On the motion of compressible inviscid fluids driven by stochastic forcing

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Driven Euler system

Field equations

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho \mathbf{u})\mathrm{d}t = 0$$

$$\mathrm{d}(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u})\mathrm{d}t + \nabla_x p(\varrho)\mathrm{d}t = \varrho \mathbf{G}(\varrho, \varrho \mathbf{u})\mathrm{d}W,$$

Stochastic forcing

$$\varrho \mathbf{G}(\varrho, \varrho \mathbf{u})\mathrm{d}W = \sum_{k=1}^{\infty} \varrho \mathbf{G}_k(\varrho, \varrho \mathbf{u})\mathrm{d}\beta_k$$

Iconic examples

$$\varrho \mathbf{G}(\varrho, \varrho \mathbf{u})\mathrm{d}W = \varrho \sum_{k=1}^{\infty} \mathbf{G}_k(x)\mathrm{d}\beta_k, \ \ \varrho \mathbf{G}(\varrho, \varrho \mathbf{u})\mathrm{d}W = \lambda \varrho \mathbf{u} \mathrm{d}\beta$$

Initial and boundary conditions

(Random) initial data

$$\varrho(0,\cdot) = \varrho_0,\ (\varrho\mathbf{u})(0,\cdot) = (\varrho\mathbf{u})_0$$

Periodic boundary conditions

$$\Omega = \mathcal{T}^N = \left([0,1]|_{\{0,1\}}\right)^N,\ N = (1), 2, 3$$

Weak (PDE) formulation

Field equations

$$\left[\int_\Omega \varrho\phi \ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \varrho\mathbf{u}\cdot\nabla_x\phi \ \mathrm{d}x\mathrm{d}t,$$

$$\left[\int_\Omega \varrho\mathbf{u}\cdot\phi \ \mathrm{d}x\right]_{t=0}^{t=\tau} - \int_0^\tau \int_\Omega \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\phi + p(\varrho)\mathrm{div}_x\phi \ \mathrm{d}x\mathrm{d}t$$

$$= \boxed{\int_0^\tau \left(\int_\Omega \varrho\mathbf{G}\cdot\phi \ \mathrm{d}x\right)\mathrm{d}W}$$

$\phi = \phi(x) -$ a smooth test function

Stochastic integral (Itô's formulation)

$$\int_0^\tau \left(\int_\Omega \varrho\mathbf{G}\cdot\phi \ \mathrm{d}x\right)\mathrm{d}W = \sum_{k=1}^\infty \int_0^\tau \left(\int_\Omega \varrho\mathbf{G}_k\cdot\phi \ \mathrm{d}x\right)\mathrm{d}\beta_k$$

Admissibility

Energy inequality

$$-\int_0^T \partial_t \psi \Big(\int_\Omega \Big[\frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho) \Big] \, \mathrm{d}x \Big) \, \mathrm{d}t$$

$$\leq \psi(0) \int_\Omega \Big[\frac{|(\varrho \mathbf{u})_0|^2}{2\varrho_0} + H(\varrho_0) \Big] \, \mathrm{d}x$$

$$+\frac{1}{2} \int_0^T \psi \Big(\int_\Omega \sum_{k \geq 1} \frac{|\mathbf{G}_k(\varrho, \varrho \mathbf{u})|^2}{\varrho} \, \mathrm{d}x \Big) \mathrm{d}t + \int_0^T \psi \mathrm{d} M_E$$

$$\psi \geq 0, \ \psi(T) = 0, \ H(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \, \mathrm{d}z$$

Relative energy inequality

Relative energy

$$\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) = \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + H(\varrho) - H'(r)(\varrho - r) - H(r)\right] \, \mathrm{d}x$$

Relative energy inequality

$$- \int_0^T \partial_t \psi \; \mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) \, \mathrm{d}t$$

$$\leq \psi(0)\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right)(0) + \int_0^T \psi \mathrm{d}M_{RE} + \int_0^T \psi \mathcal{R}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) \mathrm{d}t$$

Test functions

$$\mathrm{d}r = D_t^d r \, \mathrm{d}t + \mathbb{D}_t^s r \, \mathrm{d}W, \; \mathrm{d}\mathbf{U} = D_t^d \mathbf{U} \, \mathrm{d}t + \mathbb{D}_t^s \mathbf{U} \, \mathrm{d}W$$

Remainder

Remainder term

$$\mathcal{R}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) = \int_\Omega \varrho\Big(D_t^d \mathbf{U} + \mathbf{u}\cdot\nabla_x \mathbf{U}\Big)(\mathbf{U}-\mathbf{u})\ \mathrm{d}x$$

$$+ \int_\Omega \big((r-\varrho)H''(r)D_t^d r + \nabla_x H'(r)(r\mathbf{U} - \varrho\mathbf{u})\big)\ \mathrm{d}x$$

$$- \int_\Omega \mathrm{div}_x \mathbf{U}(p(\varrho) - p(r))\ \mathrm{d}x$$

$$+\frac{1}{2}\sum_{k\geq 1}\int_\Omega \varrho\left|\frac{\mathbf{G}_k(\varrho, \varrho\mathbf{u})}{\varrho} - [\mathbb{D}_t^s \mathbf{U}]_k\right|^2\ \mathrm{d}x$$

$$+\frac{1}{2}\sum_{k\geq 1}\int_\Omega \varrho H'''(r)|[\mathbb{D}_t^s r]_k|^2\ \mathrm{d}x + \frac{1}{2}\sum_{k\geq 1}\int_\Omega p''(r)|[\mathbb{D}_t^s r]_k|^2\ \mathrm{d}x$$

Existence theory

Local existence of strong solutions [Kim [2011]], [Breit, EF, Hofmanová [2017]]

If the initial data are smooth, then the problem admits local-in-time smooth solutions. Solutions exist up to a (maximal) positive *stopping time*. The life-span is a random variable.

Weak–strong uniqueness [Breit, EF, Hofmanová [2016]]

Pathwise uniqueness.
A weak and strong solutions defined on the same probability space and emanating from the same initial data coincide as long as the latter exists
Uniqueness in law.
If a weak and strong solution are defined on a different probability space, then their *laws* are the same provided the laws of the initial data are the same

Weak (PDE) solutions

Infinitely many weak (PDE) solutions, Breit, EF, Hofmanová [2017]

Let $T > 0$ and the initial data

$$\varrho_0 \in C^3(\Omega),\ \varrho_0 > 0,\ \mathbf{u}_0 \in C^3(\Omega)$$

be given.
There exists a sequence of *strictly positive* stopping times

$$\tau_M > 0,\ \tau_M \to \infty$$

a.s. such that the initial–value problem for the compressible Euler system possesses infinitely many solutions defined in $(0, T \wedge \tau_M)$. Solutions are adapted to the filtration associated to the Wiener process W.

Semi-deterministic approach - additive noise

"Additive noise" problem

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \varrho \sum_{k=1}^{\infty} \mathbf{G}_k \partial_t \beta_k$$

$$\varrho \sum_{k=1}^{\infty} \mathbf{G}_k \partial_t \beta_k = \varrho \mathbf{G} \mathrm{d} W$$

Additive noise, Step I

Step I

$$\partial_t(\varrho\mathbf{u}-\varrho\mathbf{G}W)+\mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u})+\nabla_x p(\varrho) = -\partial_t\varrho\mathbf{G}W = \mathrm{div}_x(\varrho\mathbf{u})\mathbf{G}W$$

Transformed system I

$$\mathbf{w} = \varrho\mathbf{u} - \varrho\mathbf{G}W$$

$$\partial_t\varrho + \mathrm{div}_x(\mathbf{w} + \varrho\mathbf{G}W) = 0$$

$$\partial_t\mathbf{w} + \mathrm{div}_x\left(\frac{(\mathbf{w}+\varrho\mathbf{G}W)\otimes(\mathbf{w}+\varrho\mathbf{G}W)}{\varrho}\right) + \nabla_x p(\varrho)$$

$$= \mathrm{div}_x(\mathbf{w}+\varrho\mathbf{G}W)\mathbf{G}W$$

Additive noise, Step II

Step II

$$\mathbf{w} = \mathbf{v} + \mathbf{V} + \nabla_x \Phi,\ \mathrm{div}_x \mathbf{v} = 0,\ \int_\Omega \mathbf{v}\ \mathrm{d}x = 0,\ \mathbf{V} = \mathbf{V}(t)$$

Transformed system II

$$\mathbf{w} = \varrho \mathbf{u} - \varrho \mathbf{G} W$$

$$\partial_t \varrho + \mathrm{div}_x (\nabla_x \Phi + \varrho \mathbf{G} W) = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{V} + \nabla_x \Phi + \varrho \mathbf{G} W) \otimes (\mathbf{v} + \mathbf{V} + \nabla_x \Phi + \varrho \mathbf{G} W)}{\varrho} \right)$$

$$+ \nabla_x p(\varrho) + \nabla_x \partial_t \Phi = \mathrm{div}_x (\nabla_x \Phi + \varrho \mathbf{G} W) \mathbf{G} W - \partial_t \mathbf{V}$$

Additive noise, Step III

Step III

Fix Φ, ϱ, $\mathbf{V}$ so that

$$\varrho(0,\cdot) = \varrho_0, \ \mathbf{V}(0) = \frac{1}{|\Omega|}\int_\Omega \mathbf{u}_0 \ \mathrm{d}x, \ \nabla_x \Phi(0,\cdot) = \mathbf{H}^\perp[\mathbf{u}_0]$$

$$\partial_t \varrho + \mathrm{div}_x(\nabla_x \Phi + \varrho \mathbf{G} W) = 0$$

$$\partial_t \mathbf{V} = \frac{1}{|\Omega|}\mathrm{div}_x(\nabla_x \Phi + \varrho \mathbf{G} W)\mathbf{G} W$$

$$\mathrm{div}_x \left(\nabla_x \mathbf{M} + \nabla_x \mathbf{M}^\perp - \frac{2}{N}\mathrm{div}_x \mathbf{M} \right)$$

$$= \mathrm{div}_x(\nabla_x \Phi + \varrho \mathbf{G} W)\mathbf{G} W - \partial_t \mathbf{V}$$

Additive noise, Step IV

Step IV

Fix $\mathbf{h}$, $\mathbb{H}$ so that

$$\mathbf{h} = \mathbf{V} + \nabla_x \Phi + \varrho \mathbf{G} W, \ \mathbb{H} = \nabla_x \mathbf{M} + \nabla_x^t \mathbf{M} - \frac{2}{N} \mathrm{div}_x \mathbf{M} \mathbb{I} \in R^{N \times N}_{0,\mathrm{sym}}$$

Tranformed system III

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{h}) \otimes (\mathbf{v} + \mathbf{h})}{\varrho} - \mathbb{H} + p(\varrho)\mathbb{I} + \partial_t \Phi \mathbb{I} \right) = 0$$

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0] - \frac{1}{|\Omega|} \int_\Omega \mathbf{u}_0 \ \mathrm{d}x$$

Additive noise, Step V

Prescribing the kinetic energy

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} = e = \Lambda - \frac{N}{2}\left(p(\varrho) + \partial_t \Phi\right), \ \Lambda = \Lambda(t)$$

Abstract Euler system

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v}+\mathbf{h}) \otimes (\mathbf{v}+\mathbf{h})}{\varrho} - \frac{1}{N}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho}\mathbb{I} - \mathbb{H} \right) = 0$$

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} = e$$

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0$$

Subsolutions

Field equations, differential constraints

$$\partial_t \mathbf{v} + \mathrm{div}_x \mathbb{F} = 0, \ \mathrm{div}_x \mathbf{v} = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{v}(T, \cdot) = \mathbf{v}_T$$

Non-linear constraint

$$\mathbf{v} \in C([0, T] \times \Omega; R^N), \ \mathbb{F} \in C([0, T] \times \Omega; R^{N \times N}_{\mathrm{sym},0}),$$

$$\frac{N}{2} \lambda_{\max} \left[\frac{(\mathbf{v} + \mathbf{h}) \otimes (\mathbf{v} + \mathbf{h})}{\varrho} - \mathbb{F} + \mathbb{M} \right] < e$$

Subsolution relaxation

Algebraic inequality

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} \leq \frac{N}{2}\lambda_{\max}\left[\frac{(\mathbf{v}+\mathbf{h})\otimes(\mathbf{v}+\mathbf{h})}{\varrho} - \mathbb{F} + \mathbb{M}\right] < e$$

Solutions

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} = e$$

$$\Rightarrow$$

$$\mathbb{F} = \frac{(\mathbf{v}+\mathbf{h})\otimes(\mathbf{v}+\mathbf{h})}{\varrho} - \frac{1}{N}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho}\mathbb{I} + \mathbb{M}$$

Augmenting oscillations

Oscillatory lemma

If

$$\varrho, e, \mathbf{h} \in C(Q; R^N), \varrho, e > 0, \ \mathbb{H} \in C(Q; R^{N\times N}_{\rm sym,0})$$

$$\frac{N}{2}\lambda_{\rm max}\left[\frac{\mathbf{h}\otimes\mathbf{h}}{\varrho} - \mathbb{H}\right] < e \text{ in } Q,$$

then there exist

$$\mathbf{w}_n \in C^\infty_c(Q; R^N), \ \mathbb{G}_n \in C^\infty_c(Q; R^{N\times N}_{\rm sym,0}), \ n = 0, 1, \dots$$

$$\partial_t \mathbf{w}_n + {\rm div}_x \mathbb{G}_n = 0, \ {\rm div}_x \mathbf{w}_n = 0 \text{ in } R \times R^N,$$

$$\frac{N}{2}\lambda_{\rm max}\left[\frac{(\mathbf{h}+\mathbf{w}_n)\otimes(\mathbf{h}+\mathbf{w}_n)}{\varrho} - (\mathbb{H}+\mathbb{G}_n)\right] < e$$

$$\mathbf{w}_n \rightharpoonup 0, \ \liminf_{n\to\infty} \int_Q \frac{|\mathbf{w}_n|^2}{\varrho} \ {\rm d}x{\rm d}t \geq \Lambda(\max_\Omega e) \int_Q \left(e - \frac{1}{2}\frac{|\mathbf{h}|^2}{\varrho}\right)^2 \ {\rm d}x{\rm d}t$$

Basic ideas of proof [DeLellis and Székelyhidi]

Basic result

Unit cube and constant coefficients ϱ, e, $\mathbf{h}$, $\mathbb{H}$

Scaling

Localizing the basic result to “small” cubes by means of scaling arguments

Approximation

Replacing all continuous functions by their means on any of the “small” cubes

Difficulties in the stochastic world

Adaptiveness

All quantities must be adapted to the filtration associated to the Wiener process W

Geometric setting

Continuous functions approximated in a similar way as in the definition of Itô's integral
Admissible directions for oscillations selected by the Kuratowski, Ryll–Nardzewski theorem

Space–time localization

Stopping the Wiener process by its Hölder norm