.1}$$

It is known (see e.g. [8]) that assuming Γ_h is logarithmically convex, the properties involved in (2.1) uniquely define the h-generalization of the Γ function in the form

$$\Gamma_h(t) = \lim_{n\to\infty} \frac{n!\, h^n (nh)^{t/h-1}}{t(t+h)\dots(t+(n-1)h)}, \qquad t \neq -jh,\ j = 0,1,2,\dots. \tag{2.2}$$

Note that the Γ_h function can be related to the Γ function by using the property

$$\Gamma_h(t) = h^{t/h-1}\Gamma(t/h). \tag{2.3}$$

While the Γ function itself is not defined at non-positive integers, the formula

$$\frac{\Gamma(t+n)}{\Gamma(t)} = (-1)^n \frac{\Gamma(1-t)}{\Gamma(1-t-n)}, \qquad t \in \mathbb{R},\ n \in \mathbb{Z}^+ \tag{2.4}$$

enables us to calculate this ratio even when the individual Γ functions are infinite. Obviously, we can use this relation also for Γ_h by virtue of the property (2.3).

The next important notion of the discrete calculus is the raising factorial (sometimes called the power function). Its h-generalization is

$$s^{(m)} = \prod_{j=0}^{m-1} (s + jh), \qquad s \in \mathbb{R}.$$

An extension of the h-power function to non-integer exponents yields the Γ_h function. Using this function we can define

$$s^{(\nu)} = \frac{\Gamma_h(s+\nu h)}{\Gamma_h(s)} \tag{2.5}$$

whenever this ratio is well-defined (see (2.2) and the note following the relation (2.4)).

Let $h > 0$ be a real scalar and let $t_n = nh$, $n = 0, 1, 2, \ldots$ be the mesh points. Assume that f is a function defined on this mesh and put $f_n = f(t_n)$. Now we are able to introduce the nabla fractional sums and differences which can be taken for discrete analogues of the corresponding integral and differential operators known from the continuous fractional calculus. In particular, we extend the well-known backward h-difference operator

$$(\nabla_h f)(t_n) = \frac{f_n - f_{n-1}}{h}, \qquad n = 1, 2, \ldots \tag{2.6}$$

to operators of non-integer orders.

Definition 2.1. Let $\nu \in \mathbb{R}^+$. We define the nabla fractional sum of f at t_n by

$$(\nabla_h^{-\nu} f)(t_n) = h \sum_{k=1}^{n} \frac{(t_n - t_{k-1})^{(\nu-1)}}{\Gamma(\nu)} f_k.$$

Further, let $m \in \mathbb{Z}^+$ be such that $m-1 < \nu \leqslant m$. Then we define the nabla fractional difference of f at t_n by

$$(\nabla_h^{\nu} f)(t_n) = (\nabla_h^m \nabla_h^{-(m-\nu)} f)(t_n).$$

Finally, we put

$$(\nabla_h^0 f)(t_n) = f_n.$$

R e m a r k 2.1. The justification of the definition of the nabla fractional sum follows from the discrete Cauchy formula converting the m-th multiple sum into a single one.

Note that both nabla fractional operators can be calculated jointly in the form

$$(\nabla_h^{\nu} f)(t_n) = h^{-\nu} \sum_{k=0}^{n-1} (-1)^k \binom{\nu}{k} f_{n-k} \tag{2.7}$$

for any real ν except for $\nu \in \mathbb{Z}^+ \cup \{0\}$, where the upper bound $n-1$ has to be replaced by ν (see [4]).

3. Some auxiliary results

The h-power function $t_n^{(\mu)}$ is defined via the ratio of two Γ_h functions, i.e. with some restrictions on the parameter μ. However, the subject of our calculations in this section is the expression $t_n^{(\mu)}/\Gamma(\mu+1)$ which is essentially $\Gamma_h(t_n+\mu h)/(\Gamma(\mu+1)\Gamma_h(t_n))$. Consequently, considering (2.3) and (2.4), it is well-defined for all $\mu \in \mathbb{R}$.

Lemma 3.1. *Let $m \in \mathbb{Z}^+$ and $\mu \in \mathbb{R}$. Then for $n \geqslant m$*

$$\nabla_h^m \Big(\frac{t_n^{(\mu)}}{\Gamma(\mu+1)}\Big) = \begin{cases} \dfrac{t_n^{(\mu-m)}}{\Gamma(\mu-m+1)}, & \mu \notin \{0,1,\ldots,m-1\}, \\ 0, & \mu \in \{0,1,\ldots,m-1\}. \end{cases} \tag{3.1}$$

P r o o f. First let $m=1$. For $\mu=0$ we get $t_n^{(0)}=1$ and its first difference is zero. For $\mu \neq 0$ it follows from (2.5) and (2.6) that

$$\begin{aligned}
\nabla_h \Big(\frac{t_n^{(\mu)}}{\Gamma(\mu+1)}\Big) &= \frac{1}{h}\Big(\frac{t_n^{(\mu)}}{\Gamma(\mu+1)} - \frac{t_{n-1}^{(\mu)}}{\Gamma(\mu+1)}\Big) \\
&= \frac{1}{h}\Big(\frac{\Gamma_h(t_n+\mu h)}{\Gamma(\mu+1)\Gamma_h(t_n)} - \frac{\Gamma_h(t_n-h+\mu h)}{\Gamma(\mu+1)\Gamma_h(t_n-h)}\Big) \\
&= \frac{1}{h}\Big(\frac{(t_n-h+\mu h)\Gamma_h(t_n-h+\mu h)}{\Gamma(\mu+1)\Gamma_h(t_n)} - \frac{(t_n-h)\Gamma_h(t_n-h+\mu h)}{\Gamma(\mu+1)\Gamma_h(t_n)}\Big) \\
&= \frac{t_n^{(\mu-1)}}{\Gamma(\mu)},
\end{aligned}$$

i.e. (3.1) holds for $m = 1$. The verification for an arbitrary positive integer $m > 1$ can be simply done by the induction principle. $\square$

Lemma 3.2. *Let $\mu \in \mathbb{R}$. Then*

$$\frac{t_n^{(\mu)}}{\Gamma_h(\mu h + h)} = \frac{(\mu h + h)^{(n-1)}}{\Gamma_h(t_n)}. \tag{3.2}$$

P r o o f. It follows from the definition of the h-power function that

$$\frac{(nh)^{(\mu)}}{\Gamma_h(\mu h + h)} = \frac{\Gamma_h(nh + \mu h)}{\Gamma_h(\mu h + h)\Gamma_h(nh)} = \frac{(\mu h + h)^{(n-1)}}{\Gamma_h(nh)}.$$

$\square$

Lemma 3.3. *Let $n \in \mathbb{Z}^+ \cup \{0\}$ and $a, b \in \mathbb{R}$. Then the Newton binomial formula*

$$\sum_{k=0}^{n} \binom{n}{k} a^{(n-k)} b^{(k)} = (a + b)^{(n)} \tag{3.3}$$

holds.

P r o o f. The essential property employed in the proof of the formula (3.3) utilizes the correspondence between h-powers and binomial coefficients. More precisely, we have

$$\frac{a^{(n)}}{\Gamma_h(nh + h)} = \frac{\Gamma(a/h + n)}{\Gamma(a/h)\Gamma(n + 1)} = (-1)^n \frac{\Gamma(1 - a/h)}{\Gamma(1 - a/h - n)\Gamma(n + 1)} = (-1)^n \binom{-a/h}{n}$$

due to the properties (2.3) and (2.4).

Then

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} a^{(n-k)} b^{(k)} &= \Gamma(n + 1) \sum_{k=0}^{n} \frac{a^{(n-k)}}{\Gamma(n - k + 1)} \frac{b^{(k)}}{\Gamma(k + 1)} \\
&= \Gamma_h(nh + h) \sum_{k=0}^{n} \frac{a^{(n-k)}}{\Gamma_h(nh - kh + h)} \frac{b^{(k)}}{\Gamma_h(kh + h)} \\
&= (-1)^n \Gamma_h(nh + h) \sum_{k=0}^{n} \binom{-a/h}{n - k} \binom{-b/h}{k} \\
&= (-1)^n \Gamma_h(nh + h) \binom{-a/h - b/h}{n} = (a + b)^{(n)}.
\end{aligned}$$

$\square$

The next property is of the key importance for our further investigations.

Theorem 3.1. *Let $\nu \in \mathbb{R}^+$ and $\mu \in \mathbb{R}$. Then*

$$\nabla_h^{-\nu}\Big(\frac{t_n^{(\mu)}}{\Gamma(\mu+1)}\Big) = \frac{t_n^{(\mu+\nu)}}{\Gamma(\mu+\nu+1)}. \tag{3.4}$$

P r o o f. The formula (3.4) can be proved straightforwardly by use of Lemma 3.2 and Lemma 3.3. We have

$$\begin{aligned}
\nabla_h^{-\nu}\Big(\frac{t_n^{(\mu)}}{\Gamma(\mu+1)}\Big) &= h\sum_{k=1}^{n}\frac{(t_n-t_{k-1})^{(\nu-1)}}{\Gamma(\nu)}\frac{t_k^{(\mu)}}{\Gamma(\mu+1)}\\
&= h^{\mu+\nu}\sum_{k=1}^{n}\frac{t_{n-k+1}^{(\nu-1)}}{\Gamma_h(\nu h)}\frac{t_k^{(\mu)}}{\Gamma_h(\mu h+h)}\\
&= h^{\mu+\nu}\sum_{k=0}^{n-1}\frac{(\nu h)^{(n-k-1)}}{\Gamma_h(nh-kh)}\frac{(\mu h+h)^{(k)}}{\Gamma_h(kh+h)}\\
&= h^{\mu+\nu}\sum_{k=0}^{n-1}\frac{1}{\Gamma_h(nh)}\binom{n-1}{k}(\nu h)^{(n-k-1)}(\mu h+h)^{(k)}\\
&= h^{\mu+\nu}\frac{(\nu h+\mu h+h)^{(n-1)}}{\Gamma_h(nh)}\\
&= h^{\mu+\nu}\frac{t_n^{(\mu+\nu)}}{\Gamma_h(\mu h+\nu h+h)} = \frac{t_n^{(\mu+\nu)}}{\Gamma(\mu+\nu+1)},
\end{aligned}$$

where on the third and last row we utilized the property (3.2) and on the fifth row the property (3.3). $\square$

Applying Lemma 3.1 and Theorem 3.1, the validity of (3.4) can be easily extended also to fractional differences.

Corollary 3.1. *Let $\nu \in \mathbb{R}^+$, $\mu \in \mathbb{R}$ and $m \in \mathbb{Z}^+$ be such that $m-1<\nu\leqslant m$. Then for $n \geqslant m$*

$$\nabla_h^{\nu}\Big(\frac{t_n^{(\mu)}}{\Gamma(\mu+1)}\Big) = \begin{cases}\dfrac{t_n^{(\mu-\nu)}}{\Gamma(\mu-\nu+1)}, & \mu-\nu \notin \{-1,\dots,-m\},\\ 0, & \mu-\nu\in\{-1,\dots,-m\}.\end{cases} \tag{3.5}$$

P r o o f. Theorem 3.1 implies that

$$\nabla_h^{\nu}\Big(\frac{t_n^{(\mu)}}{\Gamma(\mu+1)}\Big) = \nabla_h^{m}\Big(\nabla_h^{-(m-\nu)}\Big(\frac{t_n^{(\mu)}}{\Gamma(\mu+1)}\Big)\Big) = \nabla_h^{m}\Big(\frac{t_n^{(\mu+m-\nu)}}{\Gamma(\mu+m-\nu+1)}\Big).$$

Now the formula (3.5) follows immediately from Lemma 3.1. $\square$

R e m a r k 3.1. Both the properties (3.4) and (3.5) can be derived jointly provided the unifying relation (2.7) is used instead of the corresponding definition formulae. From the computational viewpoint, this procedure is essentially equivalent to the above utilized proof technique. We note that the power rule (3.4) was already proved in [1], where the case of the delta fractional sums (with $h = 1$) was considered. Although the proof technique employed in [1] seems to be applicable also in our case, we emphasize that our procedure is quite different.

4. Two-term fractional difference equation and its solutions

In this section we introduce a family of special functions to find solutions of a basic two-term nabla fractional difference equation. To motivate this introduction, we consider the first order linear difference equation

$$\nabla_h f(t_n) = \lambda f(t_n), \qquad n = 1, 2, \dots. \tag{4.1}$$

It is well-known that $F(\lambda, t_n) = (1 - h\lambda)^{-n}$ defines the solution of (4.1) which is unique up to a multiplicative constant. The function F can be taken for a discrete (nabla) analogue of the exponential function $\exp(\lambda t)$ appearing in the continuous analysis. More generally, it is a special case of the generalized exponential function considered in the frame of the time scale theory (for more details see e.g. [3]).

However, we wish to present its different generalization originating from the fact that $F(\lambda, t_n)$ is the eigenfunction of the nabla difference operator ∇_h. Then the natural extension of $F(\lambda, t_n)$ can be obtained provided the equation (4.1) involving the operator ∇_h of arbitrary positive real order is considered. On this account, assuming $h|\lambda| < 1$ and applying the binomial formula we rewrite $F(\lambda, t_n)$ into

$$F(\lambda, t_n) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \prod_{i=0}^{k-1} (t_n + ih).$$

Now we consider the m-th order ($m \in \mathbb{Z}^+$) analogue of (4.1) in the form

$$\nabla_h^m f(t_n) = \lambda f(t_n), \qquad n = m, m+1, \dots. \tag{4.2}$$

Then the fundamental set of solutions of (4.2) is formed by m functions

$$E_{m,j}(\lambda, t_n) = \sum_{k=0}^{\infty} \frac{\lambda^k}{(mk+j-1)!} \prod_{i=0}^{mk+j-2} (t_n + ih), \qquad j = 1, \dots, m, \tag{4.3}$$

where we assume $h^m|\lambda| < 1$. This claim can be verified e.g. by substituting (4.3) into (4.2). Notice also that $E_{1,1}(\lambda, t_n) = F(\lambda, t_n)$. This motivates us to the following introduction: Let $\alpha, \beta, \lambda \in \mathbb{R}$, where $h^\alpha|\lambda| < 1$. Then we can define

$$E_{\alpha,\beta}(\lambda, t_n) = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k + \beta)} t_n^{(\alpha k+\beta-1)}$$

as a discrete analogue of the Mittag-Leffler function which plays a significant role in the theory of fractional differential equations. To justify this definition, we state the main property of $E_{\alpha,\beta}$.

Theorem 4.1. *Let $\alpha, \beta, \lambda \in \mathbb{R}$, $\nu \in \mathbb{R}^+$ and $m \in \mathbb{Z}^+$ be such that $m-1 < \nu \leqslant m$. Further, let $\alpha k + \beta - \nu \notin \{0, -1, \ldots, -m+1\}$ for all $k \in \mathbb{Z}^+$. Then for $n \geqslant m$*

$$\nabla_h^\nu E_{\alpha,\beta}(\lambda, t_n) = \begin{cases} E_{\alpha,\beta-\nu}(\lambda, t_n), & \beta - \nu \notin \{0, -1, \ldots, -m+1\}, \\ \lambda E_{\alpha,\beta-\nu+\alpha}(\lambda, t_n), & \beta - \nu \in \{0, -1, \ldots, -m+1\}. \end{cases} \tag{4.4}$$

P r o o f. We have

$$\nabla_h^\nu E_{\alpha,\beta}(\lambda, t_n) = \nabla_h^\nu \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k + \beta)} t_n^{(\alpha k+\beta-1)} = \sum_{k=0}^{\infty} \lambda^k \nabla_h^\nu \Big(\frac{t_n^{(\alpha k+\beta-1)}}{\Gamma(\alpha k + \beta)}\Big),$$

which follows from (2.7). For $k \in \mathbb{Z}^+$ Corollary 3.1 implies

$$\nabla_h^\nu \Big(\frac{t_n^{(\alpha k+\beta-1)}}{\Gamma(\alpha k + \beta)}\Big) = \frac{t_n^{(\alpha k+\beta-\nu-1)}}{\Gamma(\alpha k + \beta - \nu)} \tag{4.5}$$

due to the assumption $\alpha k + \beta - \nu \notin \{0, -1, \ldots, -m+1\}$. Considering $k = 0$, Corollary 3.1 allows two possible variants. If $\beta - \nu \notin \{0, -1, \ldots, -m+1\}$, we arrive at (4.5) with $k = 0$, i.e. the first part of (4.4) holds. If $\beta - \nu \in \{0, -1, \ldots, -m+1\}$, this term vanishes. After a shift of the index k we obtain the second part of (4.4). □

Let $\nu \in \mathbb{R}^+$, $m \in \mathbb{Z}^+$, $m - 1 < \nu \leqslant m$. Last we consider the difference equation

$$\nabla_h^\nu f(t_n) = \lambda f(t_n), \qquad n = m, m+1, \ldots \tag{4.6}$$

extending (4.2) to non-integer orders. As a consequence of previous considerations, we wish to find solutions of (4.6), i.e. eigenfunctions of the operator ∇_h^ν. On this account, we put $\alpha = \nu$ in Theorem 4.1 and rewrite the relation (4.4) into the form

$$\nabla_h^\nu E_{\nu,\beta}(\lambda, t_n) = \begin{cases} E_{\nu,\beta-\nu}(\lambda, t_n), & \beta - \nu \notin \{0, -1, \ldots, -m+1\}, \\ \lambda E_{\nu,\beta}(\lambda, t_n), & \beta - \nu \in \{0, -1, \ldots, -m+1\}. \end{cases} \tag{4.7}$$

Comparing (4.7) and (4.6) we can observe that the family of the functions

$$f(t_n) = E_{\nu,\beta}(\lambda, t_n), \quad \beta = \nu,\ \nu - 1, \ldots, \nu - m + 1 \tag{4.8}$$

forms the set of solutions of the equation (4.6). The question of their linear independence as well as possible extensions of our ideas and calculations to more general fractional difference equations will be the subject of future considerations.

Finally we note that the equation (4.6) can be taken for a discretization of the fractional differential equation

$$D^\nu f(t) = \lambda f(t), \qquad t \in \mathbb{R}^+, \tag{4.9}$$

where $\nu \in \mathbb{R}^+$, $D^\nu \equiv (\mathrm{d}^m/\mathrm{d}t^m)D^{-(m-\nu)}$ is a fractional derivative operator and m is a positive integer satisfying $m - 1 < \nu \leqslant m$. It is well-known (see e.g. [10]) that expressions for the solutions of (4.9) involve the Mittag-Leffler functions $E_{\alpha,\beta}(\lambda t^\alpha)$ with the parameters α, β acquiring the same values as in our discrete case, i.e. $\alpha = \nu$ and $\beta = \nu,\ \nu - 1, \ldots, \nu - m + 1$. If we understand by $h > 0$ the stepsize and by t_n the points of the uniform grid, it is interesting to observe that letting $h \to 0$ the functions (4.8) approach the corresponding exact solutions of (4.9).

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