Computational comparison of methods for two-sided bounds of eigenvalues

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European
Finite
Element
Fair
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Lower bounds on eigenvalues

Laplace eigenvalue problem

$$
\begin{aligned}
-\Delta u_i &= \lambda_i u_i && \text{in } \Omega \\
u_i &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

Weak formulation

$$u_i \in H_0^1(\Omega): \quad (\nabla u_i, \nabla v) = \lambda_i (u_i, v) \quad \forall v \in H_0^1(\Omega)$$

Finite element method

$$V_h = \{v_h \in H_0^1(\Omega) : v_h|_K \in P_1(K), \quad \forall K \in \mathcal{T}_h\}$$

$$u_{h,i} \in V_h : \quad (\nabla u_{h,i}, \nabla v_h) = \Lambda_{h,i}(u_{h,i}, v_h) \quad \forall v_h \in V_h$$

Lower bound?

$$? \leq \lambda_i \leq \Lambda_{h,i}, \quad i = 1, 2, \ldots, m$$

Lower bounds on eigenvalues

Old problem:

Temple 1928, Kato 1949, Lehmann 1949, 1950, Harrell 1978, ...

Methods based on FEM:

1. Eigenvalue inclusions [Behnke, Mertins, Plum, Wieners 2000]
   based on [Behnke, Goerish 1994] and [Plum 1997]
2. Crouzeix–Raviart elements [Carstensen, Gedicke 2013]
3. Complementarity based [Šebestová, Vejchodský 2016]

Method 1. Eigenvalue inclusions

Input: Rough lower bounds: $\underline{\lambda}_1 \leq \lambda_1, \ldots, \underline{\lambda}_{m+1} \leq \lambda_{m+1}$,

Algorithm:

- FEM eigenpairs: $\Lambda_{h,i} \in \mathbb{R}$, $u_{h,i} \in V_h$, $i = 1, 2, \ldots, m$
- Mixed FEM problem: $\boldsymbol{\sigma}_{h,i} \in \mathbf{W}_h$, $q_{h,i} \in Q_h$, $i = 1, 2, \ldots, m$
  $\mathbf{W}_h = \{\boldsymbol{\sigma}_h \in \mathbf{H}(\operatorname{div}, \Omega) : \boldsymbol{\sigma}_h|_K \in \mathbf{RT}_k(K) \quad \forall K \in \mathcal{T}_h\}$
  $Q_h = \{q_h \in L^2(\Omega) : q_h|_K \in P_k(K) \quad \forall K \in \mathcal{T}_h\}$

$$
\begin{aligned}
(\boldsymbol{\sigma}_{h,i}, \mathbf{w}_h) + (q_{h,i}, \operatorname{div} \mathbf{w}_h) &= 0 && \forall \mathbf{w}_h \in \mathbf{W}_h, \\
(\operatorname{div} \boldsymbol{\sigma}_{h,i}, \varphi_h) &= (-u_{h,i}, \varphi_h) && \forall \varphi_h \in Q_h,
\end{aligned}
$$

Method 1. Eigenvalue inclusions

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- FEM eigenpairs: $\Lambda_{h,i} \in \mathbb{R}$, $u_{h,i} \in V_h$, $i = 1, 2, \dots, m$
- Mixed FEM problem: $\boldsymbol{\sigma}_{h,i} \in \mathbf{W}_h$, $q_{h,i} \in Q_h$, $i = 1, 2, \dots, m$
- For $n = 1, 2, \dots, m$ do

$$\gamma = \| u_{h,n} + \operatorname{div} \boldsymbol{\sigma}_{h,n} \|_{L^2(\Omega)}, \quad \rho = \underline{\lambda}_{n+1} + \gamma$$

$$\mathbf{M}_{ij} = (\nabla u_{h,i}, \nabla u_{h,j}) + (\gamma - \rho)(u_{h,i}, u_{h,j})$$

$$\mathbf{N}_{ij} = (\nabla u_{h,i}, \nabla u_{h,j}) + (\gamma - 2\rho)(u_{h,i}, u_{h,j}) + \rho^2 (\boldsymbol{\sigma}_{h,i}, \boldsymbol{\sigma}_{h,j}) + (\rho^2/\gamma)(u_{h,i} + \operatorname{div} \boldsymbol{\sigma}_{h,i}, u_{h,j} + \operatorname{div} \boldsymbol{\sigma}_{h,j})$$

$$\mu_1 \leq \cdots \leq \mu_n : \quad \mathbf{M} \mathbf{y}_i = \mu_i \mathbf{N} \mathbf{y}_i, \quad i = 1, 2, \dots, n$$

If $\mathbf{N}$ is s.p.d. and if $\mu_i < 0$ then

$$\ell^{\text{incl}}_{j,n} = \rho - \gamma - \rho / (1 - \mu_{n+1-j}) \leq \lambda_j, \quad j = 1, 2, \dots, n.$$

end for

$$\ell^{\text{incl}}_{j} = \max\{\ell^{\text{incl}}_{j,n},\ n = j, j+1, \dots, m\} \leq \lambda_j, \quad j = 1, 2, \dots, m$$

Method 2. Crouzeix–Raviart elements

Crouzeix–Raviart finite elements

$V_h^{\mathrm{CR}} = \{v_h \in P_1(\mathcal{T}_h) : v_h \text{ continuous in midpoints of all } \gamma \in \mathcal{E}_h\}$
Find $0 \neq u_{h,i}^{\mathrm{CR}} \in V_h^{\mathrm{CR}}$, $\lambda_{h,i}^{\mathrm{CR}} \in \mathbb{R}$:

$$(\nabla u_{h,i}^{\mathrm{CR}}, \nabla v_h) = \lambda_{h,i}^{\mathrm{CR}} (u_{h,i}^{\mathrm{CR}}, v_h) \quad \forall v_h \in V_h^{\mathrm{CR}}.$$

Lower bound (no round-off errors)

$$\ell_i^{\mathrm{CR}} = \frac{\lambda_{h,i}^{\mathrm{CR}}}{1 + \kappa^2 \lambda_{h,i}^{\mathrm{CR}} h_{\max}^2} \leq \lambda_i \quad \forall i = 1, 2, \ldots$$

where

- $\kappa^2 = 1/8 + j_{1,1}^{-2} \leq 0.1932$
- $h_{\max} = \max_{K \in \mathcal{T}_h} \operatorname{diam} K$

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Crouzeix–Raviart finite elements

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Lower bound (inexact solver: $\mathbf{A}\tilde{\mathbf{u}}_i^{\mathrm{CR}} \approx \tilde{\lambda}_{h,i}^{\mathrm{CR}} \mathbf{B}\tilde{\mathbf{u}}_i^{\mathrm{CR}}$)

$$\tilde{\ell}_i^{\mathrm{CR}} = \frac{\tilde{\lambda}_{h,i}^{\mathrm{CR}} - \|\mathbf{r}\|_{\mathbf{B}^{-1}}}{1 + \kappa^2 \left(\tilde{\lambda}_{h,i}^{\mathrm{CR}} - \|\mathbf{r}\|_{\mathbf{B}^{-1}} \right) h_{\max}^2} \leq \lambda_i \quad \forall i = 1, 2, \ldots$$

where

- $\kappa^2 = 1/8 + j_{1,1}^{-2} \leq 0.1932$
- $h_{\max} = \max_{K \in \mathcal{T}_h} \operatorname{diam} K$
- $\mathbf{r} = \mathbf{A}\tilde{\mathbf{u}}_i^{\mathrm{CR}} - \tilde{\lambda}_{h,i}^{\mathrm{CR}} \mathbf{B}\tilde{\mathbf{u}}_i^{\mathrm{CR}}$

Provided

- $\|\mathbf{r}\|_{\mathbf{B}^{-1}} < \tilde{\lambda}_{h,i}^{\mathrm{CR}}$
- $\tilde{\lambda}_{h,i}^{\mathrm{CR}}$ is closer to $\lambda_{h,i}^{\mathrm{CR}}$ than to any other discrete eigenvalue $\lambda_{h,j}^{\mathrm{CR}}$, $j \neq i$

Method 2. Crouzeix–Raviart elements

Upper bound

- $\mathcal{T}_h^*$ is the red refinement of $\mathcal{T}_h$
- $u_{h,i}^* = \mathcal{I}_{\mathrm{CM}} \tilde{u}_{h,i}^{\mathrm{CR}}$ for $i = 1, 2, \ldots, m$
- $\mathbf{S}, \mathbf{Q} \in \mathbb{R}^{m \times m}$ with entries $\mathbf{S}_{j,k} = (\nabla u_{h,j}^*, \nabla u_{h,k}^*)$ and $\mathbf{Q}_{j,k} = (u_{h,j}^*, u_{h,k}^*)$
- $\mathbf{S}\mathbf{y}_i = \Lambda_i^* \mathbf{Q}\mathbf{y}_i, \quad i = 1, 2, \ldots, m$
- $\Lambda_1^* \leq \Lambda_2^* \leq \cdots \leq \Lambda_m^*$
- $\lambda_i \leq \Lambda_i^* \quad$ for $i = 1, 2, \ldots, m$

Method 3. Complementarity based

- FEM eigenpairs: $\Lambda_{h,i} \in \mathbb{R}$, $u_{h,i} \in V_h$, $i = 1, 2, \dots, m$
- Flux reconstruction: $\mathbf{q}_{h,i} = \sum_{\mathbf{z} \in \mathcal{N}_h} \mathbf{q}_{\mathbf{z},i}$
- Local mixed FEM: $\mathbf{q}_{\mathbf{z},i} \in \mathbf{W}_{\mathbf{z}}$, $d_{\mathbf{z},i} \in P_1^*(\mathcal{T}_{\mathbf{z}})$

$$
\begin{aligned}
(\mathbf{q}_{\mathbf{z},i}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} - (d_{\mathbf{z},i}, \operatorname{div} \mathbf{w}_h)_{\omega_{\mathbf{z}}} &= (\psi_{\mathbf{z}} \nabla u_{h,i}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} \quad && \forall \mathbf{w}_h \in \mathbf{W}_{\mathbf{z}} \\
-(\operatorname{div} \mathbf{q}_{\mathbf{z},i}, \varphi_h)_{\omega_{\mathbf{z}}} &= (r_{\mathbf{z},i}, \varphi_h)_{\omega_{\mathbf{z}}} \quad && \forall \varphi_h \in P_1^*(\mathcal{T}_{\mathbf{z}})
\end{aligned}
$$

where

- $\omega_{\mathbf{z}}$ is the patch of elements around vertex $\mathbf{z} \in \mathcal{N}_h$
- $\mathcal{T}_{\mathbf{z}}$ is the set of elements in $\omega_{\mathbf{z}}$
- $\mathbf{W}_{\mathbf{z}} = \{\mathbf{w}_h \in \mathbf{H}(\operatorname{div}, \omega_{\mathbf{z}}) : \mathbf{w}_h|_K \in \mathbf{RT}_1(K)\ \forall K \in \mathcal{T}_{\mathbf{z}}$ and $\mathbf{w}_h \cdot \mathbf{n}_{\omega_{\mathbf{z}}} = 0$ on $\Gamma^{\mathrm{ext}}_{\omega_{\mathbf{z}}}\}$
- $P_1^*(\mathcal{T}_{\mathbf{z}}) = \begin{cases} \{v_h \in P_1(\mathcal{T}_{\mathbf{z}}) : \int_{\omega_{\mathbf{z}}} v_h \,\mathrm{d}x = 0\} & \text{for } \mathbf{z} \in \mathcal{N}_h \setminus \partial\Omega \\ P_1(\mathcal{T}_{\mathbf{z}}) & \text{for } \mathbf{z} \in \mathcal{N}_h \cap \partial\Omega \end{cases}$
- $r_{\mathbf{z},i} = \Lambda_{h,i} \psi_{\mathbf{z}} u_{h,i} - \nabla \psi_{\mathbf{z}} \cdot \nabla u_{h,i}$

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-(\operatorname{div} \mathbf{q}_{\mathbf{z},i}, \varphi_h)_{\omega_{\mathbf{z}}} &= (r_{\mathbf{z},i}, \varphi_h)_{\omega_{\mathbf{z}}} && \forall \varphi_h \in P_1^*(\mathcal{T}_{\mathbf{z}})
\end{aligned}
$$

- Error estimator: $\eta_i = \| \nabla u_{h,i} - \mathbf{q}_{h,i} \|_{L^2(\Omega)}$
- Lower bound: $\ell_1^{\mathrm{cmpl}} = \left(-\eta_1 + \sqrt{\eta_1^2 + 4\Lambda_{h,1}} \right)^2 / 4$

$$
\ell_i^{\mathrm{cmpl}} = \Lambda_{h,i} \left(1 + \underline{\lambda}_1^{-1/2} \eta_i \right)^{-1}, \quad i = 2, 3, \ldots
$$

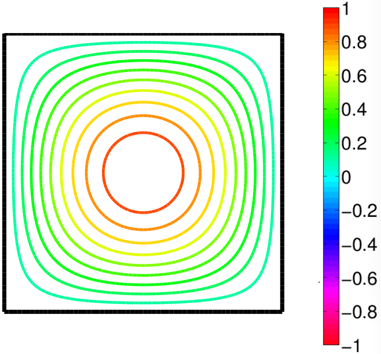
- Provided $\Lambda_{h,i} \leq 2 \left(\lambda_i^{-1} + \lambda_{i+1}^{-1} \right)^{-1}$

Example 1. Square

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega = (0,\pi)^2$$
$$u_i = 0 \qquad \text{on } \partial\Omega$$

The first eigenpair:

$$\lambda_1 = 2, \quad u_1(x,y) = \sin(x)\sin(y)$$



Example 1. Square – speed of convergence

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega = (0,\pi)^2$$
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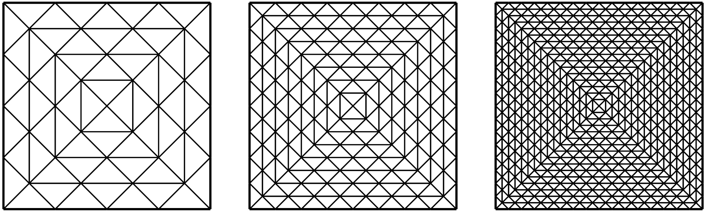
Uniformly refined meshes:



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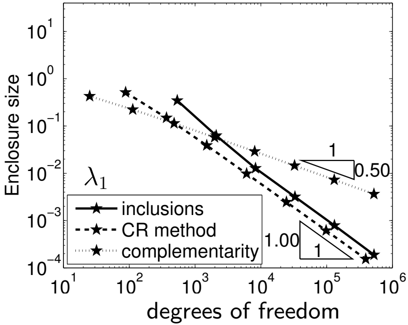
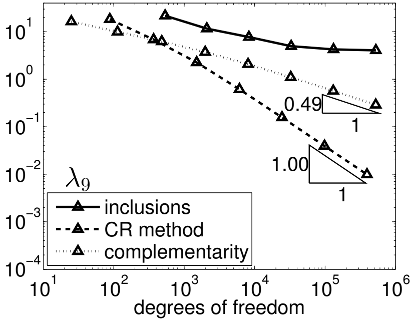
Eigenvalue enclosure sizes:





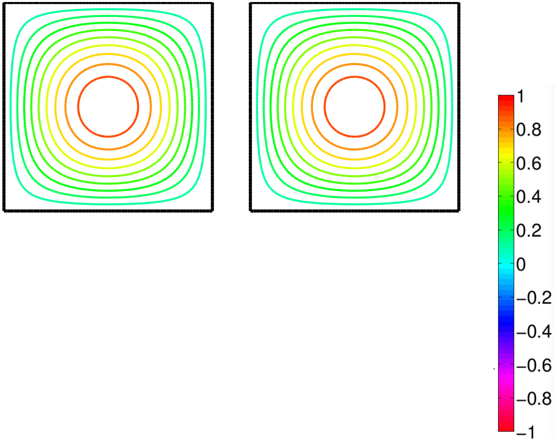
Example 1. Square – best bounds

	a priori lower bound	the largest lower bound	exact eigenvalue	the smallest upper bound
λ_1	1.652893	1.999982	2	2.000006
λ_2	4.132231	4.999429	5	5.000034
λ_3	4.132231	4.999549	5	5.000034
λ_4	6.611570	7.997871	8	8.000100
λ_5	8.264463	9.996874	10	10.000162
λ_6	8.264463	9.996874	10	10.000162
λ_7	10.743802	12.994457	13	13.000281
λ_8	10.743802	12.994457	13	13.000281
λ_9	14.049587	16.991093	17	17.000457
λ_{10}	14.049587	16.991093	17	17.000457

Two squares

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega = (0,\pi)^2 \cup (5\pi/4, 9\pi/4) \times (0,\pi)$$
$$u_i = 0 \quad \text{on } \partial\Omega$$

$\lambda_1 = 2$



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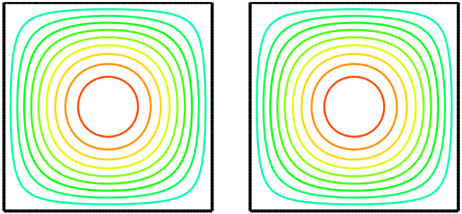
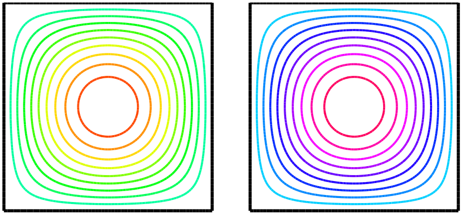
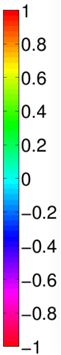


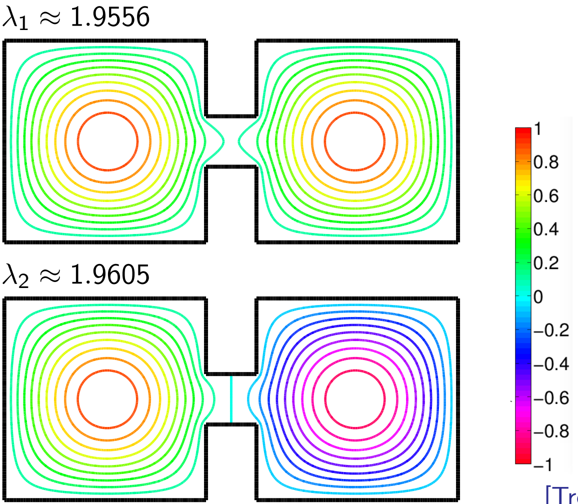
$\lambda_2 = 2$





Example 2. Dumbbell

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega = (0,\pi)^2 \cup [\pi, 5\pi/4] \times (3\pi/8, 5\pi/8)$$
$$u_i = 0 \quad \text{on } \partial\Omega \qquad \cup (5\pi/4, 9\pi/4) \times (0,\pi)$$



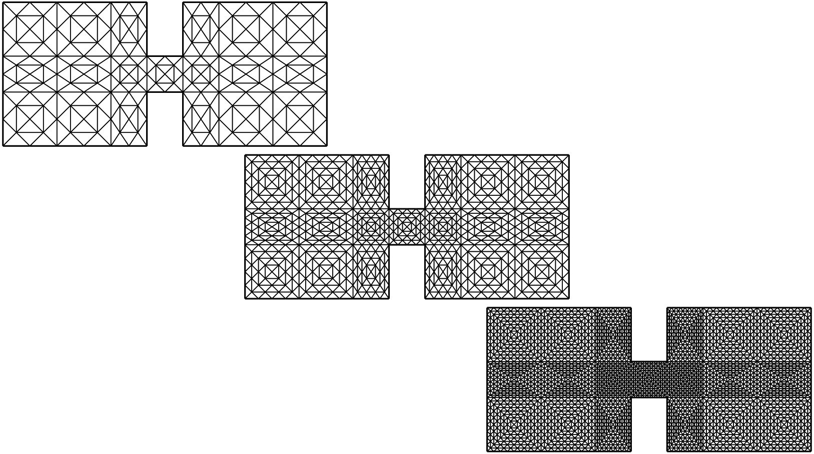
[Trefethen, Betcke 2006]

Example 2. Dumbbell – speed of convergence

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega = \text{dumbbell}$$

$$u_i = 0 \qquad \text{on } \partial\Omega$$

Uniformly refined meshes:



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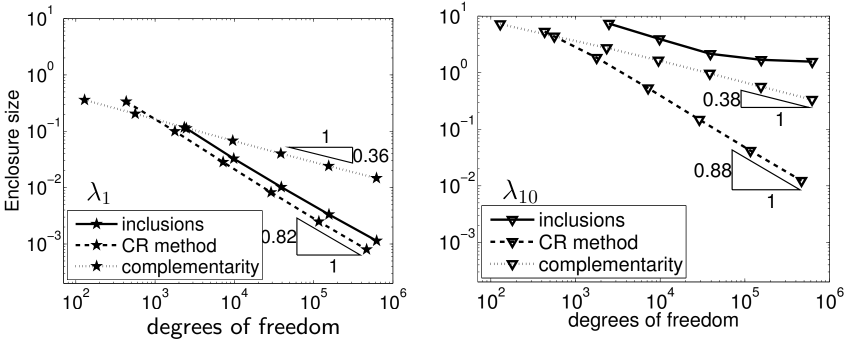

Example 2. Dumbbell – best bounds

	a priori lower bound	the largest lower bound	the smallest upper bound
λ_1	1.197531	1.955284	1.955879
λ_2	1.790123	1.960219	1.960760
λ_3	2.777778	4.798073	4.801187
λ_4	4.160494	4.827345	4.830269
λ_5	4.197531	4.995027	4.996958
λ_6	4.790123	4.995043	4.996972
λ_7	5.777778	7.982102	7.987241
λ_8	5.938272	7.982176	7.987308
λ_9	7.160494	9.347872	9.358706
λ_{10}	8.111111	9.502020	9.512035

Conclusions

	1. Inclusions	2. CR elements	3. Complementarity
convergence	**	***	*
generality	***	*	**
a priori info	*	***	**
DOFs needed	*	**	***
algebraic err.	**	***	***
adaptivity	*	**	***

Conclusions

	1. Inclusions	2. CR elements	3. Complementarity
convergence	**	***	*
generality	***	*	**
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	10*	14*	14*

Thank you for your attention

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