Sobolev embeddings and interpolations

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This is a second iteration of a text, which is intended to be an introduction into Sobolev embeddings and interpolations. The goal is to show the main ideas of the proofs, so that the reader can derive himself/herself particular formulas in cases that are not explicitly treated in textbooks. Hence, emphasis is put on methods rather than on a collection of results. Some additional information can also be found in [1, 2, 3, 4]. Note that in fact, the only tool behind all the estimates is the elementary Hölder inequality.

Contents

1 Preliminaries

The word *embedding* is used in the situation of two Banach spaces U and V, endowed with respective norms $\|\cdot\|_U$ and $\|\cdot\|_V$, and such that

$$\left.\begin{array}{l} V \subset U\,, \\ \exists C > 0 \ \ \forall v \in V: \ \|v\|_U \le C\|v\|_V\,. \end{array}\right\} \tag{1.1}$$

If (1.1) holds, then we say that V *is embedded in* U.

The embedding is said to be *compact*, if every bounded set $A \subset V$ is *precompact* in U, that is,

$$\forall \varepsilon > 0 \ \ \exists a_1, \dots, a_n \in A \ \ \forall a \in A \ \ \exists k \in \{1, \dots, n\}: \ \ \|a - a_k\|_U < \varepsilon\,. \tag{1.2}$$

The following theorem represents a basic tool in the theory of compact embeddings in function spaces.

Theorem 1.1 ***(Arzelà-Ascoli)*** *Let X, Y be Banach spaces and let $A \subset X, B \subset Y$ be compact sets. Let $C(A;B)$ be the Banach space of all continuous mappings from A into B. Let $K \subset C(A;B)$ be an* equicontinuous *set, that is,*

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall f \in K \;\; \forall x, y \in A : \|x - y\|_X < \delta \Rightarrow \|f(x) - f(y)\|_Y < \varepsilon\,.$$

Then K is compact in $C(A;B)$. Conversely, every relatively compact set in $C(A;B)$ is equicontinuous.

Proof. Let $K \subset C(A;B)$ be equicontinuous, and let $\varepsilon > 0$ be given. We find $\delta > 0$ such that for all $f \in K$ we have $\|f(x) - f(y)\|_Y < \varepsilon/4$ whenever $\|x - y\|_X < \delta$. Since A is compact, there exist $x_1, \dots, x_p \in A$ such that for every $x \in A$ there exists $i \in \mathcal{I} := \{1, \dots, p\}$ such that $\|x - x_i\|_X < \delta$. Furthermore, B is compact, hence there exist $y_1, \dots, y_q \in B$ such that for every $y \in B$ there exists $j \in \mathcal{J} := \{1, \dots, q\}$ such that $\|y - y_j\|_Y < \varepsilon/4$.

For $z \in \mathcal{J}^p$, $z = \{z_1, \dots, z_p\}$, we now denote

$$K_z \;=\; \Big\{f \in K\,;\, \forall i \in \mathcal{I}:\; \|f(x_i) - y_{z_i}\|_Y < \frac{\varepsilon}{4}\Big\}\,.$$

Set $M := \{z \in \mathcal{J}^p \,:\, K_z \neq \emptyset\}$. The set M is indeed finite and we have $K = \bigcup_{z \in J} K_z$, hence we may fix one representative $f_z \in K_z$ for each $z \in J$. For any $f \in K_z$ and $x \in A$ we find x_i such that $\|x - x_i\|_X < \delta$, and estimate

$$\begin{aligned}\|f(x) - f_z(x)\|_Y &\leq \|f(x) - f(x_i)\|_Y + \|f(x_i) - y_{z_i}\|_Y + \|f_z(x_i) - y_{z_i}\|_Y + \|f_z(x) - f_z(x_i)\|_Y \\ &< \varepsilon\,,\end{aligned}$$

which we wanted to prove. Since every finite set of mappings in $C(A;B)$ is equicontinuous, the fact that that relatively compact sets are equicontinuous follows easily. ■

2 Admissible domains

We fix an open connected bounded set $\Omega \subset \mathbb{R}^N$, where $N \in \mathbb{N}$ is an integer, and denote by $\bar{\Omega}$ its closure and by $\partial\Omega$ its boundary. We assume that the following condition holds (see Fig. 1)

(L) There exist $\delta > 0$ and $m \in \mathbb{N}$, and for each $k = 1, \dots, m$ there exists an open convex sets $\Delta_k \subset \mathbb{R}^{N-1}$, a Lipschitz continuous function $a_k : \Delta_k \to \mathbb{R}$, and a rotation A_k in $\mathbb{R}^N$ (represented by an $N \times N$ matrix, still denoted by A_k, such that $A_k^{-1} = A_k^T$ and $\det A_k = 1$), such that

- (i) $\partial\Omega \subset \bigcup_{j=1}^m A_k(G_k)$,
- (ii) $G_k = \{y \in \mathbb{R}^N\,;\, y = (y', y_N), y' \in \Delta_k, y_N \in (a_k(y') - \delta, a_k(y') + \delta)\}$,
- (iii) $G_k^- = \{y \in G_k\,;\, y_N \in (a_k(y') - \delta, a_k(y'))\}$,
- (iv) $G_k^0 = \{y \in G_k\,;\, y_N = a_k(y')\}$.
- (v) $\Omega \cap A_k(G_k) = A_k(G_k^-)$,
- (vi) $\partial\Omega \cap A_k(G_k) = A_k(G_k^0)$,

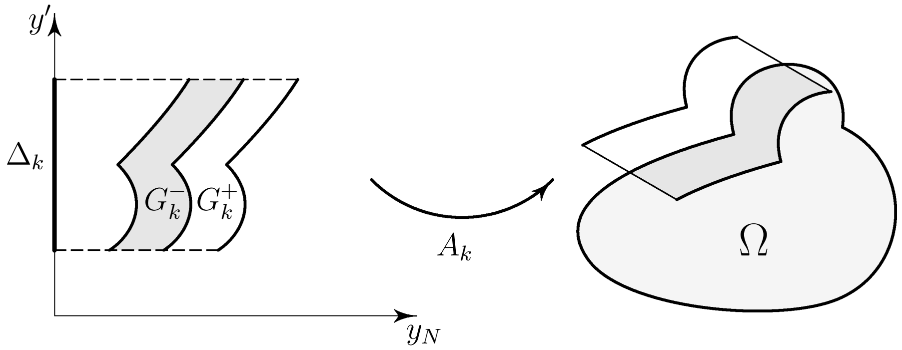

Figure 1. A domain with Lipschitzian boundary.

If **(L)** (i)–(vi) hold, then we say that Ω has *Lipschitzian boundary.*

As an example, consider the spaces $C(\bar\Omega)$ of continuous real functions defined on $\bar\Omega$, endowed with the norm

$$\|f\|_{C,0} = \sup\{|f(x)|\,;\, x \in \bar\Omega\}\,,$$

and $C^1(\bar\Omega)$ of continuously differentiable real functions on $\bar\Omega$, endowed with the norm

$$\|f\|_{C,1} = \sup\left\{|f(x)| + \sum_{k=1}^{N}\left|\frac{\partial f}{\partial x_i}(x)\right|\,;\, x \in \bar\Omega\right\}.$$

Proposition 2.1 *If Ω has Lipschitzian boundary, then the space $C^1(\bar\Omega)$ is compactly embedded in $C(\bar\Omega)$.*

Proof. Condition (1.1) is automatically satisfied. Furthermore, let $K \subset C^1(\bar\Omega)$ be bounded. Hence, there exists $M > 0$ such that

$$\forall f \in K \;\; \forall x \in \bar\Omega : |f(x)| + \sum_{k=1}^{N}\left|\frac{\partial f}{\partial x_i}(x)\right| \le M\,.$$

We are thus in the situation of Theorem 1.1 with $X = \mathbb{R}^N$, $Y = \mathbb{R}$, $A = \bar\Omega$, $B = [-M, M]$, provided we check that K is equicontinuous. Let $x, y \in \bar\Omega$ be arbitrarily chosen. We find a Lipschitz continuous function $\xi : [0,1] \to \bar\Omega$ and a constant $C > 0$ such that $\xi(0) = x$, $\xi(1) = y$, $|\xi'(\sigma)| \le C|x-y|$ a. e. (this is possible by the hypotheses on Ω), and use the chain rule to estimate

$$\begin{aligned}
|f(x) - f(y)| &= \left|\int_0^1 \frac{d}{d\sigma} f(\xi(\sigma))\,d\sigma\right| \\
&= \left|\int_0^1 \langle \nabla f(\xi(\sigma)), \xi'(\sigma)\rangle\,d\sigma\right| \\
&\le MC|x-y|\,,
\end{aligned}$$

where we denote by $\langle\cdot,\cdot\rangle$ the canonical scalar product in $\mathbb{R}^N$. The relative compactness now follows from Theorem 1.1. ∎

3 Spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$

Let $\Omega \subset \mathbb{R}^N$ be any open set. We denote as usual by $L^p(\Omega)$ the space of measurable functions $u : \Omega \to \mathbb{R}$, for which the norm $|u|_{p,\Omega}$ is finite, where

$$|u|_{p,\Omega} = \begin{cases} \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p} & \text{if } 1 \le p < \infty\,, \\ \sup\operatorname{ess}_{x\in\Omega} |u(x)| & \text{if } p = \infty\,. \end{cases} \tag{3.1}$$

The spaces $L^p(\Omega)$ with the above norms are *Banach spaces*. We say that $v \in L^p(\Omega)$ is a *generalized partial derivative* of $u \in L^p(\Omega)$ with respect to x_i, $i \in \{1, \dots, N\}$, if for every Lipschitz continuous function $\varphi : \Omega \to \mathbb{R}$ with *compact support in* Ω, that is,

$$\exists K = \bar{K} \subset \Omega \ \ \forall x \in \Omega \setminus K : \ \varphi(x) = 0\,, \tag{3.2}$$

we have

$$\int_\Omega u(x) \frac{\partial \varphi}{\partial x_i}(x)\, dx \ = \ - \int_\Omega v(x)\, \varphi(x)\, dx\,. \tag{3.3}$$

By [4, Chap. 2, Sect. 2.2], condition (3.3) is fulfilled if and only if u is absolutely continuous along almost all lines parallel to the x_i-axis and v coincides with $\partial u/\partial x_i$ almost everywhere.

The Sobolev space $W^{1,p}(\Omega)$ is defined as the subspace of $L^p(\Omega)$ of all functions u, which together with all generalized partial derivatives $\partial u/\partial x_i$ belong to $L^p(\Omega)$. With the norm

$$\|u\|_{1;p,\Omega} = |u|_{p,\Omega} + \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|_{p,\Omega}\,, \tag{3.4}$$

$W^{1,p}(\Omega)$ is also a Banach space.

The following result is crucial for the proof of embedding theorems, and its proof can be found in [4, Chap. 2, Sect. 3.6]. We fix an open bounded connected set $\Omega \subset \mathbb{R}^N$ with Lipschitzian boundary, and an open ball $B \subset \mathbb{R}^N$ such that $\bar{\Omega} \subset B$. We define $W^{1,p}_B$ to be the subset of $W^{1,p}(\mathbb{R}^N)$ consisting of all functions vanishing outside B. The norms in $L^p(\mathbb{R}^N)$, $W^{1,p}(\mathbb{R}^N)$ will simply be denoted by $|\cdot|_p$, $\|\cdot\|_{1;p}$, respectively.

Theorem 3.1 *There exists a linear* prolongation operator $E_p : W^{1,p}(\Omega) \to W^{1,p}_B$ *such that for every* $u \in W^{1,p}(\Omega)$ *we have*

(i) $E_p u(x) = u(x)$ *for a. e.* $x \in \Omega$*;*

(ii) *There exists a constant* $c_p > 0$ *such that for every* $u \in W^{1,p}(\Omega)$ *we have*

$$\|E_p u\|_{1;p} \le c_p \|u\|_{1;p,\Omega}\,;$$

(iii) *For every* $u \in W^{1,p}(\Omega)$ *we have*

$$|E_p u|_p \le c_p |u|_{p,\Omega}\,;$$

4 Some inequalities

This section collects some auxiliary inequalities that are needed in the sequel.

Proposition 4.1 ***(Young's inequality)*** *Let $f : [0,\infty) \to [0,\infty)$ be an absolutely continuous increasing function, $f(0) = 0$. Then for every $x, y \geq 0$ we have (see Fig. 2)*

$$xy \ \leq \ \int_0^x f(u)\,du + \int_0^y f^{-1}(v)\,dv\,, \tag{4.1}$$

where f^{-1} is the inverse function to f.

Proof. Substituting $v = f(u)$ we have, with the convention $\int_x^{f^{-1}(y)} = -\int_{f^{-1}(y)}^x$ if $f^{-1}(y) < x$, that

$$\begin{aligned} \int_0^x f(u)\,du + \int_0^y f^{-1}(v)\,dv \ &= \ \int_0^x (f(u) + uf'(u))\,du + \int_x^{f^{-1}(y)} uf'(u)\,du \\ &\geq \ xf(x) + x(y - f(x)) \ = \ xy\,. \end{aligned}$$

■

For $1 < p < \infty$, we denote by p' the *conjugate exponent*

$$p' \ = \ \frac{p}{p-1}\,. \tag{4.2}$$

Reciprocally, p is the conjugate of p' and we have

$$\frac{1}{p} + \frac{1}{p'} \ = \ 1\,, \quad p' - 1 \ = \ \frac{1}{p-1}\,. \tag{4.3}$$

As an immediate consequence of Proposition 4.1 we obtain, putting $f(x) = x^{p-1}$, that

$$xy \ \leq \ \frac{1}{p}x^p + \frac{1}{p'}y^{p'} \tag{4.4}$$

for every $x, y \geq 0$ and $1 < p < \infty$.

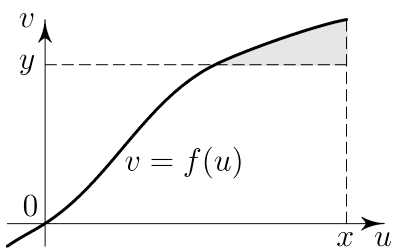

Figure 2. Young's inequality.

Proposition 4.2 ***(Hölder's inequality)*** *Let $\Omega \subset \mathbb{R}^N$ be any open set and let $1 \le p \le \infty$ be arbitrary. Then for every $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ we have*

$$\int_\Omega f(x)\, g(x)\, dx \;\le\; |f|_{p,\Omega}\, |g|_{p',\Omega}\,, \tag{4.5}$$

with the convention $1' = \infty\,,\ \infty' = 1\,.$

Proof. The case $p = 1$ or $p = \infty$ is obvious. For $1 < p < \infty$ we set

$$F(x) = \frac{f(x)}{|f|_{p,\Omega}}\,, \quad G(x) = \frac{g(x)}{|g|_{p',\Omega}}\,.$$

By (4.4) we have

$$|F(x)|\,|G(x)| \;\le\; \frac{1}{p}|F(x)|^p + \frac{1}{p'}|G(x)|^{p'} \;=\; \frac{|f(x)|^p}{p\,|f|^p_{p,\Omega}} + \frac{|g(x)|^{p'}}{p'\,|g|^{p'}_{p',\Omega}}\,,$$

hence

$$\int_\Omega F(x)\, G(x)\, dx \;\le\; \int_\Omega |F(x)|\,|G(x)|\, dx \;\le\; \frac{1}{p} + \frac{1}{p'} \;=\; 1\,,$$

which we wanted to prove. ■

Proposition 4.3 ***(Minkowski's inequality)*** *Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ be open sets, and let $f : X \times Y \to [0, \infty)$ be a measurable function. Then for every $1 < p < \infty$ we have*

$$\left(\int_Y \left(\int_X f(x,y)\, dx\right)^p dy\right)^{1/p} \;\le\; \int_X \left(\int_Y f^p(x,y)\, dy\right)^{1/p} dx\,. \tag{4.6}$$

Proof. For $y \in Y$ and $R > 0$ set

$$F(y) = \int_X f_R(x,y)\, dx\,, \quad g(y) = F^{p-1}(y)\,,$$

where

$$f_R(x,y) \;=\; \begin{cases} \min\{R, f(x,y)\} & \text{if } \max\{|x|, |y|\} < R\,, \\ 0 & \text{if } \max\{|x|, |y|\} \ge R\,. \end{cases}$$

Then

$$\begin{aligned} \int_Y F^p(y)\, dy \;&=\; \int_Y F(y)\, g(y)\, dy \;=\; \int_X \left(\int_Y f_R(x,y)\, g(y)\, dy\right) dx \\ &\overset{\text{Hölder}}{\le}\; \int_X \left(\int_Y f_R^p(x,y)\, dy\right)^{1/p} \left(\int_Y g^{p'}(y)\, dy\right)^{1/p'} dx \\ &=\; \int_X \left(\int_Y f_R^p(x,y)\, dy\right)^{1/p} dx \left(\int_Y F^p(y)\, dy\right)^{1/p'}\,, \end{aligned}$$

hence

$$\left(\int_Y F^p(y)\,dy\right)^{1/p} \;\le\; \int_X \left(\int_Y f_R^p(x,y)\,dy\right)^{1/p} dx \;\le\; \int_X \left(\int_Y f^p(x,y)\,dy\right)^{1/p} dx\,,$$

and we obtain the result from Fatou's Lemma letting R tend to $+\infty$. ■

Note that replacing X by a finite set $\{1,\dots,n\}$, $f(x,y)$ by $f_k(y)$, $k=1,\dots,n$, and $\int_X dx$ by $\sum_{k=1}^n$, the Minkowski inequality reads

$$\left|\sum_{k=1}^n f_k\right|_{p,Y} \;\le\; \sum_{k=1}^n |f_k|_{p,Y}\;, \tag{4.7}$$

which is nothing but the triangle inequality for the norm $|\cdot|_{p,Y}$.

The following example shows that the Minkowski inequality cannot be reversed.

Example 4.4 Consider $X=Y=(0,1)$, and $f(x,y)=((x-y)^+)^{-1/p}$ for some $p>1$. Then

$$\begin{aligned}\left(\int_Y \left(\int_X f(x,y)\,dx\right)^p dy\right)^{1/p} &= \left(\int_0^1 \left(\int_y^1 (x-y)^{-1/p}\,dx\right)^p dy\right)^{1/p} &&= \tfrac{1}{p-1}\,p^{1-1/p},\\ \int_X \left(\int_Y f^p(x,y)\,dy\right)^{1/p} dx &= \int_0^1 \left(\int_0^x (x-y)^{-1}\,dx\right)^{1/p} dy &&= +\infty\,.\end{aligned}$$

Remark 4.5 In the same way we prove that for every $1\le q<p<\infty$ we have

$$\left(\int_Y \left(\int_X f^q(x,y)\,dx\right)^{p/q} dy\right)^{1/p} \;\le\; \left(\int_X \left(\int_Y f^p(x,y)\,dy\right)^{q/p} dx\right)^{1/q}. \tag{4.8}$$

We set in this case

$$F(y)=\int_X f_R^q(x,y)\,dx\,,\quad g(y)=F^{(p/q)-1}(y)\,,$$

and estimate $\int_Y F^{p/q}(y)\,dy$ similarly as in the proof of Proposition 4.3.

The proof of the Minkowski inequality is related to the so-called *reverse Hölder inequality*:

$$\int_\Omega f(x)\,g(x)\,dx \;\le\; C\,|g|_{p',\Omega}\quad \forall g\in L^{p'}(\Omega) \;\Longrightarrow\; |f|_{p,\Omega} \;\le\; C\,. \tag{4.9}$$

To prove this statement, it suffices to choose $g(x)=\operatorname{sign}(f_R(x))\,|f_R(x)|^{p-1}$ with f_R defined analogously as in the proof of Proposition 4.3, use the fact that

$$\int_\Omega |f_R(x)|^p\,dx \le \int_\Omega f(x)\,g(x)\,dx \le\; C\,|g|_{p',\Omega} = C\,|f_R|_{p,\Omega}^{p/p'}\,,$$

and let R tend to ∞.

Proposition 4.6 ***(Young's inequality II for convolutions)*** *Let $1\le p,q,r\le\infty$ be given such that*

$$\frac{1}{p}+\frac{1}{r}=1+\frac{1}{q}\,. \tag{4.10}$$

For $u \in L^p(\mathbb{R}^N)$, $v \in L^r(\mathbb{R}^N)$, *and* $x \in \mathbb{R}^N$ *set*

$$w(x) \;=\; \int_{\mathbb{R}^N} u(y)\, v(x-y)\, dy\,.$$

Then $w \in L^q(\mathbb{R}^N)$ *and*

$$|w|_q \;\le\; |u|_p\, |v|_r\,. \tag{4.11}$$

Proof. The case $q=\infty$ follows immediately from Hölder's inequality. Hence, assume that $q<\infty$, and set $\alpha = r/q \in (0,1]$. To make the use of the Minkowski inequality more transparent, we write $\int_X dx$, $\int_Y dy$ instead of $\int_{\mathbb{R}^N} dx$, $\int_{\mathbb{R}^N} dy$. Then, using the fact that $1-\alpha = r/p'$ and that

$$\int_Y |v(x-y)|^r\, dy \;=\; \int_Y |v(y)|^r\, dy$$

for a. e. $x \in X$, we obtain

$$\begin{aligned}
|w|_q \quad &= \quad \left(\int_X \left| \int_Y u(y)\, v(x-y)\, dy \right|^q dx \right)^{1/q} \\
&\le \quad \left(\int_X \left(\int_Y |u(y)|\, |v(x-y)|^\alpha\, |v(x-y)|^{1-\alpha}\, dy \right)^q dx \right)^{1/q} \\
&\overset{\text{Hölder}}{\le} \quad \left(\int_X \left(\int_Y |u(y)|^p\, |v(x-y)|^{p\alpha}\, dy \right)^{q/p} dx \right)^{1/q} \left(\int_Y |v(y)|^{p'(1-\alpha)}\, dy \right)^{1/p'} \\
&\overset{\text{Minkowski}}{\le} \quad \left(\int_Y \left(\int_X |u(y)|^q\, |v(x-y)|^{q\alpha}\, dx \right)^{p/q} dy \right)^{1/p} |v|_r^{1-\alpha} \\
&= \quad \left(\int_Y |u(y)|^p\, dy \right)^{1/p} \left(\int_X |v(x)|^{q\alpha}\, dx \right)^{1/q} |v|_r^{1-\alpha} \\
&= \quad |u|_p\, |v|_r\,.
\end{aligned}$$

■

We devote the next section to the *Hardy-Littlewood inequality*, the proof of which is quite involved and requires a certain number of auxiliary steps. The proof we give here is a modification of the one from [2].

5 Hardy-Littlewood inequality

We state the Hardy-Littlewood inequality in the following form.

Proposition 5.1 *Let* $1 < p, q, r < \infty$ *be such that* $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$. *Then there exists a constant* $H_{pr} > 0$ *such that for every* $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$ *we have*

$$\iint_{\mathbb{R}^2} f(x)\, g(y)\, |x-y|^{-1/r}\, dx\, dy \;\le\; H_{pr} |f|_p\, |g|_q\,. \tag{5.1}$$

An explicit estimate for H_{pr} will be given in (5.18) below.

We first fix an even locally integrable function $h : \mathbb{R} \to [0, \infty)$, which is non-decreasing in $(-\infty, 0)$ and non-increasing in $(0, +\infty)$, and establish the following easy result.

Lemma 5.2 *For $a, b > 0$ and $r, s \in \mathbb{R}$ set*

$$\varphi_{ab}(r, s) \;=\; \int_{-a+r}^{a+r} \int_{-b+s}^{b+s} h(x - y)\, dy\, dx\,. \tag{5.2}$$

Then for all $r, s \in \mathbb{R}$ we have $\partial\varphi_{ab}/\partial r \geq 0$, $\partial\varphi_{ab}/\partial s \leq 0$ for $r < s$, $\partial\varphi_{ab}/\partial r \leq 0$, $\partial\varphi_{ab}/\partial s \geq 0$ for $r > s$, $\varphi_{ab}(r, r) = \varphi_{ab}(0, 0)$.

Proof. We obviously have $\varphi_{ab}(r, s) = \varphi_{ab}(r - s, 0) = \varphi_{ab}(0, s - r)$ for all r, s, hence it suffices to prove that $\partial\varphi_{ab}/\partial r(r, 0) \leq 0$ for $r > 0$, $\partial\varphi_{ab}/\partial s(0, s) \leq 0$ for $s > 0$. We have

$$\begin{aligned}
\frac{\partial\varphi_{ab}}{\partial r}(r, 0) \;&=\; \int_{-b}^{b} (h(a + r - y) - h(-a + r - y))\, dy \\
&=\; \int_{-b}^{b} (h(a + r + y) - h(-a + r - y))\, dy \\
&=\; \int_{a-b}^{a+b} (h(r + z) - h(r - z))\, dz \\
&=\; \int_{|a-b|}^{a+b} (h(r + z) - h(r - z))\, dz\,.
\end{aligned}$$

For a. e. $z > 0$ we have $h(r + z) \leq h(r - z)$, and the assertion follows. The argument for $\partial\varphi_{ab}/\partial s(0, s)$ is identical. ■

The idea of the proof of Proposition 5.1 is based on approximations of the functions f and g by step functions, and for each step function we use a rearrangement formula which will be proved by induction (see Fig. 3). The induction step is carried out in the following way.

Lemma 5.3 *Let h be as in Lemma 5.2, and let $m, n \in \mathbb{N} \cup \{0\}$ be given. Let $a_0, \ldots, a_n$, $b_0, \ldots, b_m$, $r_0, \ldots, r_n$, $s_0, \ldots, s_m$ be sequences such that $a_i > 0$, $b_j > 0$ for all $i = 0, \ldots, n$, $j = 0, \ldots, m$, and $r_i - r_{i-1} \geq a_i + a_{i-1}$, $s_j - s_{j-1} \geq b_j + b_{j-1}$ for all $i = 1, \ldots, n$, $j = 1, \ldots, m$.*

(i) *If $r_{i_0} - r_{i_0-1} = a_{i_0} + a_{i_0-1}$ for some $i_0 \in \{1, \ldots, n\}$, then there exist $a_i^* > 0$ and $r_i^* \in \mathbb{R}$ for $i = 0, \ldots, n-1$ such that $r_i^* - r_{i-1}^* \geq a_i^* + a_{i-1}^*$ for all $i = 1, \ldots, n-1$, and*

$$\sum_{i=0}^{n-1} a_i^* = \sum_{i=0}^{n} a_i\,, \qquad \sum_{i=0}^{n-1}\sum_{j=0}^{m} \varphi_{a_i^* b_j}(r_i^*, s_j) \;=\; \sum_{i=0}^{n}\sum_{j=0}^{m} \varphi_{a_i b_j}(r_i, s_j)\,. \tag{5.3}$$

(ii) *If $s_{j_0} - s_{j_0-1} = b_{j_0} + b_{j_0-1}$ for some $j_0 \in \{1, \ldots, m\}$, then there exist $b_j^* > 0$ and $s_j^* \in \mathbb{R}$ for $j = 0, \ldots, m-1$ such that $s_j^* - s_{j-1}^* \geq b_j^* + b_{j-1}^*$ for all $j = 1, \ldots, m-1$, and*

$$\sum_{j=0}^{m-1} b_j^* = \sum_{j=0}^{m} b_j\,, \qquad \sum_{i=0}^{n}\sum_{j=0}^{m-1} \varphi_{a_i b_j^*}(r_i, s_j^*) \;=\; \sum_{i=0}^{n}\sum_{j=0}^{m} \varphi_{a_i b_j}(r_i, s_j)\,. \tag{5.4}$$

Proof. We prove only part (i), the rest is similar. If $r_{i_0} - r_{i_0-1} = a_{i_0} + a_{i_0-1}$, then $r_{i_0-1} + a_{i_0-1} = r_{i_0} - a_{i_0}$, hence for every j we have

$$\varphi_{a_{i_0-1}b_j}(r_{i_0-1}, s_j) + \varphi_{a_{i_0}b_j}(r_{i_0}, s_j) \;=\; \varphi_{a^*_{i_0-1}b_j}(r^*_{i_0-1}, s_j)\,,$$

where

$$r^*_{i_0-1} = \frac{1}{2}(a_{i_0} - a_{i_0-1} + r_{i_0} + r_{i_0-1})\,, \qquad a^*_{i_0-1} = a_{i_0} + a_{i_0-1}\,.$$

We now set

$$r^*_i = r_i, a^*_i = a_i \ \text{ for } \ i = 0, \ldots, i_0 - 2\,, \quad r^*_i = r_{i-1}, a^*_i = a_{i-1} \ \text{ for } \ i = i_0, \ldots, n-1\,.$$

Then (5.3) is automatically fulfilled by construction. It remains to check that

$$\begin{aligned}
r^*_{i_0} - r^*_{i_0-1} - a^*_{i_0} - a^*_{i_0-1} \;&=\; r_{i_0+1} - \frac{1}{2}(a_{i_0} - a_{i_0-1} + r_{i_0} + r_{i_0-1}) - a_{i_0+1} - a_{i_0} - a_{i_0-1}\\
&=\; r_{i_0+1} - r_{i_0} - a_{i_0+1} - a_{i_0} + \frac{1}{2}(r_{i_0} - r_{i_0-1} - a_{i_0} - a_{i_0-1}) \;\geq\; 0\,,\\
r^*_{i_0-1} - r^*_{i_0-2} - a^*_{i_0-1} - a^*_{i_0-2} \;&=\; \frac{1}{2}(a_{i_0} - a_{i_0-1} + r_{i_0} + r_{i_0-1}) - r_{i_0-2} - -a_{i_0} - a_{i_0-1} - a_{i_0-2}\\
&=\; r_{i_0-1} - r_{i_0-2} - a_{i_0-1} - a_{i_0-2} - \frac{1}{2}(r_{i_0} - r_{i_0-1} - a_{i_0} - a_{i_0-1})\\
&\geq\; 0\,,
\end{aligned}$$

and the proof is complete. ∎

Lemma 5.4 *Let h be as in Lemma 5.2, and let $a_0, \ldots, a_n$, $b_0, \ldots, b_m$, $r_0, \ldots, r_n$, $s_0, \ldots, s_m$ be as in Lemma 5.3. Set*

$$A \;=\; \sum_{i=0}^{n} a_i\,, \qquad B \;=\; \sum_{j=0}^{m} b_j\,.$$

Then

$$S \;:=\; \sum_{i=0}^{n}\sum_{j=0}^{m} \varphi_{a_i b_j}(r_i, s_j) \;\leq\; \varphi_{AB}(0,0)\,.$$

Proof. We proceed by induction over $N = n + m$. For $N = 0$ we have $\varphi_{a_0b_0}(r_0, s_0) = \varphi_{a_0b_0}(r_0 - s_0, 0) \leq \varphi_{a_0b_0}(0,0)$ by Lemma 5.2. Suppose now that the statement is proven for some $N \geq 0$, and consider n, m such that $n + m = N + 1$. We will assume for definiteness that

$$r_n \geq s_m$$

(the opposite case is fully analogous). We distinguish two cases:

(i) $n = 0$. Then we set $\hat{s}_j = s_j + s_m - s_{m-1} - b_m - b_{m-1} \geq s_j$ for $j = 0, \ldots, m-1$, $\hat{s}_m = s_m$. Then $\hat{s}_j - \hat{s}_{j-1} = s_j - s_{j-1}$ for $j = 1, \ldots, m-1$, $\hat{s}_m - \hat{s}_{m-1} = b_m + b_{m-1}$. By Lemma

5.2 we have $\varphi_{a_0b_j}(r_0,s_j) \le \varphi_{a_0b_j}(r_0,\hat{s}_j)$ for all $j=0,\dots,m$. By Lemma 5.3 there exist $s_0^*,\dots,s_{m-1}^*$ and $b_0^*,\dots,b_{m-1}^*$ such that

$$\sum_{j=0}^{m-1} b_j^* \;=\; \sum_{j=0}^{m} b_j\,, \qquad \sum_{j=0}^{m-1} \varphi_{a_0b_j^*}(r_0,s_j^*) \;=\; \sum_{j=0}^{m} \varphi_{a_0b_j}(r_0,\hat{s}_j)\,.$$

We have $m-1=N$, and the induction hypothesis yields

$$\sum_{j=0}^{m} \varphi_{a_0b_j}(r_0,s_j) \;\le\; \sum_{j=0}^{m-1} \varphi_{a_0b_j^*}(r_0,s_j^*) \;\le\; \varphi_{AB}(0,0)\,.$$

(ii) $n>0$. Set

$$\begin{aligned}\hat{r}_n &= \max\{s_m, r_{n-1}+a_n+a_{n-1}\} \;\le\; r_n\\ \hat{r}_i &= r_i \qquad \text{for } i=0,\dots,n-1\,.\end{aligned}$$

By Lemma 5.2, we have

$$\hat{S} \;:=\; \sum_{i=0}^{n}\sum_{j=0}^{m} \varphi_{a_ib_j}(\hat{r}_i,s_j) \;\ge\; S\,.$$

If $\hat{r}_n = r_{n-1}+a_n+a_{n-1}$, then $\hat{S} \le \varphi_{AB}(0,0)$ by Lemma 5.3 and by the induction hypothesis similarly as in case (i). If $\hat{r}_n = s_m > r_{n-1}+a_n+a_{n-1}$, then set

$$\begin{aligned}\bar{s}_m &= \max\{s_{m-1}+b_m+b_{m-1}, r_{n-1}+a_n+a_{n-1}\} \;\le\; s_m\,,\\ \bar{s}_j &= s_j \qquad \text{for } j=0,\dots,m-1\,,\\ \bar{r}_n &= \bar{s}_m\,,\\ \bar{r}_i &= r_i \qquad \text{for } i=0,\dots,n-1\,,\end{aligned}$$

with the convention $s_{-1}=-\infty$, $b_{-1}=b_0$ if $m=0$. We have

$$\begin{aligned}\hat{S} \;&=\; \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \varphi_{a_ib_j}(\hat{r}_i,s_j) + \sum_{i=0}^{n-1} \varphi_{a_ib_m}(\hat{r}_i,s_m) + \sum_{j=0}^{m-1} \varphi_{a_nb_j}(s_m,s_j) + \varphi_{a_nb_m}(s_m,s_m)\\ &\overset{\text{Lemma 5.2}}{\le} \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \varphi_{a_ib_j}(\bar{r}_i,\bar{s}_j) + \sum_{i=0}^{n-1} \varphi_{a_ib_m}(\bar{r}_i,\bar{s}_m) + \sum_{j=0}^{m-1} \varphi_{a_nb_j}(\bar{s}_m,\bar{s}_j) + \varphi_{a_nb_m}(\bar{s}_m,\bar{s}_m)\\ &=\; \sum_{i=0}^{n}\sum_{j=0}^{m} \varphi_{a_ib_j}(\bar{r}_i,\bar{s}_j)\,.\end{aligned}$$

By construction, we have either $\bar{s}_m = \bar{s}_{m-1}+b_m+b_{m-1}$ or $\bar{r}_n = \bar{r}_{n-1}+a_n+a_{n-1}$, and the assertion follows again from Lemma 5.3 and the induction hypothesis.

■

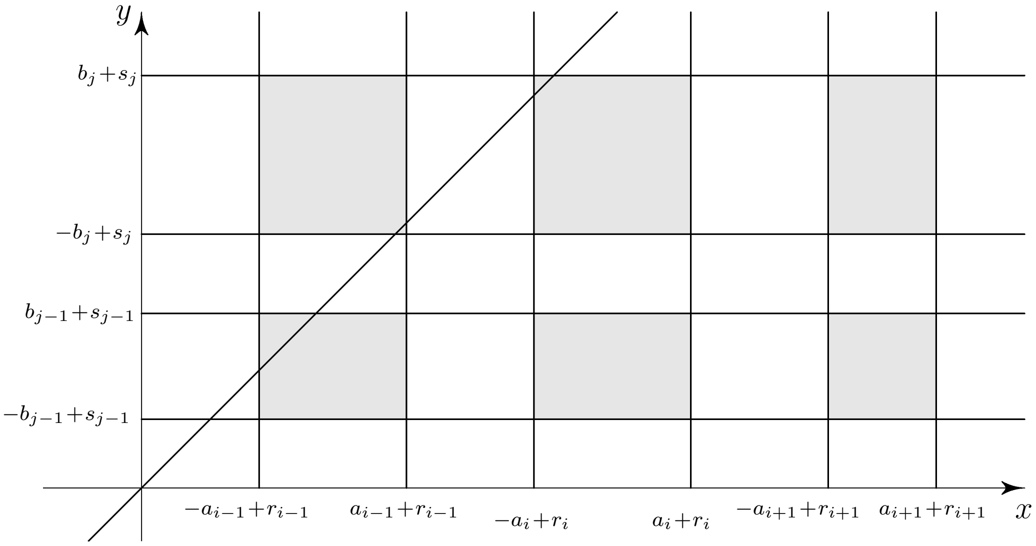

Figure 3. Illustration to Lemmas 5.3, 5.4.

Corollary 5.5 *Let* h *be as in Lemma 5.2, and let* $\{(\alpha_i, \beta_i)\,;\, i = 0, \dots, n\}$, $\{(\gamma_j, \delta_j)\,;\, j = 0, \dots, m\}$, *be two systems of intervals such that* $(\alpha_{i_1}, \beta_{i_1}) \cap (\alpha_{i_2}, \beta_{i_2}) = \emptyset$, $(\gamma_{j_1}, \delta_{j_1}) \cap (\gamma_{j_2}, \delta_{j_2}) = \emptyset$ *for all* $i_1 \neq i_2 \in \{0, \dots, n\}$, $j_1 \neq j_2 \in \{0, \dots, m\}$. *Set*

$$A = \frac{1}{2} \sum_{i=0}^{n} (\beta_i - \alpha_i)\,, \quad B = \frac{1}{2} \sum_{j=0}^{m} (\delta_j - \gamma_j)\,.$$

Then

$$\sum_{i=0}^{n} \sum_{j=0}^{m} \int_{\alpha_i}^{\beta_i} \int_{\gamma_j}^{\delta_j} h(x-y)\, dy\, dx \ \leq \ \varphi_{AB}(0,0)\,.$$

Proof. We change the ordering of the intervals (α_i, β_i), (γ_j, δ_j) in such a way that $\beta_{i-1} \leq \alpha_i$, $\delta_{j-1} \leq \gamma_j$ for all $i = 1, \dots, n$, $j = 1, \dots, m$, and set

$$\begin{aligned} r_i &= \tfrac{1}{2}(\alpha_i + \beta_i)\,, & a_i &= \tfrac{1}{2}(\beta_i - \alpha_i)\,, \\ s_j &= \tfrac{1}{2}(\gamma_j + \delta_j)\,, & b_j &= \tfrac{1}{2}(\delta_j - \gamma_j) \end{aligned}$$

for $i = 0, \dots, n$, $j = 0, \dots, m$. We have $r_i - r_{i-1} - a_i - a_{i-1} = \alpha_i - \beta_{i-1} \geq 0$, $s_j - s_{j-1} - b_j - b_{j-1} = \gamma_j - \delta_{j-1} \geq 0$, $\alpha_i = -a_i + r_i$, $\beta_i = a_i + r_i$, $\gamma_j = -b_j + s_j$, $\delta_j = b_j + s_j$, and Lemma 5.4 completes the proof. ■

The next step consists in a rearrangement formula we summarize in Lemma 5.6 below. We fix $K, L \in \mathbb{N}$ and for sequences

$$\begin{aligned} -\infty < a_0 < a_1 < \dots < a_k < +\infty\,, \qquad & 0 = f_0 \leq f_1 \leq \dots \leq f_K\,, \\ -\infty < b_0 < b_1 < \dots < b_L < +\infty\,, \qquad & 0 = g_0 \leq g_1 \leq \dots \leq g_L\,, \end{aligned}$$

consider step functions f , g of the form

$$\left.\begin{array}{rcl} f(x) &=& \sum_{i=1}^{K} f_i\,\chi_{(a_{\varrho(i)-1},a_{\varrho(i)})}(x) \;=\; \sum_{I=1}^{K} f_{\varrho^{-1}(I)}\,\chi_{(a_{I-1},a_I)}(x)\,, \\ g(y) &=& \sum_{j=1}^{L} g_j\,\chi_{(b_{\sigma(j)-1},b_{\sigma(j)})}(y) \;=\; \sum_{J=1}^{L} g_{\sigma^{-1}(J)}\,\chi_{(b_{J-1},b_J)}(y) \end{array}\right\} \tag{5.5}$$

for $x,y\in\mathbb{R}$, where χ_M is the characteristic function of a set $M\subset\mathbb{R}$, and $\varrho:\{1,\dots,K\}\to\{1,\dots,K\}$, $\sigma:\{1,\dots,L\}\to\{1,\dots,L\}$ are some permutations of indices.

We now define

$$F_k = f_k - f_{k-1} \qquad \text{for} \quad k=1,\dots,K\,. \tag{5.6}$$

Then $f_i=\sum_{k=1}^{i} F_k$ for all i, and we have

$$f(x) \;=\; \sum_{i=1}^{K}\sum_{k=1}^{i} F_k\,\chi_{(a_{\varrho(i)-1},a_{\varrho(i)})}(x) \;=\; \sum_{k=1}^{K} F_k \sum_{i=k}^{K} \chi_{(a_{\varrho(i)-1},a_{\varrho(i)})}(x)\,. \tag{5.7}$$

We further introduce for $k=1,\dots,K$ the numbers

$$a_k^* \;=\; \frac12 \sum_{i=k}^{K} (a_{\varrho(i)} - a_{\varrho(i)-1})\,, \qquad a_{K+1}^* = 0\,, \tag{5.8}$$

and for $x\in\mathbb{R}$ put

$$f^*(x) \;=\; \sum_{k=1}^{K} F_k\,\chi_{(-a_k^*,a_k^*)}(x) \;=\; \sum_{i=1}^{K} f_i\,\chi_{(-a_i^*,-a_{i+1}^*]\cup[a_{i+1}^*,a_i^*)}(x)\,. \tag{5.9}$$

Similarly, we put

$$G_\ell = g_\ell - g_{\ell-1} \qquad \text{for} \quad \ell=1,\dots,L\,. \tag{5.10}$$

Then

$$g(y) \;=\; \sum_{j=1}^{L}\sum_{\ell=1}^{j} G_\ell\,\chi_{(b_{\sigma(j)-1},b_{\sigma(j)})}(y) \;=\; \sum_{\ell=1}^{L} G_\ell \sum_{j=\ell}^{L} \chi_{(b_{\sigma(j)-1},b_{\sigma(j)})}(y)\,. \tag{5.11}$$

As before, we introduce for $\ell=1,\dots,L$ the numbers

$$b_\ell^* \;=\; \frac12 \sum_{j=\ell}^{L} (b_{\sigma(j)} - b_{\sigma(j)-1})\,, \qquad b_{L+1}^* = 0\,, \tag{5.12}$$

and for $y\in\mathbb{R}$ put

$$g^*(y) \;=\; \sum_{\ell=1}^{L} G_\ell\,\chi_{(-b_\ell^*,b_\ell^*)}(y) \;=\; \sum_{j=1}^{L} g_j\,\chi_{(-b_j^*,-b_{j+1}^*]\cup[b_{j+1}^*,b_j^*)}(y)\,. \tag{5.13}$$

We now prove the following crucial inequality.

Lemma 5.6 *Let f, g be as in (5.5), and let f^*, g^* be given by (5.9), (5.13), respectively. Let h be as in Lemma 5.2. Then we have $|f|_p = |f^*|_p$, $|g|_q = |g^*|_q$ for all $p \ge 1$, $q \ge 1$, and*

$$\iint_{\mathbb{R}^2} f(x)\, g(y)\, h(x-y)\, dy\, dx \;\le\; \iint_{\mathbb{R}^2} f^*(x)\, g^*(y)\, h(x-y)\, dy\, dx\,. \tag{5.14}$$

Proof. The fact that the L^p norms of f and f^* coincide, follows immediately from (5.5) and (5.9), taking into account the fact that for all i we have $a_{\varrho(i)} - a_{\varrho(i)-1} = 2(a_i^* - a_{i+1}^*)$. The same argument works for g and g^*, indeed.

We further have by (5.7), (5.11) that

$$\iint_{\mathbb{R}^2} f(x)\, g(y)\, h(x-y)\, dy\, dx \;=\; \sum_{k=1}^{K}\sum_{\ell=1}^{L} F_k G_\ell \sum_{i=k}^{K}\sum_{j=\ell}^{L} \int_{a_{\varrho(i)-1}}^{a_{\varrho(i)}} \int_{b_{\sigma(j)-1}}^{b_{\sigma(j)}} h(x-y)\, dy\, dx\,. \tag{5.15}$$

By Corollary 5.5, we have for every k and ℓ that

$$\sum_{i=k}^{K}\sum_{j=\ell}^{L} \int_{a_{\varrho(i)-1}}^{a_{\varrho(i)}} \int_{b_{\sigma(j)-1}}^{b_{\sigma(j)}} h(x-y)\, dy\, dx \;\le\; \int_{-a_k^*}^{a_k^*} \int_{-b_\ell^*}^{b_\ell^*} h(x-y)\, dy\, dx\,,$$

and (5.14) follows from (5.9), (5.13), and (5.15). ■

We are now ready to pass to the proof of Proposition 5.1.

Proof of Proposition 5.1. We restrict ourselves to the case that f and g are non-negative step functions of the form (5.5). The general case then follows from the density of step functions in $L^p(\mathbb{R})$, $L^q(\mathbb{R})$. By Lemma 5.6 we have

$$\iint_{\mathbb{R}^2} f(x)\, g(y)\, |x-y|^{-1/r}\, dx\, dy \;\le\; \iint_{\mathbb{R}^2} f^*(x)\, g^*(y)\, |x-y|^{-1/r}\, dx\, dy\,. \tag{5.16}$$

For $y \in \mathbb{R}$ set

$$F(y) \;=\; \int_{\mathbb{R}} f^*(x)\, |x-y|^{-1/r}\, dx\,.$$

The function f^* is even, nondecreasing in $(-\infty, 0)$ and nonincreasing in $(0, +\infty)$, hence

$$|f|_p^p \;\ge\; \int_{-|x|}^{|x|} (f^*(\xi))^p\, d\xi \;\ge\; 2|x|(f^*(x))^p \qquad \forall x \in \mathbb{R}\,. \tag{5.17}$$

Choosing $\alpha = p/q'$, we thus obtain for every $y \in \mathbb{R}$ that

$$\begin{aligned} F(y) \;&\le\; \int_{\mathbb{R}} (f^*(x))^\alpha |2x|^{(1-\alpha)/p} |f|_p^{1-\alpha}\, |x-y|^{-1/r}\, dx \\ &=\; 2^{-1+1/r}\, |f|_p^{1-p/q'} \int_{\mathbb{R}} (f^*(x))^{p/q'} |x|^{-1+1/r}\, |x-y|^{-1/r}\, dx \\ &=\; 2^{-1+1/r}\, |f|_p^{1-p/q'} \int_{\mathbb{R}} (f^*(yt))^{p/q'} |t|^{-1+1/r}\, |t-1|^{-1/r}\, dt\,. \end{aligned}$$

We now use the Minkowski inequality (4.6) to estimate the $L^{q'}$ norm of F. We have

$$\begin{aligned}
|F|_{q'} &\le 2^{-1+1/r}\,|f|_p^{1-p/q'}\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}(f^*(yt))^{p/q'}|t|^{-1+1/r}\,|t-1|^{-1/r}\,dt\right)^{q'}dy\right)^{1/q'}\\
&\overset{\text{Minkowski}}{\le} 2^{-1+1/r}\,|f|_p^{1-p/q'}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}(f^*(yt))^{p}|t|^{-q'+q'/r}\,|t-1|^{-q'/r}\,dy\right)^{1/q'}dt\\
&= 2^{-1+1/r}\,|f|_p^{1-p/q'}\int_{\mathbb{R}}|t|^{-1+1/r}\,|t-1|^{-1/r}\left(\int_{\mathbb{R}}(f^*(yt))^{p}\,dy\right)^{1/q'}dt\\
&= 2^{-1+1/r}\,|f|_p\int_{\mathbb{R}}|t|^{-1/p}\,|t-1|^{-1/r}\,dt\,.
\end{aligned}$$

By Hölder's inequality, Lemma 5.6, and inequality (5.16), the left-hand side of (5.1) is estimated from above by $|g|_q|F|_{q'}$. Hence, (5.1) holds with

$$H_{pr} \;=\; 2^{-1+1/r}\int_{\mathbb{R}}|t|^{-1/p}\,|t-1|^{-1/r}\,dt\,. \tag{5.18}$$

■

6 Smooth approximation of L^p functions

We fix a smooth (C^1 is enough for our purposes) function $\varphi:\mathbb{R}^N\to[0,\infty)$ such that $\varphi(x)=0$ outside the set $B(1):=\{x\in\mathbb{R}^N\,;\,|x|\le 1\}$, and

$$\int_{B(1)}\varphi(x)\,dx \;=\; 1\,. \tag{6.1}$$

For $u\in L^p(\mathbb{R}^N)$, $x\in\mathbb{R}^N$, and a parameter $\sigma\in(0,1]$ we set

$$u^\sigma(x) \;=\; \sigma^{-N}\int_{\mathbb{R}^N}\varphi\left(\frac{x-y}{\sigma}\right)u(y)\,dy\,. \tag{6.2}$$

For all $\sigma\in(0,1]$, the function u^σ is continuously differentiable, and we have

$$\begin{aligned}
\int_{\mathbb{R}^N}|u^\sigma-u|^p(x)\,dx &= \int_{\mathbb{R}^N}\left|\int_{B(1)}\varphi(z)(u(x-\sigma z)-u(x))\,dz\right|^p dx\\
&\overset{\text{Hölder}}{\le} \left(\int_{B(1)}\varphi^{p'}(x)\,dx\right)^{p/p'}\int_{B(1)}\int_{\mathbb{R}^N}|u(x-\sigma z)-u(x)|^p\,dx\,dz\,,
\end{aligned}\tag{6.3}$$

hence

$$u^\sigma\to u\quad\text{strongly in } L^p(\mathbb{R}^N)\ \text{ as }\ \sigma\to 0+ \tag{6.4}$$

as a consequence of the Mean Continuity Theorem, see [4, Chap. 2, Sect. 1.2].

In the sequel, we will use the following relation between the L^q norm of u^σ and L^p norm of u, which follows directly from Proposition 4.6:

$$|u^\sigma|_q \;\le\; \sigma^{-N(1/p-1/q)}|\varphi|_r\,|u|_p\quad\forall q\ge p\,, \tag{6.5}$$

where r is as in (4.10).

7 Sobolev embeddings

We now state and prove the main result of this text.

Theorem 7.1 *Let* $p, q \in (1, \infty)$ *be such that*

$$\frac{1}{p} \geq \frac{1}{q} > \frac{1}{p} - \frac{1}{N},$$

and set

$$\kappa \; := \; 1 - N\left(\frac{1}{p} - \frac{1}{q}\right) \; \in \; (0,1).$$

Then there exists $C_{pq} > 0$ *such that for every* $u \in W^{1,p}(\mathbb{R}^N)$ *and every* $\sigma \in (0,1]$ *we have*

$$|u^\sigma - u|_q \; \leq \; C_{pq}\, \sigma^\kappa \, |\nabla u|_p . \tag{7.1}$$

Proof. Notice first that for every $x \in \mathbb{R}^N$ and $\sigma \in (0,1)$ we obtain, integrating by parts, that

$$\begin{aligned} \frac{\partial}{\partial \sigma} u^\sigma(x) &= \sigma^{-N} \int_{\mathbb{R}^N} \sum_{i=1}^N \frac{\partial}{\partial y_i}\left(\frac{x_i - y_i}{\sigma}\, \varphi\left(\frac{x-y}{\sigma}\right)\right) u(y)\, dy \\ &= -\sigma^{-N} \int_{\mathbb{R}^N} \left\langle \Phi\left(\frac{x-y}{\sigma}\right), \nabla u(y) \right\rangle dy, \end{aligned} \tag{7.2}$$

where we set $\Phi(\xi) = \xi \varphi(\xi)$. This yields in particular,

$$|u^\beta(x) - u^\alpha(x)| \; \leq \; \int_\alpha^\beta \sigma^{-N} \left| \int_{\mathbb{R}^N} \left\langle \Phi\left(\frac{x-y}{\sigma}\right), \nabla u(y) \right\rangle dy \right| d\sigma \tag{7.3}$$

for every $0 < \alpha < \beta \leq 1$. To estimate the difference $u^\beta - u^\alpha$ in (7.3) in the space $L^q(\mathbb{R}^N)$, we make use of the Minkowski and Young II inequalities with r as in (4.10), using the notation $\int_X$, $\int_Y$ for $\int_{\mathbb{R}^N}$ as in Proposition 4.3. More specifically, we have

$$\begin{aligned} |u^\beta - u^\alpha|_q &\leq \left(\int_X \left(\int_\alpha^\beta \sigma^{-N} \int_Y \left| \Phi\left(\frac{x-y}{\sigma}\right)\right| |\nabla u(y)|\, dy\, d\sigma \right)^q dx \right)^{1/q} \\ &\overset{\text{Minkowski}}{\leq} \int_\alpha^\beta \sigma^{-N} \left(\int_X \left(\int_Y \left| \Phi\left(\frac{x-y}{\sigma}\right)\right| |\nabla u(y)|\, dy \right)^q dx \right)^{1/q} d\sigma \\ &\overset{\text{Young II}}{\leq} \int_\alpha^\beta \sigma^{-N} \left(\int_{\mathbb{R}^N} \left| \Phi\left(\frac{y}{\sigma}\right)\right|^r dy \right)^{1/r} \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p\, dx \right)^{1/p} d\sigma \\ &\leq |\nabla u|_p \left(\int_{B(1)} |\Phi(x)|^r\, dx \right)^{1/r} \int_\alpha^\beta \sigma^{N(1/r-1)}\, d\sigma . \end{aligned} \tag{7.4}$$

We have $N(1/r - 1) = \kappa - 1$, hence

$$|u^\beta - u^\alpha|_q \; \leq \; C_{pq}\, (\beta^\kappa - \alpha^\kappa)\, |\nabla u|_p \tag{7.5}$$

with $C_{pq} = |\Phi|_r / \kappa$. Hence, for every sequence $\sigma_i \to 0+$, u^{σ_i} is a Cauchy sequence in $L^q(\mathbb{R}^N)$. By (6.4), u^{σ_i} converge to u in $L^p(\mathbb{R}^N)$, hence $u \in L^q(\mathbb{R}^N)$ and u^{σ_i} converge to u (strongly) in $L^q(\mathbb{R}^N)$. Letting α tend to 0 and replacing β by σ, we thus obtain (7.1). ∎

Corollary 7.2 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with Lipschitzian boundary, and let*

$$\frac{1}{q} > \frac{1}{p} - \frac{1}{N}\,.$$

Then the space $W^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$.

Proof. Assume first $q \ge p$, and set $u_* = E_p u$, where $E_p : W^{1,p}(\Omega) \to W_B^{1,p}$ is the prolongation operator from Theorem 3.1. By Theorems 3.1 and 7.1, there exist constants C_1 and C_2 such that for every $\sigma \in (0,1]$ we have

$$|u_*^\sigma - u_*|_q \;\le\; C_1\,\sigma^\kappa\,|\nabla u_*|_p \;\le\; C_2\,\sigma^\kappa\,\|u\|_{1;p,\Omega}\,. \tag{7.6}$$

By (6.5) and Theorem 3.1 we have

$$|u_*^\sigma|_q \;\le\; C_3\,\sigma^{\kappa-1}\,|u_*|_p \;\le\; C_3\,c_p\,\sigma^{\kappa-1}\,|u|_{p,\Omega}\,, \tag{7.7}$$

with $C_3 = |\varphi|_r$. Consequently, there exists a constant $C_4 > 0$ such that

$$|u|_{q,\Omega} \;\le\; |u_*|_q \;\le\; C_4\left(\sigma^{\kappa-1}\,|u|_{p,\Omega} + \sigma^\kappa\,\|u\|_{1;p,\Omega}\right) \tag{7.8}$$

for all $\sigma \in (0,1]$. According to (1.1), $W^{1,p}(\Omega)$ is thus embedded in $L^q(\Omega)$. To see that the embedding is compact, consider a bounded set $M \subset W^{1,p}(\Omega)$ and an arbitrary $\varepsilon > 0$. We fix $\sigma > 0$ such that, with the notation of Theorem 7.1, we have

$$C_{pq}\,\sigma^\kappa\,|\nabla u_*|_p \;<\; \frac{\varepsilon}{4} \quad \forall u \in M\,. \tag{7.9}$$

With this fixed σ, every element u_*^σ of the set $M_\sigma = \{u_*^\sigma\,;\, u \in M\}$ vanishes outside of the set $(1+\sigma)B(1) =: B(1+\sigma)$. Moreover, M_σ is bounded in $C^1(\overline{B(1+\sigma)})$, hence, by Proposition 2.1, there exist $u_1, \dots, u_n \in M$ such that

$$\forall u \in M \;\; \exists k \in \{1,\dots,n\} \;\; \forall x \in B(1+\sigma): \;\; |u_*^\sigma(x) - u_k^\sigma(x)| < \frac{\varepsilon}{4\mathrm{meas}(B(1+\sigma))}\,. \tag{7.10}$$

We then have, by (7.9), (7.10), and Theorem 7.1, that

$$|u_* - u_k^\sigma|_q \le |u_*^\sigma - u_k^\sigma|_q + \frac{\varepsilon}{4} < \frac{\varepsilon}{2}\,. \tag{7.11}$$

For $k = 1,\dots,n$ set $M_k = \{u \in M\,;\, |u_* - u_k^\sigma|_q < \varepsilon/2$, and $J = \{k \in \{1,\dots,n\}\,;\, M_k \ne \emptyset\}$. For every $k \in J$ we fix one representative $\hat{u}_k \in M_k$, so that for every $u \in M_k$ we have $|u - \hat{u}_k|_{q,\Omega} < \varepsilon$ and $M = \bigcup_{k\in J} M_k$. The proof is thus complete for $q \ge p$. Let now $q < p$. Hölder's inequality yields

$$|u_*^\sigma - u_*|_q \;\le\; (\mathrm{meas}(B(1+\sigma)))^{1/q-1/p}|u_*^\sigma - u_*|_p\,,$$

hence the above argument remains valid. ■

Corollary 7.3 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with Lipschitzian boundary, and let $p > N$. Then the space $W^{1,p}(\Omega)$ is compactly embedded in $C(\bar{\Omega})$.*

Proof. We repeat the argument of the proof of Theorem 7.1 and Corollary 7.2, putting

$$\kappa \;:=\; 1 - \frac{N}{p} \;\in\; (0,1)\,.$$

A computation analogous to (7.4) yields for every $x \in \mathbb{R}^N$ that

$$\begin{aligned}
|u_*^\beta(x) - u_*^\alpha(x)| \quad &\le \quad \int_\alpha^\beta \sigma^{-N} \int_Y \left|\Phi\left(\frac{x-y}{\sigma}\right)\right| |\nabla u_*(y)|\, dy\, d\sigma \\
&\overset{\text{Hölder}}{\le} \quad \int_\alpha^\beta \sigma^{-N} \left(\int_{\mathbb{R}^N} \left|\Phi\left(\frac{y}{\sigma}\right)\right|^{p'} dy\right)^{1/p'} \left(\int_{\mathbb{R}^N} |\nabla u_*(x)|^p\, dx\right)^{1/p} d\sigma \\
&\le \quad |\nabla u_*|_p \left(\int_{B(1)} |\Phi(x)|^{p'}\, dx\right)^{1/p'} \int_\alpha^\beta \sigma^{-N/p}\, d\sigma\,, \qquad (7.12)
\end{aligned}$$

and we proceed as above. ■

8 Limit cases and counterexamples

Proposition 8.1 *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be an open bounded connected set with Lipschitzian boundary, and let*

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{N}\,. \qquad (8.1)$$

Then the space $W^{1,p}(\Omega)$ is embedded in $L^q(\Omega)$.

Proof. We proceed in principle as in the proof of Theorem 7.1. The main difference is that the number κ is zero here and we have to proceed more carefully. We represent $x \in \mathbb{R}^N$ as $x = (x', x_N)$, $x' \in \mathbb{R}^{N-1}$, and rewrite inequality (7.3) as

$$|u^\beta(x', x_N) - u^\alpha(x', x_N)| \;\le\; \int_\alpha^\beta \sigma^{-N} \int_{\mathbb{R}^N} \left|\Phi\left(\frac{x'-y'}{\sigma}, \frac{x_N - y_N}{\sigma}\right)\right| |\nabla u(y', y_N)|\, dy'\, dy_N\, d\sigma\,. \qquad (8.2)$$

With $r = N/(N-1)$, we now repeat the computation from (7.4), restricted to the component x', to obtain

$$\begin{aligned}
&|u^\beta(\cdot, x_N) - u^\alpha(\cdot, x_N)|_q \\
&\le \quad \left(\int_{\mathbb{R}^{N-1}} \left(\int_\alpha^\beta \sigma^{-N} \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} \left|\Phi\left(\frac{x'-y'}{\sigma}, \frac{x_N-y_N}{\sigma}\right)\right| |\nabla u(y', y_N)|\, dy'\, dy_N\, d\sigma\right)^q dx\right)^{1/q} \\
&\overset{\text{Minkowski}}{\le} \int_{\mathbb{R}} \int_\alpha^\beta \sigma^{-N} \left(\int_{\mathbb{R}^{N-1}} \left(\int_{\mathbb{R}^{N-1}} \left|\Phi\left(\frac{x'-y'}{\sigma}, \frac{x_N-y_N}{\sigma}\right)\right| |\nabla u(y', y_N)|\, dy'\right)^q dx'\right)^{1/q} d\sigma\, dy_N \\
&\overset{\text{Young II}}{\le} \int_{\mathbb{R}} \int_\alpha^\beta \sigma^{-N} \left(\int_{\mathbb{R}^{N-1}} \left|\Phi\left(\frac{y'}{\sigma}, \frac{x_N-y_N}{\sigma}\right)\right|^r dy'\right)^{1/r} \left(\int_{\mathbb{R}^{N-1}} |\nabla u(x', y_N)|^p dx'\right)^{1/p} d\sigma\, dy_N \\
&\le \quad \int_{\mathbb{R}} \int_\alpha^\beta \sigma^{-N+N-1/r} |\nabla u(\cdot, y_N)|_p \left|\Phi\left(\cdot, \frac{x_N-y_N}{\sigma}\right)\right|_r d\sigma\, dy_N\,. \qquad (8.3)
\end{aligned}$$

The function $\left|\Phi\left(\cdot, \frac{x_N-y_N}{\sigma}\right)\right|_r$ vanishes if $\sigma < |x_N - y_N|$. Moreover, Φ is bounded by a constant $\Phi_0 > 0$. Hence, using the fact that $-N + N - 1/r = -2 + 1/N$, we have

$$\int_\alpha^\beta \sigma^{-N+N-1/r} \left|\Phi\left(\cdot, \frac{x_N - y_N}{\sigma}\right)\right|_r d\sigma \;\leq\; \Phi_0 \int_{|x_N-y_N|}^\infty \sigma^{-2+1/N}\, d\sigma \;=\; \Phi_0\, r\, |x_N - y_N|^{-1/r}.$$

We thus have

$$|u^\beta(\cdot, x_N) - u^\alpha(\cdot, x_N)|_q \;\leq\; \Phi_0\, r \int_{\mathbb{R}} |\nabla u(\cdot, y_N)|_p\, |x_N - y_N|^{-1/r}\, dy_N\,.$$

At this point, we use the Hardy-Littlewood inequality (5.1), with q replaced by q'. Indeed, $1/q' + 1/p + 1/r = 2$. Hence, for every function $g \in L^q(\mathbb{R})$ we have by Proposition 5.1 that

$$\int_{\mathbb{R}} |u^\beta(\cdot, x_N) - u^\alpha(\cdot, x_N)|_q\, g(x_N)\, dx_N \;\leq\; C|g|_{q'} |\nabla u|_p$$

with some constant $C > 0$, hence, by the reverse Hölder inequality (4.9), we have

$$|u^\beta - u^\alpha|_q \;\leq\; C|\nabla u|_p\,. \tag{8.4}$$

Since u^σ converge strongly to u in $L^p(\mathbb{R}^N)$ and their L^q norms are bounded, we conclude that they converge strongly in $L^q(\mathbb{R}^N)$ as well and the embedding formula follows. ■

We now show a few examples to illustrate that the embedding inequalities are (at least qualitatively) optimal.

(i) To see that the embedding in Proposition 8.1 is not compact, and that $W^{1,p}(\Omega)$ is not embedded in $L^q(\Omega)$ if

$$\frac{1}{q} < \frac{1}{p} - \frac{1}{N}, \tag{8.5}$$

it suffices to fix any open set Ω, some $x_0 \in \Omega$, find $s_0 > 0$ such that $x_0 + s_0 B(1) \subset \Omega$, and consider the family of functions

$$u_s(x) = s^{1-N/p} \varphi\left(\frac{x - x_0}{s}\right), \quad s \in (0, s_0)\,, \tag{8.6}$$

with φ as in (6.2). We have

$$|u_s|_{p,\Omega} = s|\varphi|_p\,, \quad \left|\frac{\partial u_s}{\partial x_i}\right|_{p,\Omega} = \left|\frac{\partial \varphi}{\partial x_i}\right|_p, \quad |u_s|_{q,\Omega} = s^\alpha |\varphi|_q \quad \forall s \in (0, s_0)\,,$$

where $\alpha = 1 - N(1/p - 1/q)$. In the case (8.1), we have $\alpha = 0$. Using the fact that u_s converge to 0 in $L^p(\Omega)$ as $s \to 0+$, we conclude that the family $\{u_s\}$, having constant nonzero norm in $L^q(\Omega)$, does not contain any convergent subsequence in $L^q(\Omega)$, hence the embedding is not compact. In the case (8.5), we have $\alpha < 0$, hence the family $\{u_s\}$ is unbounded in $L^q(\Omega)$ and no embedding takes place.

(ii) In another limit case

$$p = N\,, \tag{8.7}$$

the space $W^{1,p}(\Omega)$ is embedded in $L^\infty(\Omega)$ if and only if $p = N = 1$, and the embedding is not compact. For $N \ge 2$, it suffices to consider $\Omega = B(1)$, and

$$u(x) = \left(-\log\left(\frac{|x|}{2}\right)\right)^\alpha\,,$$

for any $0 < \alpha < 1 - 1/N$. Then u is unbounded, but belongs to $W^{1,N}(B(1))$. For $N = 1$, the embedding of $W^{1,1}(\Omega)$ into $C(\bar{\Omega})$ (hence $L^\infty(\Omega)$) for every bounded interval Ω is obvious. To see that it is not compact, we may consider for $n \in \mathbb{N}$ the sequence

$$u_n(x) = \begin{cases} \sin\frac{1}{x} & \text{for } x \in \left[\frac{1}{(n+1)\pi}, \frac{1}{n\pi}\right] \\ 0 & \text{otherwise.} \end{cases}$$

It is bounded in $W^{1,1}(0, 1/\pi)$, but $\sup|u_n(x) - u_m(x)| = 1$ for all $m \ne n$, hence it is not precompact in $L^\infty(0, 1/\pi)$.

(iii) The assumption on the Lipschitzian boundary is substantial. We show that there exists an open simply connected set $\Omega \subset \mathbb{R}^2$ such that $W^{1,p}(\Omega)$ is not embedded in $L^q(\Omega)$ for any $q > p \ge 1$. This set can be defined as (see Fig. 4)

$$\Omega \;=\; \left\{x = (x_1, x_2) \in \mathbb{R}^2\,;\, 0 < x_1 < 1,\, 0 < x_2 < \mathrm{e}^{\,-1/x_1}\right\}\,.$$

For any $q > p$ we set

$$u_{pq}(x) = \mathrm{e}^{\,2/(p+q)x_1}\,.$$

Then $u_{pq} \in W^{1,p}(\Omega)$, but $u_{pq} \notin L^q(\Omega)$.

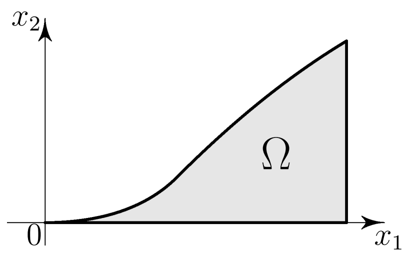

Figure 4. Non-Lipschitzian boundary.

9 Anisotropic embeddings

In evolution problems, one deals with functions which depend on a space variable $x \in \Omega$ and time $t \in \omega$, where $\omega \subset \mathbb{R}$ is an open interval corresponding to the time of the process. For $1 \le p, q < \infty$, we introduce the spaces

$$L^p(\omega\,;\, L^q(\Omega)) \;=\; \left\{u \in L^1(\Omega \times \omega)\,;\, |u|_{p,q,\Omega,\omega} := \left(\int_\omega |u(\cdot,t)|^p_{q,\Omega}\,dt\right)^{1/p} < \infty\right\}, \tag{9.1}$$

with obvious modifications for $p=\infty$ or $q=\infty$.

We state explicitly one possible embedding result for such spaces, without going into much detail in the proof, which is fully analogous to the above ones.

Theorem 9.1 *Let $\Omega\subset\mathbb{R}^N$ be an open bounded connected set with Lipschitzian boundary, let ω be a bounded open interval, and let $W^{p_0,q_0;p_1,q_1}(\omega;\Omega)$ be the space*

$$W^{p_0,q_0;p_1,q_1}(\omega;\Omega) \;=\; \Big\{u\in L^1(\Omega\times\omega)\,;\,\frac{\partial u}{\partial t}\in L^{p_0}(\omega\,;\,L^{q_0}(\Omega)),\; \frac{\partial u}{\partial x_i}\in L^{p_1}(\omega\,;\,L^{q_1}(\Omega)) \text{ for } i=1,\dots,N\Big\}.$$

If

$$\frac{p_0'}{p_1q_0}+\frac{1}{q_1}<\frac{1}{N}\,, \tag{9.2}$$

then the space $W^{p_0,q_0;p_1,q_1}(\omega;\Omega)$ is compactly embedded in $C(\bar\Omega\times\bar\omega)$. If $q_2\ge\max\{q_0,q_1\}$, $p_2\ge\max\{p_0,p_1\}$, and

$$\left(1-\frac{1}{p_0}+\frac{1}{p_2}\right)\left(\frac{1}{N}-\frac{1}{q_1}+\frac{1}{q_2}\right)>\left(\frac{1}{p_1}-\frac{1}{p_2}\right)\left(\frac{1}{q_0}-\frac{1}{q_2}\right), \tag{9.3}$$

then $W^{p_0,q_0;p_1,q_1}(\omega;\Omega)$ is compactly embedded in $L^{p_2}(\omega\,;\,L^{q_2}(\Omega))$.

Hint for the proof. Consider as before the extensions to the space $W^{p_0,q_0;p_1,q_1}(\mathbb{R};\mathbb{R}^N)$, where the norms $|\cdot|_{p_i,q_i,\Omega,\omega}$ are denoted again for simplicity as $|\cdot|_{p_i,q_i}$, $i=0,1$. For $\sigma\in(0,1]$ and $u\in W^{p_0,q_0;p_1,q_1}(\mathbb{R};\mathbb{R}^N)$, we define regularizations analogous to (6.2) in the form

$$u^\sigma(x,t) \;=\; \sigma^{-N-\lambda}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\varphi\left(\frac{x-y}{\sigma},\frac{t-s}{\sigma^\lambda}\right)u(y,s)\,dy\,ds\,, \tag{9.4}$$

where φ is a smooth nonnegative function on $\mathbb{R}^{N+1}$, which vanishes outside $B(1)\times(-1,1)$, and

$$\int_{-1}^{1}\int_{B(1)}\varphi(x,t)\,dx\,dt \;=\; 1\,.$$

The number λ is to be chosen as

$$\lambda \;=\; \frac{1+N\left(\frac{1}{q_0}-\frac{1}{q_1}\right)}{\frac{1}{p_0'}+\frac{1}{p_1}}\,. \tag{9.5}$$

Note that $\lambda>0$ by (9.3). A computation similar to (7.2)–(7.4) yields

$$\frac{\partial}{\partial\sigma}u^\sigma(x,t) \;=\; -\lambda\sigma^{-N-1}\int_{\mathbb{R}^N}\Phi_0\left(\frac{x-y}{\sigma},\frac{t-s}{\sigma^\lambda}\right)\frac{\partial u}{\partial s}(y,s)\,dy\,ds \;-\sigma^{-N-\lambda}\int_{\mathbb{R}^N}\left\langle\Phi_1\left(\frac{x-y}{\sigma},\frac{t-s}{\sigma^\lambda}\right),\nabla_y u(y,s)\right\rangle dy\,ds\,, \tag{9.6}$$

where $\Phi_0(\xi,\tau)=\tau\varphi(\xi,\tau)$, $\Phi_1(\xi,\tau)=\xi\varphi(\xi,\tau)$, hence

$$|u^\beta(x,t)-u^\alpha(x,t)| \;\le\; \lambda\mathcal{I}_0(x,t)+\mathcal{I}_1(x,t) \tag{9.7}$$

for $0<\alpha<\beta\le 1$, where

$$\left.\begin{aligned}\mathcal{I}_0(x,t) &= \textstyle\int_\alpha^\beta \sigma^{-N-1}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\left|\Phi_0\left(\frac{x-y}{\sigma},\frac{t-s}{\sigma^\lambda}\right)\right|\left|\frac{\partial u}{\partial s}(y,s)\right| dy\,ds\,d\sigma\,,\\ \mathcal{I}_1(x,t) &= \textstyle\int_\alpha^\beta \sigma^{-N-\lambda}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\left|\Phi_1\left(\frac{x-y}{\sigma},\frac{t-s}{\sigma^\lambda}\right)\right||\nabla_y u(y,s)|\,dy\,ds\,d\sigma\,.\end{aligned}\right\} \tag{9.8}$$

Let (9.3) hold. With the intention to use Young's inequality for convolutions again, we introduce the numbers r_0,s_0,r_1,s_1 by the identities

$$\frac{1}{r_0}=1-\frac{1}{q_0}+\frac{1}{q_2}\,,\quad \frac{1}{s_0}=1-\frac{1}{p_0}+\frac{1}{p_2}\,,\quad \frac{1}{r_1}=1-\frac{1}{q_1}+\frac{1}{q_2}\,,\quad \frac{1}{s_1}=1-\frac{1}{p_1}+\frac{1}{p_2}\,. \tag{9.9}$$

We use again the notation $\int_X dx$, $\int_Y dy$ for $\int_{\mathbb{R}^N} dx$, $\int_{\mathbb{R}^N} dy$, and $\int_T dt$, $\int_S ds$ for $\int_{\mathbb{R}} dt$, $\int_{\mathbb{R}} ds$. For $t\in\mathbb{R}$, we have

$$\begin{aligned}|\mathcal{I}_0(\cdot,t)|_{q_2} &\overset{\text{Minkowski}}{\le} \int_\alpha^\beta \sigma^{-N-1}\int_S\left(\int_X\left(\int_Y\left|\Phi_0\left(\frac{x-y}{\sigma},\frac{t-s}{\sigma^\lambda}\right)\right|\left|\frac{\partial u}{\partial s}(y,s)\right|dy\right)^{q_2}dx\right)^{1/q_2} ds\,d\sigma\\ &\overset{\text{Young II}}{\le} \int_\alpha^\beta \sigma^{-N-1}\int_S\left|\Phi_0\left(\frac{\cdot}{\sigma},\frac{t-s}{\sigma^\lambda}\right)\right|_{r_0}\left|\frac{\partial u}{\partial s}(\cdot,s)\right|_{q_0} ds\,d\sigma\\ &= \int_\alpha^\beta \sigma^{-N-1+N/r_0}\int_S\left|\Phi_0\left(\cdot,\frac{t-s}{\sigma^\lambda}\right)\right|_{r_0}\left|\frac{\partial u}{\partial s}(\cdot,s)\right|_{q_0} ds\,d\sigma\,,\end{aligned} \tag{9.10}$$

hence

$$\begin{aligned}|\mathcal{I}_0|_{p_2,q_2} &\overset{\text{Minkowski}}{\le} \int_\alpha^\beta \sigma^{-N-1+N/r_0}\left(\int_T\left(\int_S\left|\Phi_0\left(\cdot,\frac{t-s}{\sigma^\lambda}\right)\right|_{r_0}\left|\frac{\partial u}{\partial s}(\cdot,s)\right|_{q_0} ds\right)^{p_2}dt\right)^{1/p_2} d\sigma\\ &\overset{\text{Young II}}{\le} \int_\alpha^\beta \sigma^{-N-1+N/r_0}\left|\Phi_0\left(\cdot,\frac{\cdot}{\sigma^\lambda}\right)\right|_{s_0,r_0}\left|\frac{\partial u}{\partial s}\right|_{p_0,q_0} d\sigma\\ &= |\Phi_0|_{s_0,r_0}\left|\frac{\partial u}{\partial s}\right|_{p_0,q_0}\int_\alpha^\beta \sigma^{-N-1+N/r_0+\lambda/s_0}\,d\sigma\,.\end{aligned} \tag{9.11}$$

Similarly,

$$\begin{aligned}|\mathcal{I}_1(\cdot,t)|_{q_2} &\overset{\text{Minkowski}}{\le} \int_\alpha^\beta \sigma^{-N-\lambda}\int_S\left(\int_X\left(\int_Y\left|\Phi_1\left(\frac{x-y}{\sigma},\frac{t-s}{\sigma^\lambda}\right)\right||\nabla_y u(y,s)|\,dy\right)^{q_2}dx\right)^{1/q_2} ds\,d\sigma\\ &\overset{\text{Young II}}{\le} \int_\alpha^\beta \sigma^{-N-\lambda}\int_S\left|\Phi_1\left(\frac{\cdot}{\sigma},\frac{t-s}{\sigma^\lambda}\right)\right|_{r_1}|\nabla_y u(\cdot,s)|_{q_1}\,ds\,d\sigma\\ &= \int_\alpha^\beta \sigma^{-N-\lambda+N/r_1}\int_S\left|\Phi_1\left(\cdot,\frac{t-s}{\sigma^\lambda}\right)\right|_{r_1}|\nabla_y u(\cdot,s)|_{q_1}\,ds\,d\sigma\,,\end{aligned} \tag{9.12}$$

hence

$$|\mathcal{I}_1|_{p_2,q_2} \overset{\text{Minkowski}}{\leq} \int_\alpha^\beta \sigma^{-N-\lambda+N/r_1} \left(\int_T \left(\int_S \left|\Phi_1\left(\cdot, \frac{t-s}{\sigma^\lambda}\right)\right|_{r_1} |\nabla_y u(\cdot,s)|_{q_1}\, ds\right)^{p_2} dt\right)^{1/p_2} d\sigma$$

$$\overset{\text{Young II}}{\leq} \int_\alpha^\beta \sigma^{-N-\lambda+N/r_1} \left|\Phi_1\left(\cdot, \frac{\cdot}{\sigma^\lambda}\right)\right|_{s_1,r_1} |\nabla_y u|_{p_1,q_1}\, d\sigma$$

$$= |\Phi_1|_{s_1,r_1} |\nabla_y u|_{p_1,q_1} \int_\alpha^\beta \sigma^{-N-\lambda+N/r_1+\lambda/s_1}\, d\sigma\,. \tag{9.13}$$

Set

$$\kappa = N\left(\frac{1}{p_0'}+\frac{1}{p_1}\right)^{-1}\left(\left(1-\frac{1}{p_0}+\frac{1}{p_2}\right)\left(\frac{1}{N}-\frac{1}{q_1}+\frac{1}{q_2}\right)-\left(\frac{1}{q_0}-\frac{1}{q_2}\right)\left(\frac{1}{p_1}-\frac{1}{p_2}\right)\right).$$

Then $\kappa > 0$ by (9.3), and we have

$$-N-1+\frac{N}{r_0}+\frac{\lambda}{s_0} = -N-\lambda+\frac{N}{r_1}+\frac{\lambda}{s_1} = \kappa-1\,.$$

Combining (9.7) with (9.8), (9.11), and (9.13) yields

$$|u^\beta - u^\alpha|_{p_2,q_2} \leq C_{p_0,p_1,p_2,q_0,q_1,q_2}\,(\beta^\kappa-\alpha^\kappa)\left(\left|\frac{\partial u}{\partial t}\right|_{p_0,q_0} + |\nabla_x u|_{p_1,q_1}\right), \tag{9.14}$$

and we obtain the result similarly as in Theorem 7.1. ■

Note that the order of integration in (9.1) cannot be reversed. For $p \geq q$ we have by Remark 4.5 that $L^q(\Omega\,;\,L^p(\omega))$ is embedded into $L^p(\omega\,;\,L^q(\Omega))$, but the opposite inclusion does not hold, see Example 4.4. On the other hand, denoting

$$W^{q_0,p_0;q_1,p_1}(\Omega;\omega) = \Big\{u \in L^1(\Omega\times\omega)\,;\, \frac{\partial u}{\partial t} \in L^{q_0}(\Omega\,;\,L^{p_0}(\omega)),\ \frac{\partial u}{\partial x_i} \in L^{q_1}(\Omega\,;\,L^{p_1}(\omega)) \text{ for } i=1,\dots,N\Big\},$$

we may repeat the computations in (9.10)–(9.13) with reversed order of integration, to check that conditions (9.2) and (9.3) remain valid for the compact embedding of $W^{q_0,p_0;q_1,p_1}(\Omega;\omega)$ into $C(\bar\Omega\times\bar\omega)$ and $L^{q_2}(\Omega\,;\,L^{p_2}(\omega))$, respectively. Let us mention one important particular case which frequently occurs in applications. We omit the proof which is the same as for the other cases.

Corollary 9.2 *If* $q_2 \geq \max\{q_0,q_1\}$, *and*

$$\frac{1}{p_0'}\left(\frac{1}{N}-\frac{1}{q_1}+\frac{1}{q_2}\right) > \frac{1}{p_1}\left(\frac{1}{q_0}-\frac{1}{q_2}\right), \tag{9.15}$$

then the space $W^{q_0,p_0;q_1,p_1}(\Omega;\omega)$ *is compactly embedded in* $L^{q_2}(\Omega\,;\,C(\bar\omega))$.

Embeddings of function spaces that are "anisotropic" also in the space variables, for example

$$\frac{\partial u}{\partial x_i} \in L^{p_i}(\omega\,;\, L^{q_i}(\Omega))\,, \quad i = 1, \dots, N\,,$$

can be treated in the same way. The regularizations then have to be chosen in the form

$$u^\sigma(x,t) \;=\; \sigma^{-1-\sum\mu_i} \int_{\mathbb{R}} \int_{\mathbb{R}^N} \varphi\left(\frac{x_1 - y_1}{\sigma^{\mu_1}}, \dots, \frac{x_N - y_N}{\sigma^{\mu_N}}, \frac{t-s}{\sigma}\right) u(y,s)\,dy\,ds\,, \tag{9.16}$$

with suitably chosen exponents $\mu_1, \dots, \mu_N$.

10 Interpolations

We first recall the following classical interpolation result in L^p spaces.

Proposition 10.1 *Let $\Omega \subset \mathbb{R}^N$ be an open set (bounded or unbounded), and let $1 \le p_0 < p_1 \le \infty$ be given. If $u \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$, then $u \in L^p(\Omega)$ for all $p \in [p_0, p_1]$, and we have*

$$|u|_{p,\Omega} \;\le\; |u|_{p_0,\Omega}^{1-\alpha} |u|_{p_1,\Omega}^{\alpha}$$

for all $u \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$, where

$$\alpha = \frac{\frac{1}{p_0} - \frac{1}{p}}{\frac{1}{p_0} - \frac{1}{p_1}}\,.$$

Proof. Set $q = p_1/\alpha p$. Then $q' = p_0/(1-\alpha)p$, and we may use Hölder's inequality to obtain

$$\begin{aligned} |u|_{p,\Omega} &= \left(\int_\Omega |u(x)|^{(1-\alpha)p} |u(x)|^{\alpha p} dx\right)^{1/p} \\ &\le \left(\int_\Omega |u(x)|^{(1-\alpha)pq'} dx\right)^{1/pq'} \left(\int_\Omega |u(x)|^{\alpha pq} dx\right)^{1/pq} \\ &= |u|_{p_0,\Omega}^{1-\alpha} |u|_{p_1,\Omega}^{\alpha}\,. \end{aligned}$$

■

We now establish an interpolation formula between L^p spaces and Sobolev spaces.

Theorem 10.2 *Let $p, q, s \in (1, \infty)$ be such that*

$$\frac{1}{s} > \frac{1}{q} > \frac{1}{p} - \frac{1}{N}\,,$$

and set

$$\kappa \;:=\; 1 - N\left(\frac{1}{p} - \frac{1}{q}\right)\,, \quad \gamma = N\left(\frac{1}{s} - \frac{1}{q}\right)\,.$$

Then there exists $C_{pqs} > 0$ such that for every $u \in W^{1,p}(\mathbb{R}^N) \cap L^s(\mathbb{R}^N)$ and every $\sigma \in (0,1]$ we have

$$|u|_q \;\le\; C_{pqs}\left(\sigma^{-\gamma}|u|_s + \sigma^\kappa\,|\nabla u|_p\right)\,. \tag{10.1}$$

Proof. The assertion follows from (6.5) and (7.1) provided $q \geq p$. In particular, for $q = p$ we have $\kappa = 1$, $\gamma = \gamma_0 := N(1/s - 1/p)$, and

$$|u|_p \;\leq\; C_{pps}\left(\sigma^{-\gamma_0}|u|_s + \sigma\,|\nabla u|_p\right) . \tag{10.2}$$

Let now $q < p$. By Proposition 10.1 we have

$$|u|_q \;\leq\; |u|_s^{1-\alpha}|u|_p^{\alpha}\,,$$

where

$$\alpha = \frac{\frac{1}{s} - \frac{1}{q}}{\frac{1}{s} - \frac{1}{p}}\,.$$

This yields

$$|u|_p \;\leq\; C_{pps}^{\alpha}\left(\sigma^{-\alpha\gamma_0}|u|_s + \sigma^{\alpha}\,|u|_s^{1-\alpha}\,|\nabla u|_p^{\alpha}\right) .$$

We now use inequality (4.4) with p replaced by $1/\alpha$, and with $x = \mu\sigma^{\alpha}|\nabla u|_p^{\alpha}$, $y = |u|_s^{1-\alpha}/\mu$, where we set $\mu = \sigma^{(1-\alpha)\alpha\gamma_0}$, and obtain

$$\sigma^{\alpha}\,|u|_s^{1-\alpha}\,|\nabla u|_p^{\alpha} \leq \alpha\sigma^{1+(1-\alpha)\gamma_0}\,|\nabla u|_p + (1-\alpha)\sigma^{-\alpha\gamma_0}\,|u|_s\,.$$

Hence,

$$|u|_p \;\leq\; 2C_{pps}^{\alpha}\left(\sigma^{-\alpha\gamma_0}|u|_s + \sigma^{1+(1-\alpha)\gamma_0}\,|\nabla u|_p\right) ,$$

which is precisely (10.1). ∎

We conclude this text with the famous Gagliardo-Nirenberg inequality.

Corollary 10.3 ***(Gagliardo-Nirenberg inequality)*** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with Lipschitzian boundary, and let*

$$\frac{1}{s} > \frac{1}{q} > \frac{1}{p} - \frac{1}{N}\,.$$

Set

$$\varrho \;=\; \frac{\frac{1}{s} - \frac{1}{q}}{\frac{1}{N} + \frac{1}{s} - \frac{1}{p}}\,.$$

Then there exists a constant $K_{pqs} > 0$ such that for every $u \in W^{1,p}(\Omega)$ we have

$$|u|_{q,\Omega} \;\leq\; K_{pqs}\left(|u|_{s,\Omega} + |u|_{s,\Omega}^{1-\varrho}\|u\|_{1;p,\Omega}^{\varrho}\right) . \tag{10.3}$$

Proof. As in the proof of Corollary 7.2, we set $u_* = E_p u$. By Theorem 10.2, we have

$$|u_*|_q \;\leq\; C_{pqs}\left(\sigma^{-\gamma}|u_*|_s + \sigma^{\kappa}\,|\nabla u_*|_p\right) . \tag{10.4}$$

If $|\nabla u_*|_p > |u_*|_s$, then we set

$$\sigma = \left(\frac{|u_*|_s}{|\nabla u_*|_p}\right)^{1/(\gamma+\kappa)}\,,$$

otherwise we choose $\sigma = 1$. In both cases we obtain

$$|u_*|_q \;\leq\; 2C_{pqs}\left(|u_*|_s + |u_*|_s^{\kappa/(\gamma+\kappa)}\,|\nabla u_*|_p^{\gamma/(\gamma+\kappa)}\right) . \tag{10.5}$$

We have $\kappa/(\gamma+\kappa) = 1 - \varrho$, $\gamma/(\gamma+\kappa) = \varrho$, and the desired result follows from Theorem 3.1. ∎

It is in principle possible to derive from (9.14) the corresponding interpolation inequalities also for anisotropic spaces. The general formulas then become, however, rather complicated and we omit them here.

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