Weak vs. strong solutions of problems in fluid mechanics

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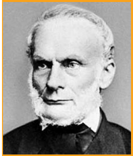
Dynamics of real fluids

No fluid is perfect

Real fluids are:

- three dimensional
- viscous
- heat conductive
- compressible
- obeying the basic laws of thermodynamics



DIE ENERGIE DER WELT IST CONSTANT;
DIE ENTROPIE DER WELT
STREBT EINEM MAXIMUM ZU

Rudolph Clausius, 1822-1888

Mathematical model

State variables

Mass density

$$\varrho = \varrho(t, x)$$

Absolute temperature

$$\vartheta = \vartheta(t, x)$$

Velocity field

$$\mathbf{u} = \mathbf{u}(t, x)$$

Thermodynamic functions

Pressure

$$p = p(\varrho, \vartheta)$$

Internal energy

$$e = e(\varrho, \vartheta)$$

Entropy

$$s = s(\varrho, \vartheta)$$

Transport

Viscous stress

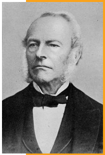
$$\mathbb{S} = \mathbb{S}(\vartheta, \nabla_x \mathbf{u})$$

Heat flux

$$\mathbf{q} = \mathbf{q}(\vartheta, \nabla_x \vartheta)$$

Field equations

Claude Louis Marie Henri Navier [1785-1836]



George Gabriel Stokes [1819-1903]

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \mathbf{f}$$

Entropy production

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

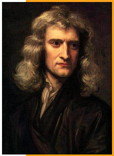
$$\sigma = (\geq) \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Constitutive relations

Joseph Fourier [1768-1830]

Fourier's law

$$\mathbf{q} = -\kappa(\vartheta)\nabla_x \vartheta$$

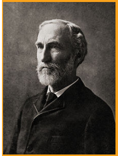


Isaac Newton [1643-1727]

Newton's rheological law

$$\mathbb{S} = \mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\right) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I}$$

Gibbs' relation



Willard Gibbs
[1839-1903]

Gibbs' relation:

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right)$$

Thermodynamics stability:

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Boundary conditions

Impermeability

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

No-slip

$$\mathbf{u}_{\tan}|_{\partial\Omega} = 0$$

No-stick

$$[\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

Thermal insulation

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Weak solutions to the complete system

- Equation of continuity holds in the sense of distributions (renormalized equation also satisfied)
- Momentum balance holds in the sense of distributions
- Entropy production equation holds in the sense of distributions, entropy production rate satisfies the inequality
- The system is augmented by

Total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right]\ \mathrm{d}x = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\ \mathrm{d}x$$

$$\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\ \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho F\ \mathrm{d}x \text{ if } \mathbf{f} = \nabla_x F(x)$$

Technical hypotheses

Pressure

$$p(\varrho,\vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4$$

$$P(0) = 0,\ P'(Z) > 0,\ P(Z)/Z^{5/3} \to p_\infty > 0 \text{ as } Z \to \infty$$

Internal energy

$$e(\varrho,\vartheta) = \frac{3}{2}\vartheta \frac{\vartheta^{3/2}}{\varrho} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{\varrho}\vartheta^4$$

Transport coefficients

$$\mu(\vartheta) \approx (1+\vartheta^\alpha),\ \alpha \in [1/2,1],\ \kappa(\vartheta) \approx (1+\vartheta^3)$$

Weak vs. strong solutions

Local existence of strong solutions

Strong (classical) solutions exist on a (possibly) short time interval for general data and globally for "small" data [Matsumura and Nishida, Valli and Zajaczkowski]

Global existence of weak solutions

Weak (distributional) solutions exist globally in time [EF and A.Novotný]

Weak strong uniqueness

A weak solution coincides with the strong solution emanating from the same initial data as long as the latter exists [EF and A.Novotný]

Conservative vs. dissipative system

Conservative character

$$\text{total mass } \int_\Omega \varrho(t,\cdot)\,\mathrm{d}x = M_0,$$

$$\text{total energy } \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F \right)(t,\cdot)\,\mathrm{d}x = E_0$$

Dissipative character

$$\text{total entropy } \int_\Omega \varrho s(\varrho,\vartheta)\,\mathrm{d}x = S(t) \nearrow S_\infty$$

Equilibrium solutions

Conservative driving force

$$\mathbf{f} = \nabla_x F, \ F = F(x)$$

Total energy conservation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) \ \mathrm{d}x = 0$$

Static solutions

$$\nabla_x p(\tilde\varrho, \overline{\vartheta}) = \tilde\varrho \nabla_x F, \ \overline{\vartheta} > 0 \text{ constant}$$

Total mass and energy

$$\int_\Omega \tilde\varrho \ \mathrm{d}x = M_0, \ \int_\Omega \left(\tilde\varrho e(\tilde\varrho, \overline{\vartheta}) - \tilde\varrho F \right) \ \mathrm{d}x = E_0$$

Total dissipation balance

Ballistic free energy

$$H_\Theta(\varrho,\vartheta) = \varrho\Big(e(\varrho,\vartheta) - \Theta s(\varrho,\vartheta)\Big)$$

Relative entropy

$$\mathcal{E}(\varrho,\vartheta,\mathbf{u}|\tilde\varrho,\overline{\vartheta})$$

$$= \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + H_{\overline{\vartheta}}(\varrho,\vartheta) - \partial_\varrho H_{\overline{\vartheta}}(\tilde\varrho,\overline{\vartheta})(\varrho-\tilde\varrho) - H_{\overline{\vartheta}}(\tilde\varrho,\overline{\vartheta})\right)\ \mathrm{d}x$$

Total dissipation balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\varrho,\vartheta,\mathbf{u}|\tilde\varrho,\overline{\vartheta}) + \int_\Omega \sigma\ \mathrm{d}x = 0$$

$\tilde\varrho,\ \overline{\vartheta}\ -$ equilibrium state

Thermodynamic stability

Positive compressibility and specific heat

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \quad \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Coercivity of the ballistic free energy

$$\varrho \mapsto H_\Theta(\varrho, \Theta) \text{ strictly convex}$$

$$\vartheta \mapsto H_\Theta(\varrho, \vartheta) \text{ decreasing for } \vartheta < \Theta \text{ and increasing for } \vartheta > \Theta$$

Long-time behavior for conservative driving forces

$$\mathbf{f} = \nabla_x F, \ \ F = F(x)$$

$$\varrho(t,\cdot) \to \tilde{\varrho} \text{ in } L^{5/3}(\Omega) \text{ as } t \to \infty$$

$$\vartheta(t,\cdot) \to \overline{\vartheta} \text{ in } L^{4}(\Omega) \text{ as } t \to \infty$$

$$(\varrho \mathbf{u})(t,\cdot) \to 0 \text{ in } L^{1}(\Omega; R^3) \text{ as } t \to \infty$$

Hypotheses

$$\int_\Omega \varrho(t,\cdot)\ \mathrm{d}x > M_0,\ \ t > 0$$

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F \right)(t,\cdot)\ \mathrm{d}x < E_0,\ \ t > 0$$

$$\int_\Omega \varrho s(\varrho,\vartheta)(t,\cdot)\ \mathrm{d}x > S_0,\ \ t > 0$$

Conclusion

$$\|\varrho(t,\cdot) - \tilde{\varrho}\|_{L^{5/3}(\Omega)} < \varepsilon,\ \ \|\vartheta(t,\cdot) - \overline{\vartheta}\|_{L^4(\Omega)} < \varepsilon \text{ for } t > T(\varepsilon)$$

$$\|\varrho\mathbf{u}(t,\cdot)\|_{L^1(\Omega;R^3)} < \varepsilon \text{ for } t > T(\varepsilon)$$

Uniform decay of density oscillations

$$\partial_t \varrho_\varepsilon + \mathbf{u}_\varepsilon \cdot \nabla_x \varrho_\varepsilon = -\mathrm{div}_x \mathbf{u}_\varepsilon \ \varrho_\varepsilon$$

$$\varrho_\varepsilon \to \varrho, \ \varrho_\varepsilon \log(\varrho_\varepsilon) \to \overline{\varrho \log(\varrho)} \text{ weakly in } L^1$$

$$d(t) = \int_\Omega \left(\overline{\varrho \log(\varrho)} - \varrho \log(\varrho) \right)(t, \cdot) \ \mathrm{d}x$$

Density oscillations decay

$$\partial_t d(t) + \Psi(d(t)) \leq 0$$

$$\Psi(0) = 0, \ \Psi(d) > 0 \text{ for } d > 0.$$

General time-dependent driving forces

$$\mathbf{f} = \mathbf{f}(t,x), \ |\mathbf{f}(t,x)| \leq \overline{F}$$

EITHER

$$E(t) \equiv \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right)(t,\cdot) \ \mathrm{d}x \to \infty \text{ as } t \to \infty$$

OR

$$|E(t)| \leq E \text{ for a.a. } t > 0$$

In the case $E(t) \leq E$, each sequence of times $\tau_n \to \infty$ contains a subsequence such that

$$\mathbf{f}(\tau_n + \cdot, \cdot) \to \nabla_x F \text{ weakly-(*) in } L^\infty((0,1) \times \Omega),$$

where $F = F(x)$ may depend on $\{\tau_n\}$

STEP 1:

Assume that $E(\tau_n) < E$ for certain $\tau_n \to \infty \Rightarrow$ total entropy remains bounded $\Rightarrow$ integral of entropy production bounded

STEP 2:

For $\tau_n \to \infty$ we have $\nabla_x p(\varrho, \vartheta) \approx \varrho \mathbf{f}$, $\vartheta \approx \overline{\vartheta}$, meaning, $\mathbf{f} \approx \nabla_x F$

STEP 3:

The energy cannot "oscillate" since bounded entropy *static solutions* have bounded total energy

Corollaries

- $$\mathbf{f} = \mathbf{f}(x) \neq \nabla_x F$$
  $$\Rightarrow$$
  $$E(t) \to \infty$$

- $$\mathbf{f} = \mathbf{f}(t,x) \ \text{(almost) periodic in time}, \ \mathbf{f} \neq \nabla_x F, \ F = F(x)$$
  $$\Rightarrow$$
  $$E(t) \to \infty$$

Rapidly oscillating driving forces

Hypotheses:

$$\mathbf{f} = \omega(t^\beta)\mathbf{w}(x), \mathbf{w} \in W^{1,\infty}(\Omega; R^3), \ \beta > 2$$

$$\omega \in L^\infty(R), \ \sup_{\tau > 0} \left| \int_0^\tau \omega(t) \ {\rm d}t \right| < \infty$$

Conclusion:

$$(\varrho \mathbf{u})(t, \cdot) \to 0 \text{ in } L^1(\Omega; R^3) \text{ as } t \to \infty$$

$$\varrho(t, \cdot) \to \overline{\varrho} \text{ in } L^{5/3}(\Omega) \text{ as } t \to \infty$$

$$\vartheta(t, \cdot) \to \overline{\vartheta} \text{ in } L^4(\Omega) \text{ as } t \to \infty$$

Rapidly oscillating growing driving forces

Hypotheses:

$$\mathbf{f} = t^{\delta}\omega(t^{\beta})\mathbf{w}(x), \mathbf{w} \in W^{1,\infty}(\Omega; R^3)$$

$$\boxed{\delta > 0,\ \ \beta - 2\delta > 2 \text{ or } \delta \leq 0,\ \ \beta - \delta > 2}$$

$$\omega \in L^{\infty}(R),\ \sup_{\tau > 0} \left| \int_0^{\tau} \omega(t)\ \mathrm{d}t \right| < \infty$$

Conclusion:

$$(\varrho\mathbf{u})(t, \cdot) \to 0 \text{ in } L^1(\Omega; R^3) \text{ as } t \to \infty$$

$$\varrho(t, \cdot) \to \overline{\varrho} \text{ in } L^{5/3}(\Omega) \text{ as } t \to \infty$$

$$\vartheta(t, \cdot) \to \overline{\vartheta} \text{ in } L^4(\Omega) \text{ as } t \to \infty$$

Dissipative boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \ \mathbf{q} \cdot \mathbf{n} = d(x)(\vartheta - \tilde{\vartheta})$$

Time periodic forcing

$$\mathbf{f}(t+\omega, \cdot) = \mathbf{f}(t, \cdot)$$

Time periodic solutions

$$\varrho(t+\omega, \cdot) = \varrho(t, \cdot), \ \vartheta(t+\omega, \cdot) = \vartheta(t, \cdot), \ \mathbf{u}(t+\omega, \cdot) = \mathbf{u}(t, \cdot)$$